COLLEGE PHYSICS

BY

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GENERAL PHYSICS

CHAPTER I

MOTION : LAWS OF MOTION : MECHANICS

MOTION

1. **Rest and Motion** :—A body is said to be *at rest* when it does not change its position with time. When a body changes its position with time, it is said to be *in motion*. To know whether the position of an object changes with time or not, an absolutely fixed point in space is necessary. But no such stationary point is found in the universe. The earth which is generally taken as the reference position for all bodies is always in motion. It has a rotational motion about its own axis and also a motion round the sun. The sun is also not at rest ; it has a whirling motion amongst the galaxy of stars and the latter are also moving with respect to each other. So in practice an absolutely fixed reference position cannot be obtained and as such absolute rest and absolute motion have no meaning in reality. So the state of rest or of motion of a body is always considered relative to some framework of reference which is taken as fixed.

2. **Kinds of Motion** :—The motion of a body may be either *translatory* or *rotatory* or both. The translatory motion may again be *rectilinear* or *curvilinear*. A body is said to be in *translatory motion* when it moves in such a way that its component parts have such identical motion that the line joining any two points of the body always moves parallel to itself. When the translational motion takes place along a straight line, it is said to be *rectilinear* and when it occurs along a curved path it is called *curvilinear*.

When a body turns about a fixed point or an axis, it is said to be in *rotatory motion*. A stone tied at the end of a string held in the hand and whirled round and the motion of a flywheel about a shaft are typical examples of rotatory motion.

3. **Important Terms** :—

Displacement.—The displacement of a moving body in a given time is its change of position in a definite direction during that time. So displacement is a physical quantity which has a magnitude as well as a direction. Such quantities having both magnitude and direction are known as *Vector* quantities.

Speed—The rate at which a moving body describes its path is called its *speed*. So speed has only *a magnitude and no direction*.

Velocity.—The velocity of a body is its rate of displacement. It may also be defined as the rate of change of position in a definite direction. So velocity has a magnitude as well as a direction.

The velocity of a moving body is said to be *uniform* when it always moves along the same straight line in the same sense describing equal distances in equal intervals of time.

When the velocity of a body is non-uniform but takes place in the same direction, its *average velocity* is given by the total distance passed over by the body in a given interval of time divided by the time interval.

The *unit of velocity* in the C.G.S. system is *one centimetre per second* and that in the F.P.S. system is *one foot per second*.

The *dimensional formula* of velocity is given by LT^{-1} (vide Chapter XIV).

Acceleration.—The acceleration of a body under motion is the rate of change of its velocity. It is said to be *uniform* when equal changes in velocity occur in equal intervals of time. In other cases, it is variable.

Acceleration has a magnitude as well as a direction. The *unit of acceleration* in the C.G.S. system is one centimetre per second per second (1 cm./sec.^{2}) and that in the F.P.S. system is one foot per second per second (1 ft./sec.^{2}).

The *dimensional formula* of acceleration is LT^{-2}(vide Chapter XIV).

4. Equations of Motion :—

(a) Distance traversed in t secs. by a body moving with uniform velocity v.

If the body moves with a uniform velocity v, then, by definition v is the distance traversed by the body in each unit of time.

Hence, in 1 unit of time the distance traversed is v ;
,, 2 units ,, ,, ,, ,, $2v$;
,, 3 ,, ,, ,, ,, ,, $3v$;
and so ,, t ,, ,, ,, ,, ,, tv ;

Therefore, if s be the distance traversed in time t,

$$s = vt.$$

(b) Velocity v acquired in time t secs. by a body moving with a uniform acceleration f.

Suppose u is the velocity at the beginning of an interval of time t. Since the uniform acceleration of the body is f, the velocity of the body is increased in each second by a velocity of f per sec.

$\therefore$ *At the end of* 1 *sec. the velocity is* $u+f$;
" " 2 *secs.* " " $u+2f$;
" " 3 " " " $u+3f$;
and so, " " t " " " $u+tf$;
Hence, $v=u+ft$ ... ... (1)

(c) Distance traversed in t secs. by a body moving with a uniform acceleration f.

Let the body move along a straight line with a uniform acceleration f, and let u and v be its velocities at the beginning and end of any interval t during its motion, s being the distance traversed.

As the acceleration is uniform, and the velocity gradually changes from u to v, the average velocity during the time should therefore be something intermediate between u and v. Let V denote its velocity at the middle of the time considered i.e. at time $\frac{t}{2}$, so that, $V=u+f\times\frac{t}{2}$.

Now, x seconds before this middle instant the velocity is $V-fx$ and in an extremely small interval of time τ there, the distance travelled by the body is practically $(V-fx)\,\tau$. In the same small interval, x seconds later than the middle instant, the distance travelled by the body will be $(V+fx)\,\tau$. The total distance covered by the body during these two equal small intervals is therefore,

$$(V-fx)\tau+(V+fx)\tau=2V\tau$$

which is the same as if the body moved with the velocity V during both these intervals. The whole time interval t can be imagined to be divided into such pairs of equal small intervals equidistant from the middle instant and as for each pair the above reasoning holds, the actual distance (s) to be travelled during time t will be the same as if the body moved with a uniform velocity V from beginning to end. In other words, V represents the true average velocity of the body.

Hence $s=V.t=\left(u+f\frac{t}{2}\right)t=ut+\frac{1}{2}f.t^2$ ... ... (2)

(d) Velocity of a body acquired in distance s under a uniform acceleration f.

From Eq. (1), $v^2=(u+ft)^2=u^2+2uft+f^2t^2=u^2+2f(ut+\frac{1}{2}ft^2)$ [from Eq. (2)] $=u^2+2f.s$ ... ... (3)

(e) The distance traversed in any particular second :—

The distance traversed in the nth sec. $=$ the distance traversed in n seconds—the distance traversed in $(n-1)$ seconds $=$

$$(un+\tfrac{1}{2}fn^2)-\{u(n-1)+\tfrac{1}{2}f(n-1)^2\}=u+\frac{2n-1}{2}f.$$

LAWS OF MOTION

5. Newton's Laws of Motion :—Three fundamental laws of motion were enunciated by Sir Isaac Newton in 1686. They constitute the very basis of the science of Dynamics and so also of the science of Astronomy. These laws are almost axiomatic ; but nevertheless, the exactness with which the positions and motions of all *earthly* and *celestial* bodies can be predicted from calculations based on them, lends the strongest support to the truth of these laws.

(i) The First Law.—*Everybody continues in its state of rest or of uniform motion in a straight line except in so far as it be compelled by any external impressed force to change that state.*

(ii) The Second Law.—*The 'change of motion' i.e. the rate of change of momentum is proportional to the impressed force, and takes place in the direction in which the force acts.*

(iii) The Third Law.—*To every action there is an equal and opposite reaction.*

6. The First Law of Motion :—

The law embodies two aspects :

(1) The first aspect of the law provides us with the fundamental law of inert material bodies, which may be called the **law of inertia**, according to which, inert bodies have no tendency *of their own* to alter their states whether the state be a state of rest or a state of uniform motion in a straight line. The former tendency is referred to as the *inertia of rest*, and the latter, the *inertia of motion.*

(2) The second aspect of the law provides us with the definition of **force**. The idea of force (or what it does) has really been derived from this aspect of the law. As an inert body must continue in its state of rest or in its state of uniform rectilinear motion, as the case may be, unless impressed forces act on it to change its state, we find from this that a force is that which tends to set a body in motion or to alter the state of motion of a body on which it acts.

Force—It is not possible for any inert body to change its state by itself whether the state be of rest or of motion. The change, whatever it is, can only be effected by some external cause which is termed *force*. Hence *a force is that, which acting on a body, changes or tends to change the state of rest or of uniform motion of the body, in a straight line.* The definition of force is evidently derived from Newton's first law of motion.

7. The Second Law of Motion :—

Momentum*.—It is a property a moving body possesses by virtue of its mass and velocity conjointly, and *is measured by the product of mass and velocity.*

For instance, the momentum possessed by a 400-ton train moving with a velocity of $\frac{1}{2}$ mile per minute is equal to the momentum possessed by a 200-ton train moving with a velocity of 1 mile per minute. For, $400 \times \frac{1}{2} = 200 \times 1$.

The great havoc sometimes done by a cyclone is due to the great momentum of the mass of air. The mass of air may be small but its velocity is very great, and so the momentum (*i.e.* mass × velocity) is large.

By taking the hammer at a distance before striking a nail in order to drive it into a piece of wood, a greater velocity of the hammer is acquired and consequently a greater momentum is obtained.

[**N.B.** It must be noticed that **momentum at any instant = mass × velocity** at that instant (and not mass × speed), *i.e.* momentum is a vector quantity and should also be noted that there is no connection between the *momentum of a moving body* and the moment of a force.]

The Units of Momentum.—Unit momentum is the momentum possessed by unit mass moving with unit velocity.

The C.G.S. unit of momentum *is the momentum possessed by a mass of 1 gm. moving with a velocity of 1 cm. per sec.*

The F.P.S. unit of momentum *is the momentum possessed by a mass of 1 lb. moving with a velocity of 1 ft. per sec.*

8. (A) Measurement of Force :—The second law of motion gives us a method of *measuring force.*

Let a *constant force* P continuously act on a particle of mass m and let u be the velocity and f the acceleration at any instant of time during the action of the force. Then, by Newton's second law of motion,

$$\begin{aligned} P &\propto \text{rate of change of momentum } (mu) \text{, of the particle} \\ &\propto (m \times \text{rate of change of } u) \text{, since the mass } m \text{ is constant} \\ &\propto mf \\ \therefore\ P &= k \times mf, \text{ where } k \text{ is constant.} \end{aligned}$$

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* Newton used the expression 'change of motion' instead of 'change of momentum.' 'Motion of a body', he states, 'is the quantity arising out of its *mass* and '*velocity conjointly*'. The idea contained in this expression is this. A marble swiftly moving over the floor has more '*quantity of motion*' in it than when moving slowly. But a heavy roller, though moving slowly, has more 'quantity of motion' in it than the swiftly moving marble. That is, the product of the *mass* and the *velocity* is a measure of the quantity of motion. Evidently, Newton meant by 'motion' what we call *momentum*.

Now, if we choose our unit of force as that which acting continuously produces unit acceleration in unit mass, we have, $m=1$, $f=1$ when $P=1$. Hence k must be equal to 1 and we get,

$$P=mf \qquad \ldots \qquad \ldots \qquad (1)$$

Hence, we may write, force = mass × acceleration. If ds be an infinitely small displacement in an infinitely small time dt, the velocity v, in the language of calculus, will be written as, $v=\frac{ds}{dt}$ and hence the acceleration as $f=\frac{dv}{dt}=\frac{d^2s}{dt^2}$.

So the force will be expressed as $P=m\frac{d^2s}{dt^2}$.

Note.—The equations of motion, the simple deductions of which have already been given, may also be easily deduced using the method of calculus.

If v be the uniform velocity and s the space traversed in t secs., then, since, $v=\frac{ds}{dt}$ or, $ds=vdt$, by integration we get,

$$\int ds=v\int dt \text{ ; or, } s=vt+c, \ (c=\text{constant of integration}).$$

When $t=0$, $s=0$. $\therefore$ $c=0$. Hence, $s=vt$ $\cdots$ (1)

Again if u = initial velocity when $t=0$, v = final velocity when $t=t$, and f = acceleration (uniform) acting on the body,

then $f=\frac{dv}{dt}$; or, $dv=f\,dt$.

Integrating, $\int dv=f\int dt$; or $v=ft+c$.

When $t=0$, $v=u$. $\therefore$ $c=u$. Hence, $v=u+ft$ ... (2)

Again since,

$$v=\frac{ds}{dt}=u+ft, \text{ we get, } ds=udt+ft.dt.$$

$\therefore$ Integrating we get, $s=ut+\frac{1}{2}ft^2+c$.

When $t=0$, the space traversed is zero, i.e. $s=0$.

$\therefore$ $c=0$. Hence, $s=ut+\frac{1}{2}ft^2$ ... ... (3)

Similarly, other relations can also be deduced.

(B) Important Theorems on Force.—All vector quantities like force, velocity, etc. are found to obey some laws which are described below.

(1) The Law of Parallelogram of Forces.—*If a particle is acted on simultaneously by two forces, represented in magnitude and direction by the two adjacent sides of a parallelogram drawn from a point, these forces are equivalent to a single resultant force, represented*

in magnitude and direction by the diagonal of the parallelogram passing through the same point.

Since a force has a magnitude as well as a direction, it is a *vector quantity* and as such it can be represented in magnitude and direction by a straight line.

Let the sides OA and OB of the parallelogram $OACB$ (Fig. 1) represent two forces P and Q in magnitude and direction inclined at acute angle BOA. The diagonal OC represents their resultant R in magnitude and direction. Produce OA to D, and drop CD perpendicular on OD.

Let $\angle BOA=\theta=\angle CAD$. Then we have, $OC^2=(OA+AD)^2+DC^2$

$$=OA^2+AD^2+2OA.AD+DC^2$$
$$=OA^2+AC^2+2\,OA.AD$$
$$(\because\ AC^2=AD^2+DC^2)$$
$$=OA^2+AC^2+2\,OA.AC\cos\theta$$
$$(\because\ AD=AC\cos\theta)$$

or, $\boldsymbol{R^2=P^2+Q^2+2\,PQ\cos\theta}$

If $\theta=90^\circ$, $\boldsymbol{R^2=P^2+Q^2}$

$(\because\ \cos 90^\circ=0)$.

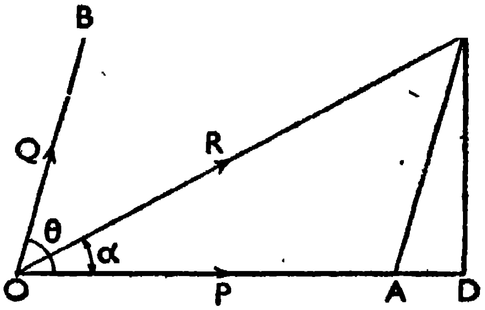


Fig. 1

The *direction of the resultant* is obtained as follows.

Let the resultant R make an angle α with one of the component forces, say, OA. Then $\tan\alpha=\dfrac{CD}{OD}=\dfrac{CD}{OA+AD}=\dfrac{Q\sin\theta}{P+Q\cos\theta}$.

Note.—Just like forces, other *vector quantities like velocity, acceleration, etc. also obey the parallelogram law.*

(ii) Triangle of Forces.—*If three forces acting at a point be represented in magnitude and direction by the sides of a triangle taken in order, they will be in equilibrium.*

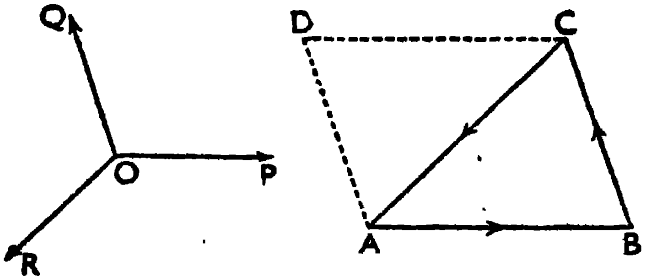


Fig 2

Suppose the forces P, Q and R, acting at O are such that they can be represented both in magnitude and direction by the sides AB, BC and CA respectively of the triangle ABC (Fig. 2). The theorem states that they shall be in equilibrium.

Proof.—Complete the parallelogram $ABCD$. BC and AD being equal and parallel, the forces represented by BC and AD are the same.

By the parallelogram of forces, the resultant of the forces AB and CD is represented by AC both in magnitude and direction. Hence the resultant of the forces AB, BC, and CA is equal to the resultant of forces AC and CA and is thus zero. Hence the forces P, Q, and R, are in equilibrium.

(iii) Converse of the Triangle of forces.—The converse of the triangle of forces is also true. This can be stated as follows : *"If three forces acting at a point be in equilibrium, they can be represented in magnitude and direction by the three sides of a triangle taken in order."*

(iv) Polygon of forces.—*If any number of forces, acting at a point, be such that they can be represented, in magnitude and direction, by the sides of a closed polygon, taken in order they shall be in equilibrium.*

(C). Resolution of a force into two Components at Right Angles to each other.—This is in practice the most important case of the resolution of a force into two components.

Suppose OC (Fig. 3) represents a force P to be resolved into two components one of which is, suppose, in the direction OA making an angle α with OC and the other is perpendicular to OA. In both the above figures, the adjacent sides OM and ON of the parallelogram $OMCN$ gives the desired components of which

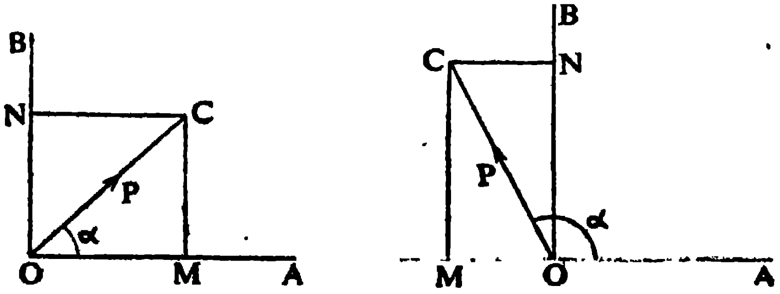


Fig. 3

$$OM = P\cos\alpha \text{ and } ON = P\sin\alpha.$$

9. The Impulse of a Force :—*The impulse of a force acting on a body for any time is the* **product of the force and the time for which the force acts.**

Suppose a particle of mass m moving at any instant with velocity u is acted on with a constant force P for time t. Now $P=mf$, if f be the acceleration produced. If the velocity of the particle at the end of an interval t measured from that instant be v, then

$$P=mf=m\frac{dv}{dt}\text{ ; or, } Pdt=mdv.$$

Integrating within proper limits, $P\int_0^t dt = m\int_u^v dv$

or, $\boldsymbol{P\times t} = m(v-u) = mv - mu$ ... ... (1)

That is, *impulse = change of momentum.*

10. Impulsive Force :—

An impulsive force is a large force acting on a body for a short time, the impulse of the force being finite but the displacement of the body during the short interval negligible. Its whole effect is given by its impulse only.

Suppose the initial position and motion of a body are known when a force begins to act on it. The effect of the force on the body will be generally wholly known, if the final position and motion of the body can be known, *i.e.* if the displacement it causes and the change of momentum it produces be known. In the case of an impulsive force the displacement being negligible, its whole effect will, therefore, be given by the change of momentum it produces on the body, *i.e.* its whole effect is given by its impulse.

For a body initially at rest, $u=0$ and therefore equation (1) of Art. 9 becomes, $\boldsymbol{Pt=mv}$. ... ... (2)

11. The Unit of Force :—From what has been shown above, the **unit of force** may be defined as, (*i*) *that force which acting on a unit mass produces unit acceleration.*

(*ii*) *that force which acting for unit time on unit mass initially at rest creates in it unit velocity.*

(*iii*) *that force which acting on any mass at rest for unit time produces in it unit momentum in the direction of the force.*

Two systems.—There are *two systems* of force-units, (*a*) the absolute, and (*b*) the gravitational. The absolute units do not vary throughout the universe; but the gravitational units of force vary from place to place because they depend upon the value of the acceleration due to gravity g, which varies, though slightly, at different places.

(a) Absolute or Dynamical Units of Force :

Dyne	Poundal
The C.G.S. absolute unit of force is called a dyne, which is the force that can produce an acceleration of one centimetre per second per second when acting on a mass of one gram.	*The F. P. S. absolute unit of force is called a poundal, which is the force that can produce an acceleration of one foot per second per second when acting on a mass of one pound.*

(b) **Gravitational Unit.**—The weight of a body is the force with which it is attracted by the earth. The acceleration with which a body falls freely is denoted by 'g' the value of which in the F.P.S. system is 32·2 ft. per sec. per sec., and in the C.G.S. system the value is 981 cms. per sec. per sec. So,

(*i*) The weight or force, of 1 lb., *i.e.* 1 lb.-wt. acting on a mass of 1 lb., produces an acceleration of 32·2 ft. per sec. per sec.

But the force of 1 poundal acting on a mass of 1 lb. produces an acceleration of 1 ft. per sec. per sec.

∴ **Weight of a pound** (also called a **pound-weight** written as **lb.-wt.**) = **32·2** (i.e. g) **poundals.**

∴ *m pounds-weight (m lbs.-wt.) = mg poundals.*

Hence, a force of 1 poundal = 1/32·2 of weight of one pound
= **wt.** of 16/32·2 oz.
= **wt.** of half an ounce nearly.

(*ii*) Again the weight or force, of one gram, which is expressed as 1 *gm.-wt.* acting on a mass of 1 gram produces an acceleration of 981 cm. per sec. per sec.

But the force of 1 dyne acting on a mass of 1 gram produces an acceleration of 1 cm. per sec. per sec.

∴ **Weight of a gram** (called a **gram-weight.**) = **981** (i.e. g) **dynes.**

∴ ***m grams-weight (m gms.-wt.) = mg dynes.***

Hence, a force of 1 dyne = 1/981 of a gram-weight.

Generally, if m lbs. be the mass of a body, the only force acting on it is its weight, W. So, by substituting W for P, and g for f in the relation $P = mf$, we get, $W = mg$,

i.e. **weight of a body (in dynes) = mass (in grams) × g ;**
(where **g = 981**)

and **weight of a body (in poundals) = mass (in lbs.) × g ;**
(where **g = 32·2**)

[**Note.**—**A force of 1 dyne can be practically realised by the weight W of one milligram ; for $W = mg = (1/1000) \times 981 = 1$ dyne (nearly).**]

The **gravitational unit of force** is the weight of unit mass. Hence—

The C.G.S. gravitational unit of force is a force equal to the weight of a gram.

The F.P.S. gravitational unit of force is a force equal to the weight of a pound.

∴ **The gravitational unit of force = g × absolute unit of force.**

[**Note**—(*i*) The weight of a pound has different values at different places of the earth due to the difference in the values of g.

(*ii*) The formula, $P=mf$, is true only when the force is expressed in absolute units, *i.e.* in poundals or dynes, and not in pounds-weight or grams-weight.

(*iii*) In solving problems using the above formula, reduced the forces into absolute units (if they are given in gravitational units *i.e.* in lbs.-wt. or gms.-wt.) by multiplying with the corresponding value of g.

(*iv*) Finally, if necessary, reduce the forces to gravitational units by dividing by 'g'.]

Relation between a Dyne and a Poundal :—

1 Poundal $=1/32{\cdot}2$ of wt. of a pound,

$=(1/32{\cdot}2)\times 453{\cdot}6$ wt. of 1 gram ($\because$ 1 pound $=453{\cdot}6$ grams.)

$=(981/32{\cdot}2)\times 453{\cdot}6$ dynes ($\because$ 1 gm.-wt. $=981$ dynes)

$=13{,}800$ dynes (in round numbers).

The **dimensional formula** of force is MLT^{-2} (vide Chapter XIV).

Examples

(1) *Express in dynes the force due to 1 ton-weight* ($g=981{\cdot}4$ cms. per sec.^{2}),

Ans. : 1 ton-weight $=2240$ lbs.-wt. $=2240\times 453{\cdot}6$ gms-wt.

$=2240\times 453{\cdot}6\times 981{\cdot}4$ dynes $=9{\cdot}97\times 10^8$ dynes.

(2) *A force equal to the weight of 10 lbs. acting on a body generates an acceleration of 4 ft. per sec. Find out the mass of the body.*

Ans. : Here $P=$ wt. of 10 lbs. $=10\times 32$ poundals ; $f=4$ ft. per sec. per sec.

$\therefore$ By the formula $P=mf$, we have $10\times 32=m\times 4$, or $m=80$ lbs.

(3) *A train weighing 400 tons is travelling at the rate of 60 miles an hour. The speed of the train is reduced to 15 miles per hour in 30 seconds. Find the average retarding force on the train.*

Ans. : 400 tons $=400\times 2240$ lbs. ; 60 miles an hour $=88$ ft. per sec.

15 miles an hour $=22$ ft. per sec.

We have by equation (d), Art. 9, $Pt=mv-mu$.

$$\therefore\quad P\times 30=(400\times 2240\times 22)-(400\times 2240\times 88).$$

$$\therefore\quad P=-\frac{400\times 2240\times 66}{30}=-1{,}971{,}200 \text{ poundals.}$$

(4) *On turning a corner a motorist rushing at 45 miles an hour finds a child on the road 100 ft. ahead. He instantly stops the engine and applies brakes so as to stop within 1 ft. of the child (supposed stationary). Calculate the time required to stop the car, and the retarding force.* (*Car and the passenger weigh 2000 lbs.*]

Ans. : Here $u=45$ miles per hour $=66$ ft. per sec.

The final velocity $v=0$, and the distance travelled before the car stops $=100-1=99$ ft.

If f be the acceleration generated by the force, we have, $v^2=u^2+2fs$;

or, $0=66^2+2f\times 99$; where $f=\dfrac{-66^2}{2\times 99}=-22$ ft. per sec.^{2}

Again, $v=u+ft$, or, $0=66-22t$; whence $t=66\div 22=3$ secs. ;

or, the time required to stop the car $=3$ secs.

$\therefore$ The retarding force, $P=mf=2000\times 22=44{,}000$ poundals.

(5) *A constant force acts for 3 secs. on a mass of 16 lbs. and then ceases to act. During the next 3 secs. the body describes 81 ft. Find out the magnitude of the force in lbs.-wt. and poundals. (Acceleration due to gravity = 32 ft. per sec. per sec.).*

Ans. : If the force P acts for t secs., the impulse $P \times t = m(v-u)$,

Here $u=0$; we have $P \times 3 = 16v$ ... ... (1)

After the force ceases to act, the body describes 81 ft. in 3 secs. So the uniform velocity during this period is $v = 81/3 = 27$ ft. per sec.

$\therefore$ From (1), $P = \frac{16 \times 27}{3} = 144$ poundals (or, $\frac{144}{32} = 4{\cdot}5$ lbs.-wt.)

Otherwise thus —

The uniform velocity during the last 2 secs. $= 81/3 = 27$ ft. per sec.
So 27 ft. is the final velocity of the first 3 secs.
Hence, considering the first period of 3 secs. we have,

$u=0$; $v=27$; $f=?$
$v=u+ft$; or, $27 = 0 + f \times 3$; $\therefore$ $f = 27/3 = 9$ ft per sec.^{2}
Hence, $P = mf = 16 \times 9 = 144$ poundals ; (or, 4·5 lbs.-wt.)

12. Pull, Push, Tension and Thrust :—There are several *ways* in which a **force** may be exerted, the most familiar of these being by *pulls* or *pushes*.

A **Pull** is usually applied along some length of a substance, as for example, along a string, or a chain ; and it is said that the string is in a *state of tension*. The *pull* is also spoken of as a *tension*. A pull may, as well, be exerted along a rigid substance say, a rod, etc.

A **Push** cannot be exerted along flexible substances like strings or threads. Pushes can only be applied to rigid substances.

A Push distributed over an area is often spoken of as a **thrust**. If any one presses a stone with his finger, the finger exerts a thrust on the stone tending to push it away.

13. The Third Law of Motion :—

If one body exerts a force on a second body, the second body exerts an equal and opposite force, called the **reaction** of it, on the first. The mutual force (per unit area) between two bodies is known as **stress**. So the third law is also sometimes called the law of **reaction** or the **law of stress.**

This law is a result of experience. It states that the action between two bodies is mutual *The law is true whether the two bodies concerned are at rest or in motion and whether they are in contact or act from a distance.* Since every force must necessarily be accompanied by an equal and oppositely directed reaction, all forces in nature are in the form of stresses between portions of matter.

Newton's third law of motion really gives us an insight into how forces act in nature. The assertion of the law is that forces never exist singly ; whenever they appear, they appear in pairs. If one of them is an action-force, the other is a co-existent equivalent opposite froce to be called its reaction-force. The question then arises "If a force acting on a body has an equal and opposite reaction, why should the body move at all ?" *The body moves because the action and the reaction do not act on the same body or on the same part of the body.* Take, for example, the case of a body falling to the earth from above. The earth exerts a gravitational attractive force on the body which being of small mass moves towards the earth. At the same time, the same body attracts the earth towards itself with an equal force which is here the reaction-force. But this reaction-force acts on the earth and not on the body. The aerth, being very massive, does not appreciably move towards the body under such a small force. Thus, in considering a mechanical problem what is needed at the beginning is to ascertain the particular body whose motion we want to consider, and then look out to ascertain which force, the action-force or the reaction-force, acts on the body.

It should be noted that (*a*) the reaction lasts only so long as the action is present ; (*b*) the action and reaction act on different bodies and never on the same body and so they can never produce equilibrium, because for equilibrium two equal and opposite forces must act on the same body.

14. The Principle of Conservation of Linear Momentum :—

When two or more bodies move under their mutual actions and reactions only, and no external forces act on the system, the sum total of their momenta along any directions is constant. The law holds both for finite and impulsive forces.

Let two bodies *A* and *B* move under their mutual action and reaction only, there being no other external forces acting on them. Then, by Newton's third law of motion, the action of *A* on *B* at any instant is equal and opposite to the reaction of *B* on *A*. Again, so long as there is action, there is also reaction. That is, the time for which the *two forces* (*action and reaction*) act is the same for both. So, the impulses of the two forces are equal and opposite. That is the change of momentum produced in *A* is equal and opposite to the change of momentum produced in *B*. In other words, the total change of momentum of *A* and *B*, taken together, is zero, which means that the sum total of momenta of *A* and *B* along any direction, is unchanged.

The result can be extended to the case of any number of bodies moving under conditions as stated above.

Illustrations.—(i) When a man jumps from a boat to the shore, it is well known that the boat experiences a backward thrust which displaces it away from the shore. It is due to the impulsive force exerted by the man. The change of momentum of the boat caused by the force is equal and opposite to that of the man.

(ii) Motion of a Shot and Gun.—When a gun is fired, the powder is almost instantaneously converted into a gas at high pressure, which by expansive action forces the shot out of the muzzle. The force on the shot at any instant, before it leaves the muzzle, is equal and opposite to that exerted on the gun backwards. The time for which both these forces (action and reaction) act being the same, their impulses are equal but opposite. So the change of momentum of the shot is equal and opposite to that of the gun. But both the shot and the gun being initially at rest, the momentum produced in the shot is equal and opposite to that in the gun.

Suppose m and M are the masses of the shot and the gun respectively, v the velocity with which the shot emerges from the muzzle and V, the recoil velocity of the gun, supposing it to be free to move. Then, $m(v-0)=M(V-0)$; or, $mv=MV$.

Example

A 14 lbs. shot is fired from a gun the mass of which is 2 tons, with a velocity of 1000 ft. per sec. Find the initial velocity of recoil of the gun.

Ans.: Let u be the initial velocity of recoil of the gun. The backward momentum of the gun is equal and opposite to the forward momentum of the shot. Now momentum of the shot $=14\times1000$ ft.-lb.-sec. units.

Momentum of the gun $=(2\times2240)\times u$ ft.-sec. units.

$$\therefore \quad (2\times2240)\times u=14\times1000$$

$$\text{or,} \quad u=\frac{14\times1000}{2\times2240}=3\cdot125 \text{ ft. per sec.}$$

15. Angular Velocity :—*When a body moves on a plane, its angular velocity about any fixed point in that plane is given by the angle that may be imagined to be described per second by the line joining the body to the point, as the body moves.* It is said to be uniform, if equal angles are described in equal times, however small the time interval may be.

If in a time, t, the angle uniformly described be θ°, then the uniform angular velocity ω is given by, $\omega=\theta/t$ *degrees per second.*

But the angular velocity is generally expressed in circular measure, i.e. *radians* per second.

In one complete revolution, four right angles are described and the circular measure of four right angles is 2π radians. Hence, if t be the time for n revolutions $\omega t=2\pi n$, or, $\omega=2\pi n/t$ radians per sec.

If any body makes *n revolutions per minute* (R. P. M.), the number of revolutions per sec. (R.P.S.) is $n/60$

$\therefore$ The angular velocity of the body,
$\omega=2\pi\times n/60=n\pi/30$ radians per sec.

Relation between Linear and Angular Velocity in Uniform Circular Motion.—Let ω be the uniform angular velocity of a particle moving round the circumference of a circle of radius r (Fig. 4). If t seconds be the time for one complete revolution.

$t = 2\pi/\omega$ sec. ($\because$ the angle turned through is 2π radians).

Again, if v be the linear velocity of the particle,

$$t = \frac{\text{circumference}}{v} = \frac{2\pi r}{v}\text{sec.}$$

Hence, $2\pi r/v = 2\pi/\omega$.

or, $v = \omega r$.........(1)

Thus, the linear velocity of any particle of a body rotating about a fixed axis is directly proportional to its direction from the axis of rotation and is obtained by the product of the angular velocity and the distance.

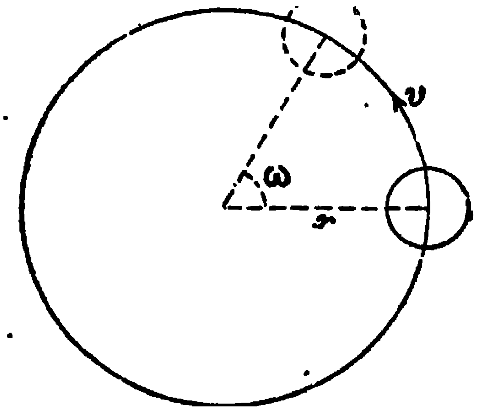


Fig. 4

Examples

1. A circus horse trots round a circular path at a speed of 8 miles an hour, being held by a rope 20 ft. long. Find the angular velocity of the rope.

Ans.: 8 miles an hour $= \frac{5280}{60 \times 60} \times 8$ ft. per sec. [$\because$ 1 mile $= 5280$ ft.]

Here $v = \frac{5280}{60 \times 60} \times 8$, $r = 20$. $\therefore$ $\frac{5280}{60 \times 60} \times 8 = 20\omega$, from eq. (1) above :

or, $\omega = 0{\cdot}58$ radians per sec.

2. A flywheel rotates about a fixed axis at the rate of 150 revolutions per minute ; find the angular velocity of any point on the wheel. What is the linear velocity, if the radius of the wheel is 3½ ft. ?

Ans.: R. P. M. of wheel $= 150$. Angle described per minute $= 150 \times 2\pi$ radians, where $\pi = 22/7$ $\therefore$ Angular velocity, $\omega = \frac{150 \times 2\pi}{60} = 5\pi$ radians per sec.

Linear velocity, $r\omega = 3\frac{1}{2} \times 5\pi = 55$ ft./sec.

16. Rotational Effect of a Force :—

Whenever an unbalanced force acts on a body free to move about a point or particular line (known as the *axis*) the body begins to rotate round that point or axis. This rotational effect of a force is best illustrated by the swinging of a door about the hinges. This effect is measured by a physical quantity which is known as the ***moment of the force***.

MOMENTS

(i) Moment of Mass.—The moment of a mass about a given point or plane is the product of the mass and the distance of the mass, from the point (or plane).

Centre of Mass.—The centre of mass of a given body or a system of bodies rigidly connected together, is a point such that if a plane is passed through it, the *mass-moments* (moments of masses) on one side of the plane is equal to the mass-moments on the other side. The centre of mass of all regularly shaped bodies lies at their geometrical centres.

(ii) Moment of a force about a given point.—The moment of a force about a given point is the product of the force and the length of the perpendicular drawn from the given point upto the line of action of the force. The length of the perpendicular drawn from the given point upon the line of action of the force is called the *arm of the moment*. The moment, therefore, never vanishes unless (*a*) either the force vanishes, or (*b*) the arm of the moment is zero, i.e. the line of action of the force passes through the point about which the moment is taken.

Centre of Gravity.—The centre of gravity of a body is such a point that if a plane is imagined through it the moments of the forces acting on its constituent particles due to gravity on one side of the plane is equal to the similar moments on the other side. It is the point through which the weight of the whole body acts downwards. C.G. of a body is identical with its centre of mass. It is to be realised that the term centre of gravity has no significance in free space where gravity is non-existent. Centre of mass has, however, a significant meaning under all circumstances.

If r be the distance of a constituent point-mass m of a rigid body from any arbitrary point, then the total moment of force for the body due to gravity about that point is given by Σmgr. Now supposing the whole weight of the body is concentrated at its centre of gravity, whose distance is R from the above point, the same moment can be expressed as MgR, where M is the total mass of the body.

$$\therefore \quad MgR = \Sigma mgr\ ;$$

$$\text{or,} \quad R = \frac{\Sigma mr}{M} = \frac{\Sigma mr}{\Sigma m}.$$

Since the body is a rigid one, the same calculation applies to all its co-ordinate axes. So if X, Y, Z be the co-ordinates of the centre of gravity from an arbitrary origin,

$$X = \frac{\Sigma mx}{\Sigma m}, \quad Y = \frac{\Sigma my}{\Sigma m} \text{ and } Z = \frac{\Sigma mz}{\Sigma m}.$$

When the body is homogeneous, we can express these results as,

$$X = \frac{\int x.dm}{\int dm}, \quad Y = \frac{\int y.dm}{\int dm} \text{ and } Z = \frac{\int z.dm}{\int dm}.$$

Example

Calculate the distance of the C. G. of a homogeneous solid cone from its vertex.

Ans. : If ρ be the density of the cone and O its vertex, then the mass of circular elementary strip of thickness dx and of radius y at a distance x from O is $\rho.\pi y^2.dx$ (Fig. 5). So the distance of the centre of gravity from O is given by,

$$X=\frac{\int_0^h x.\rho\pi y^2 dx}{\int_0^h \rho\pi y^2 dx}.$$

But $y=x\tan\theta$.

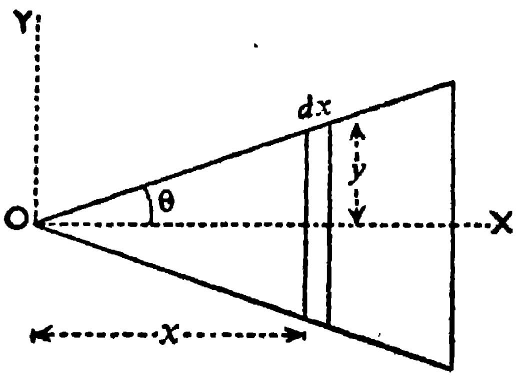


Fig. 5

$$X=\frac{\int_0^h x^3\tan^2\theta\, dx}{\int_0^h x^2\tan^2\theta\, dx}=\frac{h^4/_4}{h^3/_3}=\frac{3h}{4}.$$ (as θ is constant).

(iii) Effect of a Force applied to a Body.—From Newton's first law of motion it follows that the effect of a force acting on a body is to make it move if it is at rest or to change its motion if it is already in uniform motion. Now motion may be either translatory or rotatory. The question then arises whether a force externally impressed on a body will produce translatory or rotatory motion or both. The nature of the resulting motion depends ***on the position or point of application of the force to the body and on the condition in which the body is placed.*** If the body is free and the line of action of the force passes through the centre of mass of the body, the resulting motion will be translatory. If the line of action of the force does not pass through the centre of mass of the body, the force produces translation of the body accompanied by rotation.

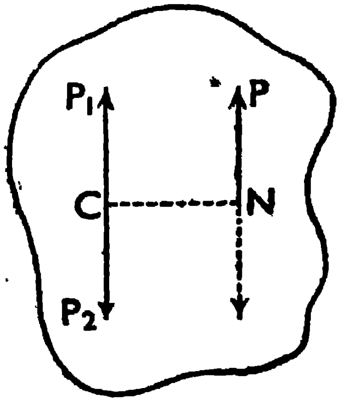


Fig. 5(a)

To illustrate this last point, let us consider a plane lamina (a body of small thickness, e.g. a piece of sheet tin) whose centre of mass, suppose, is at C [Fig. 5(a)]. Let a force P act on the body in the direction shown in the figure and ON be the perpendicular drawn from C upon the line of action of P. To find the effect of P upon the body, imagine two equal and opposite forces P_1, P_2 acting in the same line applied at the point C, each being equal to P and parallel to the line of action of P. These two self-neutralising forces P_1, P_2 do not in any way alter the conditions under which P was applied. P_1 acting along CP_1, causes translation of the body in its own direction, whereas P and P_2 together rotate the body in an *anti-clockwise fashion.* So in considering the effect of a force upon a body, not only the magnitude and direction of the force are important, as pointed out by Newton's second law of motion, but the ***position or point of application of the force to the body is also important.***

(a) Physical meaning of the Moment of a Force about a Point or an Axis.—If a body is fixed at a given point in it or constrained to move about an axis, no translatory motion of the body is possible. Let the plane lamina shown in Fig. 6 resting on a smooth table and fixed at the point O by means of a nail or hinge, represent such a body.

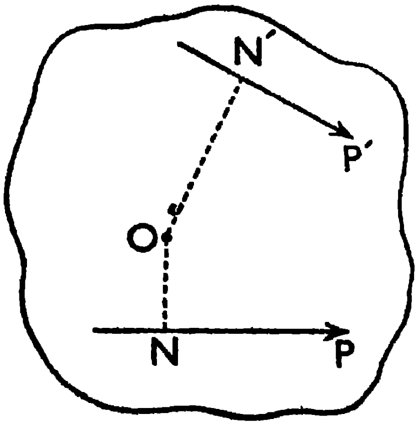


Fig. 6

The effect of a force P acting on the body as shown in the figure would be to cause it to turn about the point O as centre and this effect would not be zero unless (1) the force P were zero, or (2) the force P passed through O when ON would be zero. The magnitude of the **turning effect**, or ***moment***, will depend on (a) the magnitude of P, and (b) the length of the perpendicular ON drawn from O upon the line of action of P. The turning action will be proportional to P, when the arm ON is constant and proportional to the arm ON when P is constant and so it is proportional to the product ($P \times ON$), called the moment of P about O, and is taken as a fit measure of the tendency of P to turn the body about O. The moment of a force is also called a **torque.**

(b) Positive and Negative Moments.—The moment of a force about a point or axis is a vector quantity. In Fig. 6, ***the moment*** of the force P about O, represents ***a turning effect*** tending to rotate the body about O in an anti-clockwise direction. Such anti-clockwise (or contra-clockwise) moments are, by convention, called ***positive moments.*** The moment of P' about the same point O is such that it tends to rotate the body in the clockwise direction. Such a moment tending to cause a clockwise turning effect is called a ***negative moment.***

(c) Algebraic Sum of Moments.—The algebraic sum of the moments of a set of forces about a given point is the sum of the moments of the forces, each moment being given its proper sign, positive or negative, as defined above.

(iv) Principle of Moments.—If a number of coplanar forces acting on a rigid body have a resultant, the algebraic sum of their moments about any point in their plane is equal to the moment of their resultant. If the body is at rest under the action of several forces in the same plane, the algebraic sum of the moments of the forces about any point in their plane is zero. That is, the sum of the contra-clockwise moments is equal to the sum of the clockwise moments.

(v) **The Couple.**—Two *equal and unlike* parallel forces, whose lines of action are not the same, form a *couple*. The perpendicular distance between the lines of two forces forming a couple is called the **arm** of the couple. The **moment** of a couple is the product of one of the two forces forming the couple and the arm of the couple. *A couple acting on a body exerts a turning effect on it* and the moment of the couple, known also as the **torque**, measures this turning effect. An anti-clockwise moment is conventionally taken as positive and a clockwise moment as negative. In Fig. 7, the couple (P,P) having arm AB tends to produce rotation of a body in the clockwise direction and it thus illustrates a negative couple.

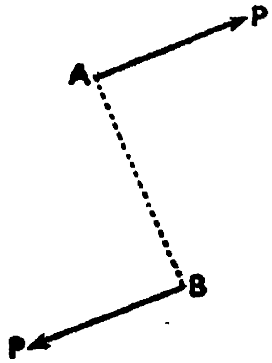


Fig. 7—A Clockwise Couple.

(vi) **Theorems on Couple.**—The algebraic sum of the moments of the two forces forming a couple about any point in their plane is constant and equal to the moment of the couple. The effect of a couple on a rigid body is unaltered, if it be transferred to any plane parallel to its own, the arm remaining parallel to its original direction. Any number of couples in the same plane acting upon a rigid body are equivalent to a single couple whose moment is equal to the algebraic sum of the moments of the couples. A single force and a couple acting in the same plane upon a rigid body cannot produce equilibrium. *To balance a couple, a couple of equal and opposite moment acting in the same plane or in a parallel plane is necessary.*

17. Circular Motion.—*If a particle is constrained to move in a circular path of radius r with a uniform speed v, it must have at any instant an acceleration of magnitude v^2/r directed towards the centre of the path.*

Let a particle be constrained to move along a circular path of centre O and of radius r with a uniform speed v (Fig. 8). At any point P of its path, the velocity is v along the tangent PT. After an infinitely small time t, the position of the particle being P', its velocity should be v along the new tangent $P'T'$. Join P and P'—to the centre and produce OP' to B intersecting PT at A. Now if the particle P were left to its inertial motion along PT without being subjected to any other external force, it would have reached the point A in time t, where $PA = vt$. The velocity v (along AT) at A could then be resolved into two components $v \cos \theta$ acting along AB parallel to $P'T'$ and $v \sin \theta$ acting radially outwards along OR (*i.e.* AR). Since t is very small, θ is also very small and as such A, P and P' are very close points. So $v \cos \theta = v$, and $v \sin \theta = v\theta$. Now as the particle is not allowed to move along PT and is rather constrained to follow the circular path, the particle which should have been at A due to its inertial motion, takes up the position P' attended with velocity v along $P'T'$ parallel to AB. So here only the cosine component of the velocity exists. The sine component, $v\theta$, is annulled by applying a radially inward force (hence an acceleration) of suitable magnitude on the particle.

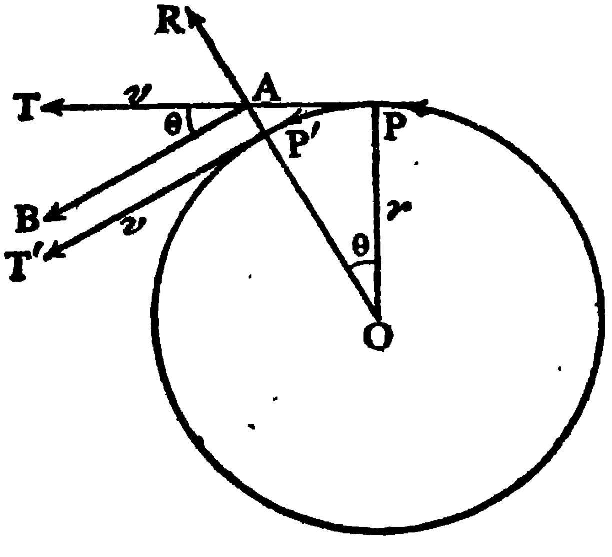


Fig. 8

At P there is no radial component of the tangential velocity, v. But at A, after t second, the radial component of the velocity $= v\theta$.

$\therefore$ The rate of change of outward radial velocity $= \frac{v\theta}{t}$ $= \frac{v}{t} \cdot \frac{\text{arc } PP'}{r} = \frac{v}{r} \cdot \frac{\text{arc } PP'}{t} = \frac{v^2}{r} = \omega^2 r =$ outward radial acceleration ($\because v = \omega r$).

This acceleration is the *centrifugal acceleration* of the particle arising out of its inertial motion and is directed radially away from centre. So in order to annul the effect of this acceleration and to make the particle move uniformly in a circular orbit, an equal and oppositely directed acceleration must act on the particle (due to some external force). This radially inward acceleration is known as the centripetal acceleration and its magnitude is v^2/r or $\omega^2 r$.

Centripetal and Centrifugal Forces.—When a body of mass m moves in a circle of radius r with constant speed v, it is always subject to an acceleration v^2/r directed to the centre of the path. Obviously then, there must be a force mv^2/r constantly acting *on the body* towards the centre of the path to constrain it to move in a circle. This force is known as the **centripetal** (Latin *peto*, to seek) **force**.

By Newton's third law of motion, an equal and *opposite* force, its reaction, is called into play. This force of reaction acts on the body at the centre and is directed away from the centre. It is known as the **centrifugal** (Latin *fugio*, to fly) **reaction**.

The **centripetal** (*i.e.* centre-seeking) force is exerted on the revolving body by another body at the centre **towards** itself, along the radius while the **centrifugal** reaction is exerted by the revolving body on the body at the centre and is directed **away** from the centre, the magnitude being equal but the direction just opposite. The centripetal force, in nature, may arise in different cases due to different reasons, namely, mechanical tension, gravitational force of attraction, magnetic or electric forces, etc.

The centrifugal reaction is sometimes loosely called the *centrifugal force*. But as has been already indicated, the latter is the force due to the centrifugal acceleration arising out of the inertial motion of the body moving uniformly in a circle.

(i) Take, for example, the case of a man whirling in a circle at a constant speed v, a stone of mass m tied to one end of a string, the other end, at a distance r, being held by him (Fig. 9). A centripetal force mv^2/r continuously acts on the stone towards the centre of the circular path. The centrifugal reaction acts on the body at the centre, *i.e.* the hand, it being equal in a magnitude but just opposite in direction. This is experienced by the hand and the man thinks as if the stone will fly outwards if he releases his grip. The tension T of the string is equal to either of these two forces and is given by $T=\frac{mv^2}{r}$.

It should, however, be noted that if the string be released or cut all on a sudden, then the rotating body flies off tangentially to the circular path and not away from the centre along the radius. This is because, as soon as the string is cut, the centripetal force, ceases to act and the motion of the stone continues due to inertia and takes place in the direction in which the stone was moving at the instant, *i.e.* in a tangential line. As soon as the centripetal force goes, the velocity component responsible for the centrifugal acceleration together with the other component, maintains the constancy of the tangential velocity in magnitude and direction.

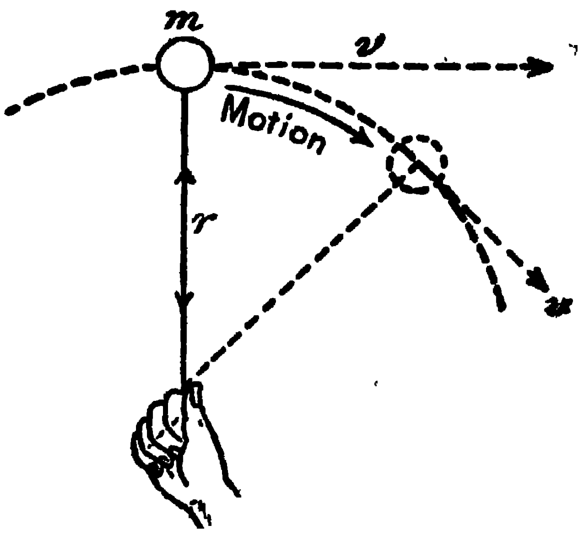


Fig. 9

N.B. ***Every cyclist must have noticed that the mud from a bicycle tyre flies off tangentially when there is no sufficient adhesive force (=centripetal force) between it and the tyre to keep it moving in a circle.***

(ii) A bucket containing water may be swing round in a vertical plane without the water falling down, if the motion is rapid enough. When the bucket is upside down, the weight of the bucket and water acting downwards is balanced by the centrifugal force acting vertically upwards.

Example

A stone of weight 1 lb. swings round in a circle at the end of a string 4 ft. long and takes ½ second for every complete revolution. Calculate the stretching force in the string.

Ans.: The magnitude of the stretching force $=\dfrac{mv^2}{r}$.

Velocity of the stone, $v=\dfrac{\text{distance}}{\text{time}}=\dfrac{2\pi r}{t}$.

$$\therefore \quad \frac{mv^2}{r}=\frac{m\times 4\pi^2 r^2}{r\times t^2}=\frac{1\times 4\times 9{\cdot}87\times 4}{1/4}=631{\cdot}68 \text{ poundals}$$

$$=\frac{631{\cdot}68}{32{\cdot}2}\text{ lbs.-wt.}=19{\cdot}61\text{ lbs.-wt.}$$

(iii) Motion of a body in a vertical circle.—Suppose a body of mass m is rotated in a vertical circle with the help of a string of length l, to one end of which the body is attached. Let O be the centre of the circle of radius l, so described. The body in course of each rotation will once occupy the lowermost position at B when its kinetic energy is maximum and so also its velocity. When the body occupies the uppermost point of the vertical circle, its potential energy is maximum and the $K.E.$ is minimum. So its velocity is also minimum. Thus the velocity of the body during rotation is non-uniform.

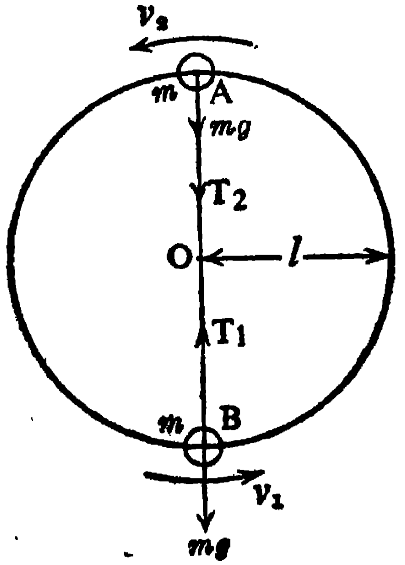


Fig. 10

Let v_1 and v_2 be these maximum and minimum velocities and T_1 and T_2 be the tensions in the string while the body is at B and A respectively (Fig. 10).

$$\text{Then} \quad T_1-mg=\frac{mv_1^2}{l} \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

$$\text{and} \quad T_2+mg=\frac{mv_2^2}{l} \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

When the body is at A, its total energy is,

$$E=\tfrac{1}{2}mv_2^2+2mgl$$

the potential energy being measured with respect to the position B.

When the body is at B, its energy is wholly kinetic.

So, $E_B = \frac{1}{2} mv_1^2$.

$\therefore$ From the principle of conservation of energy,

$$\tfrac{1}{2} mv_1^2 = \tfrac{1}{2}mv_2^2 + 2mgl \;;$$

or, $v_1^2 - v_2^2 = 4gl$ ... ... (3)

The velocity (v_2) of the body at A may have a limiting value below which the tension on the string vanishes, i.e. $T_2 = 0$, and the string becomes slack. This limiting value of v_2 is obtained from (2),

When $mg = mv_2^2/l$; or, $v_2^2 = gl$.

$\therefore$ Substituting in (3), $v_1^2 = 5gl$ ... ... (4)

So the horizontal velocity with which a body, freely suspended from a string of length l, must be projected to describe a vertical circle is given by, $v_1 = \sqrt{5gl}$.

17(a). SOME MORE ILLUSTRATIONS OF CENTRIPETAL AND CENTRIFUGAL FORCES :—

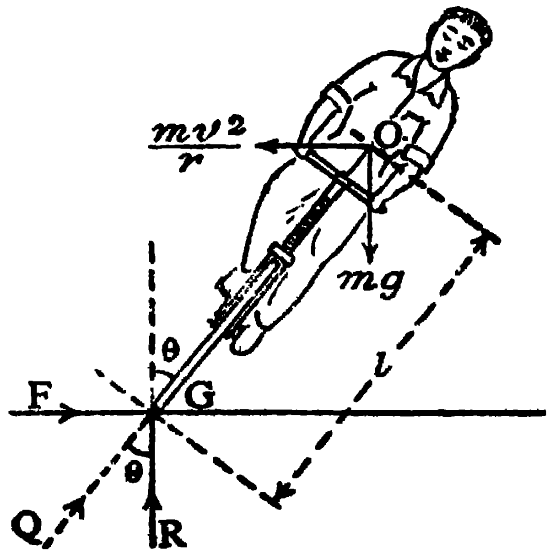


Fig. 11

(i) **Motion of a Bicycle in Curved Path.**—Motion of bicycle rider in a circular path is also an example of the ***centripetal*** and ***centrifugal force.*** A cyclist turning a corner has to incline his body inwards, i.e. towards the centre of the curved path (Fig. 11) for a safe journey.

At that time the forces acting are, (a) mg, the total weight of the machine and the rider, acting vertically downwards through O, the $C.G.$ of the system ; (b) the centrifugal force mv^2/r, acting horizontally through O, where r is the radius of the curved path and v the speed ; (c) the reaction Q of the ground acting at G directed along GO. This force provides the two component forces, a force F, being the horizontal component which acts along the ground and the component R which acts vertically to the ground. The couple formed by R and mg is balanced by the couple formed by F and mv^2/r.

The higher the speed, the greater will be the centrifugal force and so the couple due to it, and the rider will have to bend his body more *inwards* to increase the opposite couple.

Inclination of the Cycle.—If θ = *inclination of the cycle with the vertical*, l = distance *along the cycle* from O, the $C.G.$ of the system, to the ground G, the couple formed by the centrifugal force being equal to that formed by the weight of the system, for a steady motion,

$$\frac{mv^2}{r}\times l\cos\theta = mg\times l\sin\theta\ ;\quad \text{or, } \tan\theta = \frac{v^2}{r.g}.$$

Thus for a given value of r, when v increases, θ must increase. So if a cyclist rides with great speed along a curve of small radius he must incline inwards to the required extent to avoid a fall. Side-slip shall occur, if θ is either too large or too small for the speed v for a given value of r.

Example

A cyclist is describing a curve of 50 ft. radius at a speed of 10 miles per hour. Find the inclination to the vertical of the plane of the cycle, assuming the rider and the cycle to be in one plane.

Ans. : Use the relation, $\tan\theta = v^2/rg$, of the above article. Here $v = \frac{44}{3}$ ft./sec., $r = 50$ ft., and $g = 32$ ft./sec.^{2}. Hence, $\tan\theta = \frac{44\times 44}{3\times 3\times 50\times 32}$; or, $\theta = 7°40'$.

(ii) The Banking of Tracks.—(*a*) A racing track for motor cars is constructed in such a way that it is banked inwards, such that a stationary car would have a tendency of slipping towards the centre of the track.

In this case the resolved part of the weight of the car along the inclined ground supplies the centripetal force necessary to keep the car moving and the other resolved part normal to the ground balances the upward reaction of the ground.

(*b*) While a railway line takes a bend, the outer rail is placed a little higher than the inner one, so that a train moving on it may have its floor inclined to the horizontal.

The wheels of the carriage are provided with flanges on the inner side for both the wheels in a pair, so as not to allow the wheels to move sidewise and cause derailment. If the rails are on the same level while taking a bend, the tendency of the train to move in a straight line produces a pressure on the curved rails, the reaction of which at the flanges supplies the necessary centripetal force for the motion on the curved path. Such huge friction between the flanges and the rails may wear out the flanges quickly. To avoid this, the outer rail is so raised as to reduce the friction between the rails and the flanges to nil, the inclination depending on the radius of the bend as also on the average speed of the train at the bend.

Let $ABCD$ (Fig. 12) be the section of the carriage (mass $=m$) through the line shown by GO, joining the centre of gravity, G, of the carriage, and the centre of the circular track (radius $=r$). Suppose the outer rail is raised over the inner so that the floor AB of the carriage is inclined at the angle θ to the horizontal AE, (when there is no lateral pressure exerted by the flanges of the wheels on the rails) when the carriage is moving with speed v.

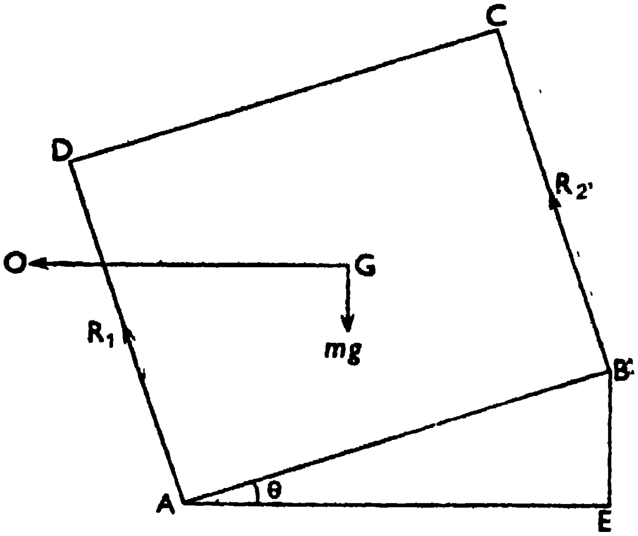


Fig. 12

In such a case, the reactions of the inner and outer rails will be normal to AB and are shown by R_1 and R_2. Resolving vertically,

$$(R_1+R_2)\cos\theta=mg \text{ (the wt. of the carriage)} \quad \ldots \quad \ldots \quad (1)$$

Resolving horizontally,

$$(R_1+R_2)\sin\theta=\frac{mv^2}{r} \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

Dividing (2) by (1), $\tan\theta=v^2/rg$ ... ... (3)

If trains pass round the curve at different speeds, pressures exerted by the flanges on the rails cannot be eliminated completely.

Examples

1. *A railway carriage of mass 15 tons is moving with a speed of 45 m.p.h. on a circular track of 2420 ft. radius. If the rails are 4½ ft. apart, find by how much the outer rail should be raised over the inner rail so that there is no side thrust on the rails.*

Ans.: Use the relation, $\tan\theta=v^2/rg$ of the above article, when $v=45$ m.p.h. $=66$ ft./sec., $r=2420$ ft. and $g=32$ ft./sec.^{2}.

$$\therefore \quad \tan\theta=\frac{66\times66}{2420\times32}=\frac{9}{160}. \quad \text{Since } \theta \text{ is small, } \tan\theta=\theta=\sin\theta=\frac{9}{160}.$$

The height of the outer rail above the inner $=4\frac{1}{2}\times\frac{9}{160}=\frac{81}{320}$ ft. $=3$ inches nearly.

2. *A rail wagon has its c.g. at a height of 3 ft. above the rails, which are 4 ft. apart. What would be the safe maximum speed at which it could travel round an unbanked curve of radius 108 ft.?* (*Gau. Univ.*)

($g=32$ ft./sec.^{2})

Ans. : Here the centrifugal force $\frac{mv^2}{r}$ acts horizontally through the c.g. and mg acts vertically downwards through it.

So the moment of centrifugal force about the outer rail $= \frac{mv^2}{r} \times$ height of c.g.

$$= 3mv^2/r \text{ (clockwise)}.$$

Again, the moment of mg about the same rail $= mg \times$ half the distance between the rails

$$= mg \times 2 \text{ (anti-clockwise)}.$$

∴ At the safe maximum speed these two moments just balance,

$$\therefore \quad 2mg = 3mv^2/r \, ;$$

$$\text{or,} \quad v^2 = \frac{2gr}{3} = \frac{2 \times 32 \times 108}{3}$$

$$\therefore \quad v = 48 \text{ ft. per sec.}$$

(iii) The Centrifugal Drier.—This affords another example of the application of a centrifugal force. This is used in laundries. The wet clothes (which are to be dried) are placed in a cylindrical wire case which is caused to rotate at high speed. The water becomes separated from the clothing and moves off, as the adhesive force between it and the material of the clothings is not sufficient to keep it moving uniformly in a circle.

(iv) Cream Separator.—A given volume of cream has smaller mass than the same volume of skimmed milk, and so a smaller force is required to hold it in a circle of given radius. Hence, if cream particles and milk particles are set in rapid rotation, the milk particles will have greater tendency to move to the outside of the vessel, the cream particles remaining nearer the centre.

(v) Flattening of the Earth.—Initially the earth consisted of a mass of fused matter. Because due to the revolution of the earth about its axis, a centrifugal force is generated which is greatest at the equator and decreases gradually towards and finally vanishes at a pole, the earth bulged at the equator and got flattened at the poles. The model shown in Fig. 13 is commonly used in the laboratory to exhibit a similar effect. It consists of a spindle having some circular rings made of thin metallic strips mounted around it so as to form the surface of a sphere. This body is fixed at

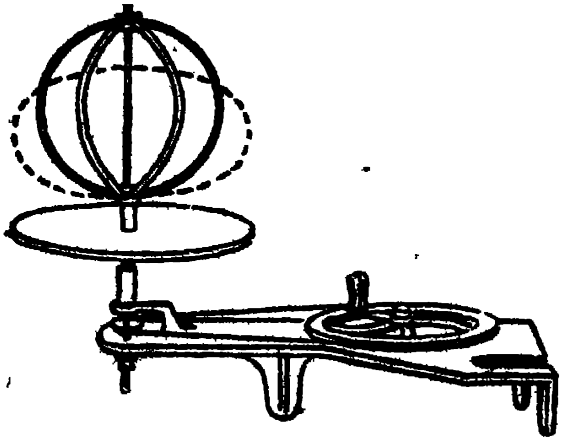


Fig. 13

the bottom to the spindle, while the top of it ends in a collar which fits on the spindle and can slide up and down. When the spindle is rapidly rotated (by fitting it vertically on the axle of a horizontal whirling table) each particle of the strips forming the rings tends to move outwards due to the centrifugal force and causes the collar to slide down the rod. The body takes up a form flattened at the top and bottom *i.e.* along the axis and bulged in the middle as shown by the dotted contour.

(vi) **Watt's Speed Governor.**—The governor of an engine, first designed by James Watt, is a self-acting device by which the supply of power to an engine is automatically regulated such that the mean speed of the engine may remain constant at the rated value or within certain limits, when the load on the engine varies.

The ***conical governor*** (Fig. 14) consists of two heavy metal balls *A* and *B* at the end of two arms hinged at the top of a vertical spindle *V* which is driven by the main shaft of engine and rotates with it. Often the two arms are connected by two links (as shown in the figure) to a sleeve which, while it rotates with the spindle *V*, can slide up and down on it with the rise and fall of the balls. This to-and-fro motion of the sleeve is used to regulate the supply of steam to the engine.

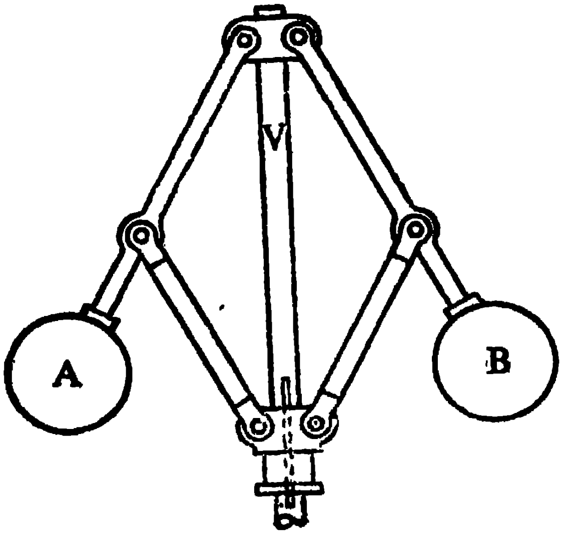


Fig. 14—
A Conical Governor.

Theory of the Conical Governor.—Suppose *A* is a ball revolving about the vertical axis *OY* being suspended from the point O_1 by a slender rod, the joint at O_1 being such as to permit of free angular movement of AO_1 about O_1 in the plane *AOY* (Fig. 15). As *A* revolves at a steady speed, AO_1 describes the surface of a cone, whose vertex is at *O* where AO_1 intersects *OY*. Let *h* be the height of *O* above the level of *A* and let *r* be the radius of the base of the cone. The forces acting on *A* in the plane *AOY* are its weight *W*, acting downwards, the centrifugal force F acting radially outwards and the tension T along AO_1, neglecting the weight of the rod. For steady motion the moments balance one another. Taking moments about O, $F.h = W.r$

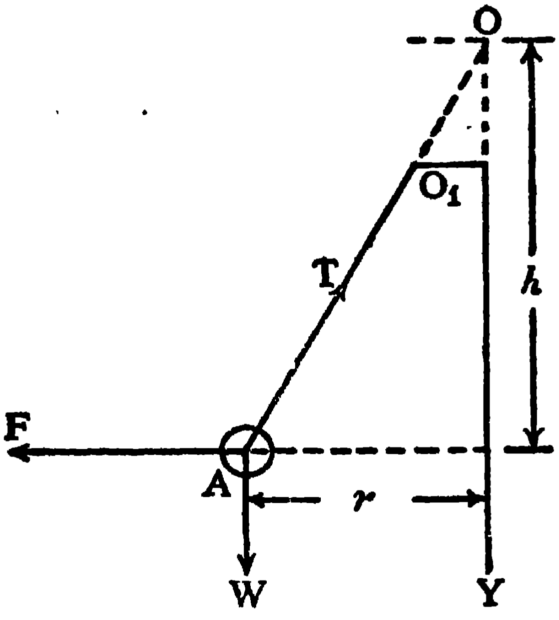


Fig. 15

where $F=\frac{mv^2}{r}, \frac{W}{g}\times\frac{(\omega r)^2}{r}$ $=\frac{W\omega^2 r}{g}$, taking m=mass of A, v=linear velocity and ω= angular velocity of it$=2\pi/T$.

$$\therefore \frac{W.\omega^2 r}{g}\times h=W.r;$$

$$\text{or,}\quad h=\frac{g}{\omega^2}.$$

That is, *the height of a conical governor is inversely proportional to the square of the angular velocity.*

[When the engine speeds up due to any decrease of load, the increased centrifugal force acting on the balls causes the balls to rise up (since the *height* h of the conical governor varies inversely as the square of the speed) and so the sleeve also rises up. If the speed of the engine falls due to any increase of load, the balls fall in level due to the decreased centrifugal force and so the sleeve moves down. The sleeve has a groove (Fig. 14) on it to receive the forked end of a lever through which, and other levers and links, if necessary, the sliding motion of the sleeve is transmitted and converted into the movement of a *valve*, which regulates the supply of steam, gas, or charge, as the case may be, and thereby keeps to speed of an engine constant.]

Note :—The theory of a conical pendulum is the same as above. Its time period T is given by the relation, $\omega^2=\left(\frac{2\pi}{T}\right)^2=g/h$;

$$\text{or,}\quad \omega=\frac{2\pi}{T}=\sqrt{g/h}\,;\quad \text{or,}\quad T=2\pi\sqrt{\frac{h}{g}}.$$

(vii) A Body loses Weight due to the Earth's Rotation.—In its diurnal rotation, as the earth rotates about its polar axis NS, a body on its surface also rotates in a circle with the same angular speed as that of the earth (Fig. 16). As a result of it, the body is subjected to a centrifugal force tending to make the body fly outwards along the radius of its own circle. A part of the weight of the body (which is a force directed towards the centre of the earth arising out of gravitational forces) is used up in counter-balancing the centrifugal force and the remainder appears as the weight of the body. Thus, by the centrifugal forces generated by the rotation of the earth, a body on the earth's surface loses a part of its weight. The loss of weight due to this cause is greatest at the equator and gradually diminishes towards and finally vanishes at each pole.

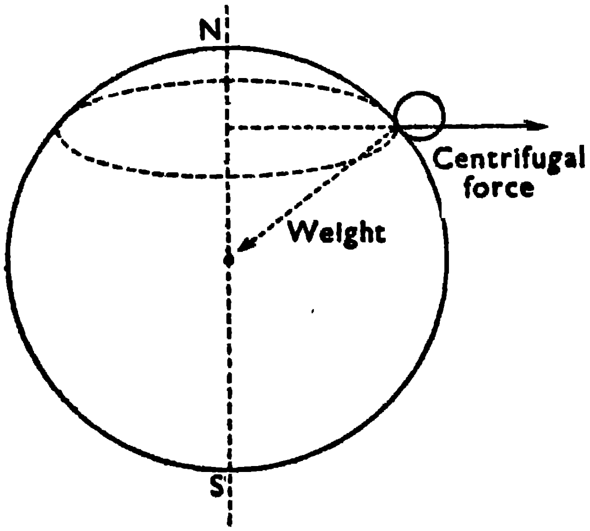


Fig. 16

(viii) The Centrifugal Pump.—It is a rotatory pump with continuous discharge and can be worked over a wide range of speeds. It is suitable where a large volume of liquid is to be discharged against low heads and is widely used in irrigation. In a centrifugal pump pressure energy is imparted to a mass of liquid, water ordinarily, by the rotation of an impeller wheel. The wheel is formed of a number of curved blades (Fig. 17) which entangles the liquid and revolves in a suitable casing. The liquid passes from a suction pipe into the centre or *eye*, as it is called, of the impeller. As the wheel is rotated, say, by an electric motor or any other device, the liquid acquires a high whirling velocity, resulting in an increase of pressure in a radial direction outwards and a tendency to outward flow due to *centrifugal action*. Thus the velocity is reduced and the pressure increased. If the speed of rotation be sufficiently high, the increase in pressure becomes large enough to more than balance the static head (provided it is low) against which it is to act and the flow takes place. This reduces the pressure and causes the fluid to rise in the suction pipe and enter the wheel at its centre. The flow takes the liquid into an outer shell called the *volute* chamber which leads to the *discharge outlet* of the pump.

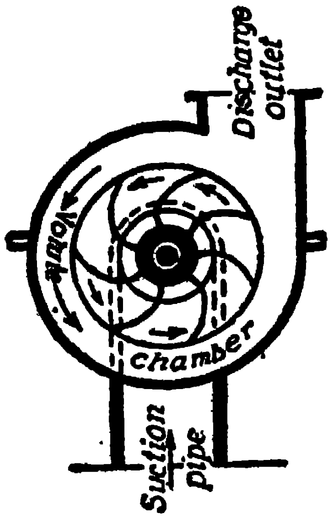


Fig. 17—
A Centrifugal Pump.

Example

Calculate the apparent weight of a body of one ton mass at the equator, the radius of the earth being 4000 miles.

Ans.: The apparent weight$=m(g-v^2/r)$; 4000 miles$=4000\times5280$ ft.; and 1 ton$=2240$ lbs.

$$\therefore\quad \frac{mv^2}{r}=\frac{m\times4\pi^2\times r^2}{r\times t^2}=\frac{m\times4\pi^2\times r}{t^2}=\frac{2240\times4\times9{\cdot}87\times4000\times5280}{(24\times60\times60)^2}\ \text{(here } t=24 \text{ hours)}$$

$$=250{\cdot}2 \text{ poundals}=\frac{250{\cdot}2}{32{\cdot}2}\text{ lbs.-wt.}=7{\cdot}77\text{ lbs.-wt.}$$

Hence the apparent wt. of the body $=2240-7{\cdot}77=2232{\cdot}23$ lbs.-wt.

Questions (Chapter I)

1. Explain clearly how the idea of 'inertia' of a body is deduced from Newton's First Law of Motion.

2. State Newton's Second Law of Motion and explain how it enables you to measure forces.

3. State the laws of motion which are associated with the name of Newton and add explanatory notes leading to the definition of force and of the units for its measurement.

4. State Newton's Laws of Motion and show how from the first we obtain a definition of force and from the second a measure of force.

5. What are the units in common use for expressing a force ?

6. Explain the distinction between the absolute measure and the gravitational measure of force and show how one may be expressed in terms of the other.

Express the weight of 10 kgms. in dynes and the value of a dyne in gm..-wt.

[*Ans.*: 10,000g dynes ; $1/g$ gm.-wt.].

7. Find the total force exerted on a vertical wall by the water from a fire-hose which delivers water with a horizontal velocity of 18 metres per sec. from a circular nozzle of 5 cm. diameter. The water is assumed not to rebound.

[*Ans.*: $6{\cdot}364\times10^7$ dynes.]

8. Find the uniform force required to stop in a distance of 10 yards a 3 ton lorry running on a level road at the speed of 15 m.p.h. Find also the time during which the force acts.

[*Ans.*: 1694 lbs.-wt. ; 30/11 secs.]

9. State Newton's Third Law of Motion and explain it carefully. Show how this leads us to the principle of conservation of momentum.

10. A 10 gm. bullet is shot from a 5 Kgm. gun with a speed of 400 metres/sec. What is the backward speed of the gun ?

[*Ans.*: 80 cms./sec.]

11. Define momentum and impulse. State and illustrate by examples the Principle of Conservation of Momentum.

A 10-gm. bullet is fired from a kilogram gun suspended to move freely. This bullet now enters a block of wood of mass 990 gms. If the speed of the bullet is 500 metres/sec., find the speed of recoil of the gun and the velocity imparted to the block.

[*Ans.*: 5 metres/sec. ; 5 metres/sec.]

12. Explain why a force is needed to keep a body moving uniformly in a circle. Calculate this force in terms of the mass of the body, its uniform speed and the radius of the circle.

13. What would have been the length of the year, if the earth were at half its present distance from the sun ?

[*Ans* : 129 days]

14. Explain the following statement bringing out the scientific principles involved—'If a small can filled with water is rapidly swung in a vertical circle, the water does not fall down.'

15. Write a note on centrifugal and centripetal forces.

16. What are 'Centripetal' and 'Centrifugal' forces and how do they act on a body moving in a circular orbit ? Discuss in details their importance to man.

17. What are 'Centripetal' and 'Centrifugal' forces and how are they directed ? Derive the magnitude of centrifugal force, and give three examples of its application.

A motor-cyclist goes round a circular race course at 120 m.p.h. How far from the vertical must he lean inwards to keep his balance, (*a*) if the track is 1 mile long, (*b*) if it is 880 yds. long ?

$$\left[\textit{Ans.}:\ (a)\ \tan^{-1}\frac{121}{105};\ (b)\ \tan^{-1}\frac{242}{105}.\right]$$

18. Explain why a force is required to keep a body moving in a circle at constant speed. What is the name of this force ? What happens to the moving body when the force is withdrawn.

A ball at the end of a string is whirled at constant speed in a horizontal plane. If the radius of the circle is 4 ft. and the speed of the ball is 10 ft./sec. calculate the magnitude of the radial acceleration.

[*Ans.* : 25 ft./sec.^{2}]

19. Why must a cyclist lean inwards to keep his balance when he is going round a circular course at high speed ? Deduce the relation between the speed, inclination and the radius of the course.

20. What is the proper angle for banking a road around a curve of 200 ft. radius to allow for speeds of 40 m.p.h. ?

[*Ans.* : 28·27°]

21. Write a note on a Centrifugal Pump. (C. U. 1963)

CHAPTER II

WORK : ENERGY : POWER

18. Work :—Work is said to be done *by* or *against* a force, when its point of application moves in or opposite to the direction of the force and is measured by the product of the force and the displacement of the point of application of the direction of the force. The work may also be determined by the product of the displacement and the component of the force in the direction of the displacement.

When a man raises a weight, the force which he exerts does work *against* the force of gravity which acts downwards. Work is done by a horse when it draws a carriage against the force of friction, called into play between the carriage and the ground, which opposes the motion.

Suppose a force P acts on a body at A in the direction AX an it moves to B in a given time (Fig. 18).

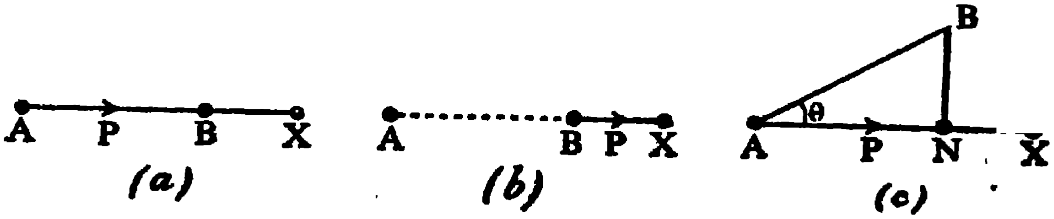


Fig. 18

(*i*) If the displacement AB is in the direction AX [Fig. 18, (*a*)], the work done by P, is $W = P \times AB$, and is called positive work.

(*ii*) If the displacement BA is in a direction opposite to the direction of P [Fig. 18, (*b*)], the displacement measured in the direction of $P = -AB$, and the work done by the force P is, $W = -P \times AB$ and is called negative work. This work is done against P.

(*iii*) If the displacement AB is in a direction different from the line of action of P, say, making an angle θ with AX [Fig. 18, (*c*)], then the displacement measured in the direction of P is, $AN = AB \cos\theta$, where BN is a normal from B on AX. Therefore, work done by P is $W = P \times AN = P \times AB \cos\theta = AB \times P \cos\theta$. That is work = force × component of the displacement of the point of application of the force in the direction of the force = displacement × component of the force along the direction of the displacement.

N.B. It should be noted from above that *no work* is done by or against a force at *right angles* to its own direction, because θ in this case is 90°.

19. The Unit of Work :—*Unit work is done when unit force moves its point of application, in its own direction, through unit distance.* As the unit of force is measured in the two systems, ***absolute*** and *gravitational*, so the unit of work may also be measured in the above two systems :

(*a*) *The absolute unit of work* in the C.G.S. system is one **Erg** ; *it is the work done when a force of one dyne moves its point of application through a distance of one centimetre in its own direction.*

The absolute unit for work in the F.P.S. system is one **Foot-Poundal** ; *it is the work done when a force of one poundal moves its point of application in its own direction through a distance of one foot.*

(*b*) *The gravitational unit of work* in the C.G.S. system is the **Gram-Centimetre** ; *it is the work done in lifting a mass of one gram through a vertical distance of one centimetre.*

[For practical purpose the unit chosen by engineers is the **Kilogram-Metre.**]

The gravitational unit of work in the F.P.S. system is the **Foot-Pound** (ft.-lb.) ; *it is the work done in raising a mass of one pound through a vertical distance of one foot.*

Since the weight of a gram is nearly 981 dynes, 1 gram-centimetre = 981 ergs.

1 erg = 1 dyne-cm. ; 1 foot-poundal = 421,390 ergs.

Note :—The erg being very small, **three additional units of work** (or energy) are used by electrical engineers for practical purposes, *viz.*—

(*i*) **The Joule** = 10^7 ergs.

(*ii*) The **Watt-Hour** = 3,600 Joules = $(3{,}600 \times 10^7)$ ergs *i.e.* one Joule per second for one hour.

(*iii*) The **Kilo-Watt-hour** (kWh) = 3,600,000 Joules = $(1000 \times 3600 \times 10^7)$ ergs, *i.e.* 1000 Joules per second for one hour = 36×10^{12} ergs.

The Kilo-Watt-hour (kWh) is the legal supply unit adopted by the Board of Trade of England and is called the **Board of Trade Unit (B.O.T. Unit).**

Conversion of Foot-Poundal into Ergs :

$$1 \text{ poundal} = \frac{1}{32{\cdot}2} \text{ of the wt. of 1 lb.} = \left(\frac{1}{32{\cdot}2} \times 453{\cdot}6\right) \text{ grams-weight}$$

$$= \left(\frac{1}{32{\cdot}2} \times 453{\cdot}6 \times 981\right) \text{ dynes ; and 1 foot} = 30{\cdot}48 \text{ cms.}$$

$$\text{Hence, 1 foot-poundal} = \frac{30{\cdot}48 \times 453{\cdot}6 \times 981}{32{\cdot}2} \text{ ergs} = (4{\cdot}2139 \times 10^5)$$

ergs approximately.

Relation between the two Units of Work.—Since the gravitational unit of force is g times the absolute unit of force,

Gravitational unit of work $= g \times$ absolute unit of work.

Since the weight of a pound is 32·2 poundals.

1 foot-pound $= 32{\cdot}2$ foot-poundals $= 32{\cdot}2 \times 421{,}390$ ergs
$= 1{\cdot}356 \times 10^7$ ergs $=$ **1·356 Joules.**

Since work $=$ force $\times$ distance, the **dimensional formula of work or energy** is given by ML^2T^{-2} (vide chapter XIV).

20. Power :—The power or activity of an agent, e g. a dynamo or an engine, is the rate at which it does work, *i.e.* the work done by it in unit time, when the work is done continuously.

When we consider the time taken by an agent to perform any work we need to consider what is called the **power** of the agent. The average power used in an operation is the ratio of $\frac{\text{total work done}}{\text{time taken}}$.

The Unit of Power.—(*a*) The C.G.S. **absolute unit** of power, **is one erg per second.**

This being too small for practical purposes, **two additional units** are employed in electrical engineering, *viz.*—

(*i*) The Watt $= 1$ Joule per sec. $= 10^7$ ergs per sec.

(*ii*) The Kilowatt $= 1{,}000$ watts.

(*b*) The **F.P.S. absolute unit** of power is **one foot-poundal per second.**

The **Gravitational unit** of power is **one foot-pound per second.**

The dimensional formula of power is $[ML^2T^{-2}]/[T] = ML^2T^{-3}$.

Horse-power.—It is the British practical unit of power and is commonly used in Mechanical Engineering.

One Horse-Power (H.P.) $= 33{,}000$ ft.-lbs. per min. $= 550$ ft.-lbs. per sec.

In order to find out the working capacity of a horse, James Watt, the inventor of the Steam Engine, carried out an experiment in which a weight of 150 lbs. was lifted from the bottom of a coal pit by a horse through a vertical distance of 220 ft. in one minute. Thus, the work done was $(150 \times 220) = 33{,}000$ ft.-lbs. in one minute $= 550$ ft.-lbs. in one second.

So James Watt adopted this as a unit of power, which he termed one **Horse-power** (H.P.).

21. Conversion of Horse-power into Watts :—

Now 1 foot-pound $= (1{\cdot}356 \times 10^7)$ ergs, and **550 ft.-lbs.** $= \mathbf{(746 \times 10^7)}$ **ergs.**

Hence 1 H.P. = 550 ft.-lbs. per sec. = 746×10^7 ergs per sec.
= **746 watts** ($\because$ 1 watt = 10^7 ergs per sec.)

and 1 Kilowatt = $\frac{1000}{746}$ = 1·34 H.P.

Conversion of Kilowatt-hour into Foot-pounds.—
Since 1 Kilowatt = 1·34 H.P. = (1·34 × 550) ft.-lbs. per sec.
and Work = Power × Time in seconds,
we have, 1 **Kilowatt-hour** = (1·34 × 550) × (60 × 60) ft.-lbs.
= **2,653,200 foot-pounds.**

[**Remember.**—The amount of work done by an average horse is only $\frac{3}{4}$ H.P. The average amount of work done by an active man is $\frac{1}{7}$ H.P. The power of a motor car engines varies from 6 to 30 H.P.; that of a jeep from 20 to 80 H.P.; those of gas engines from $\frac{1}{2}$ to 270, while the power of a large battle cruiser may reach upto 120,000 H.P.].

22. Distinction between Work and Power :—As power is the rate of doing work, it involves a time-unit and its average value is measured by the ratio of the work done to the time taken in doing the work, if the work is done continuously.

That is, power = $\frac{\text{work}}{\text{time}}$. Some examples of power are, 1 H.P. = 550 ft.-lbs. per sec.; 1 watt = 10^7 ergs per sec., etc. Thus from the above, Work = Power × Time.

So 'watt-hour' or 'kilowatt-hour' which are products of 'power' and time 'intervals' are units of work.

Examples

(1) *A man whose weight is 10 stones runs up a flight of stairs carrying a load of 10 lbs. to a height of 20 ft. in 10 seconds. Find the mean power of the man during this interval.*

Ans : 10 stones = 14 × 10 = 140 lbs.
Total work done in 10 secs. = (140 + 10) × 20 = 3000 ft.-lbs.

$\therefore$ The work done per sec. = $\frac{3000}{10}$ = 300 ft.-lbs. So, power = $\frac{300}{550}$ = 0·545 H.P.

(2) *A man weighing 140 lbs. takes his seat in a lift which weighs 2 tons. He is taken to the 3rd floor which is at a height of 57 ft. from the ground floor in 2 minutes. Calculate the work done and the power required in this process.* [1 ton = 2240 lbs.]

Ans. : The total weight of the man and the lift = 140 + 2240 × 2 = 4620 lbs.
The work done in raising 4620 lbs. through 75 ft.

= force × distance = 4620 × 75 = 346,500 ft.-lbs.

The unit of power in the F.P.S. system is one horse-power, which is 550 ft.-lbs. per second. $\therefore$ Power = rate of doing work,

= $\frac{346{,}500}{2 \times 60}$ ft.-lbs. per second = $\frac{346{,}500}{2 \times 60 \times 550}$ H.P. = 5·25 H.P.

23. Mechanical Energy :—The energy required by a body for doing mechanical work is known as its mechanical *energy* ; it is measured by the total work the body can do under the circumstances (*position, configuration, or motion*) in which it is placed.

Obviously, the unit of energy should be identical with that of work. Therefore, *erg, foot-pound, Joule,* etc. which are units of work are also **units of energy**.

The falling water at Niagara is used to drive dynamos which generate electricity. Hence the elevated water of the falls has got energy. A wound spring moves the hands of a watch, and so it has energy. Wind has energy, for work is done by it when it drives a boat.

24. Distinction between Energy and Power :—The *energy* of a body indicates the total amount of work the body, under the circumstances in which it is placed, can do and has no reference to the time in which that work is to be done while *power* denotes the rate at which work is done and is irrespective of the total work done.

25. The Two Forms of Mechanical Energy :—Mechanical energy may have either of the two froms, *potential* and *kinetic.*

(a) Potential Energy.—A body may possess energy by virtue of its position or configuration ; such energy is called potential energy and is measured by the amount of work the body can do in passing from its present position or configuration to some standard position or configuration, usually called the *zero-position* or *configuration.*

(*i*) *Potential energy due to position.*

A lifted weight, like a **Pile-driver**, does work when falling down under the force of gravity. So it has potential energy. Water stored up in elevated reservoirs in municipal water supply, formations of ice on a mountain top, are also similar instances of potential energy. For bodies raised above the surface of the earth, the earth's surface is usually taken as the zero-position.

(*ii*) *Potential energy due to configuration.*

A coiled spring as in the case of a watch or a gramophone, a bent or compressed spring, compressed air, etc. have potential energy for, in coming back to the normal configuration (condition), each one of them can do work.

Potential Energy of a Raised Body.—Consider a body raised above the earth's surface. In this raised *position* the body has potential energy.

Let m = mass of the body ; g = acceleration due to gravity ; h = vertical height through which the body is raised from the ground level.

The potential energy = work done in raising the body $= mg \times h = mgh$. If m be taken in pounds and h in feet, then the potential energy,

P.E. $= mgh$ **ft.-poundals** (where $g = 32{\cdot}2$) $= mh$ **ft.-pounds.**

If m be taken in grams and h in centimetres,

P.E. $= mgh$ **ergs.** (where $g = 981$) $= mh$ **gm.-cms.**

Note :—It must be realised that the potential energy due to gravity is independent of the path along which it is raised to a height 'h', because mg acts vertically downwards and the effective vertical distance h does not depend on the path.

(b) Kinetic Energy.—A body in motion possesses energy due to its motion ; such energy is known as *kinetic energy* and is measured by the amount of work the body can perform against external impressed forces before its motion is stopped. The bullet fired from a rifle, the rotating fly-wheel of an engine, a falling body, a swinging pendulum, a cannon ball in motion have all got kinetic energy.

Kinetic Energy of a body moving with velocity v.—Consider a body in *motion*. At the instant of consideration, let the velocity of the body be v.

Let the mass of the body be m and suppose it is brought to rest by a constant force P resisting its motion, which produces in the body an acceleration $(-f)$, given by $P = mf$.

Let s be the distance traversed by the body before it comes to rest.

We have, $0 = v^2 + 2(-f)s$.

$\therefore$ $f.s = \frac{1}{2}v^2$. Therefore, the $K.\ E.$ of the body = work done till coming to rest $= P \times s = mf \times s = m \times f.s = m\ \dfrac{v^2}{2} = \dfrac{1}{2}mv^2$.

Hence, the kinetic energy of a body moving with a velocity v is equal to half of the product of the mass and the square of the velocity.

This can also be deduced by the calculus method. The velocity v of a moving body is due to a force, $P = mf = m.\ dv/dt$. Now if an elementary displacement ds be considered in the direction of P, the energy involved, dW (evidently kinetic) is,

$$dW = P.ds = m\frac{dv}{dt}.\ ds = m.\ \frac{ds}{dt}.\ dv = mv.\ dv.$$

$$\therefore \text{ The total K. E.} = W = m\int v\,dv = \tfrac{1}{2}\,mv^2 + c.$$

If the body starts from rest, $v = 0$, and $W = 0$.

Hence $c = 0$. $\therefore$ K.E. $= \frac{1}{2}\,mv^2$ ... (1)

If the body initially possesses a velocity u and at the end of the journey acquires the final velocity v, the kinetic energy is given by,

$$W = m \int_u^v v.\,dv = \frac{1}{2} m (v^2 - u^2) \qquad \ldots \qquad (2)$$

Note. If m be taken in pounds and v in feet per second ; the kinetic energy, K. E. $= \frac{1}{2}mv^2$ **ft.-poundals** (lb. $\times$ ft.^{2}/sec.^{2} = ft. $\times$ lb. $\times$ ft./sec.^{2} = **ft.-poundals**)

$= \frac{1}{2}mv^2/g$ **ft.-pounds** (where $g = 32{\cdot}2$).

If m be taken in grams and v in cms. per second,

K.E. $= \frac{1}{2}mv^2$ **ergs** (gm. $\times$ cms.^{2}/sec.^{2} = cms. $\times$ gm. $\times$ cms./sec.^{2} = cms. $\times$ dynes = **ergs**) $= \frac{1}{2}\,mv^2/g$ **gm.-cms.** (where $g = 981$).

N.B. Kinetic energy of rotating bodies will be dealt with in Chapter V.

26. Potential Energy and State of Equilibrium :—*The state of stable equilibrium of* a body corresponds to a **minimum** *of potential energy*, because the centre of gravity of a body, when in stable equilibrium, occupies the lowest possible position and any displacement tends to raise the position of the centre of gravity and thus increases the potential energy of the body. When *the potential energy of the body is* **maximum** any displacement will give rise to a couple tending to move the body further, and thus, in the position *the equilibrium of the body* is **unstable**. Again, when the body is *in the state of* **neutral** *equilibrium*, its *potential energy will remain* **constant** for any small displacement.

27. Transformation of Energy and the Principle of Conservation of Energy :—If a body is at some height above the ground, it has got some gravitational potential energy. If it is now allowed to fall freely through a distance, it loses an amount of potential energy equivalent to the work done by the weight of the body, but gains an equal amount of kinetic energy. Just before the body strikes the ground, it has only kinetic energy equal to the amount of potential energy in the beginning, that is, the potential energy is wholly transformed into kinetic energy. Again, when a clock is wound up, its spring possesses potential energy exactly equal to the amount of work done in winding it up. This potential energy, which is stored up in the spring, is continuously converted into the kinetic energy of the moving hands of the clock as the spring becomes unwound. In the same way the potential energy of the compressed spring in a spring-gun is converted into the kinetic energy of the bullet. These are only different cases of **transformation of energy**, according to which a body may lose mechanical energy in one form, but gain an *equal amount* of mechanical energy in another form. The above examples only illustrate a fundamental principle in **Dynamics,** known as *the principle of conservation of energy*, which may be stated as follows—

"*If a body or a system of bodies move under a conservative system of forces the sum of its kinetic and potential energies remains constant throughout.*"

Some explanation of what has been called a *conservative system of forces* as stated above is necessary at this stage. In Dynamics, a system of forces acting on a body is said to be *conservative* if the work done by the forces, as the body moves, depends only on the initial and the final positions of the body but not on any intermediate position, or on the path of motion, or on the velocity or the direction of motion of the body at any instant. From this definition it is clear that the gravitational force and electrical or magnetic forces are all *conservative.* But, on the other hand, frictional forces are not conservative. Thus for a body sliding down a *rough* inclined plane, the sum of potential and kinetic energies is not constant but gradually diminishes, which can be mathematically proved. Similar is the case when two actual bodies collide. What happens then to the energy which is lost in such cases of non-conservative forces ? This leads us to the formulation of the more general form of the Principle of Conservation of Energy in Science which may be called the **Universal Principle of Conservation of Energy** and may be stated as follows—

"*Energy cannot be created, nor can it be destroyed, but may be transformed from one kind into another or into a number of other forms. The sum total of energies in this universe is constant.*"

This generalised form of the principle enables us to understand what happens to the mechanical energy lost in cases of *dissipative* or *non-conservative forces.* The generalised principle states that the mechanical energy lost is not really destroyed but reappears in some other form or forms in equivalent quantity. Thus, the loss of mechanical energy in the case of a body sliding down an inclined plane or that in the case of two bodies making a collision, as already referred to, is really not a loss, for an equivalent energy reappears in each case mostly as heat and partly as sound. When a falling body touches the ground, the mechanical energy is reduced to zero but is transformed in equivalent quantity mostly into heat and partly into sound. Thus, we find that whatever be the system of forces acting on a body, conservative or non-conservative, the total energy of the system will be found to remain constant, if we take into account all the different forms of energy to which energy is transformable namely, mechanical, thermal, magnetic, electrical, acoustical and light energies. Sometimes it becomes really difficult to trace out the different forms into which energy transforms itself and makes us doubt the validity of the principle but when closely examined, it will be found that the situation arises not due to any defect in the universal character of the principle but due to our inadequate knowledge of the transformations. Consider the various transformations of energy in the case of an ordinary steam engine connected to a dynamo for the generation of electricity. When the coal burns, we get *heat energy*. The heat does work in changing water to steam which then expands. The expanding steam exerts force and causes the piston to move, and thus runs the engine. Thus, the heat energy is transformed into *mechanical energy*, and when the engine drives a dynamo, which generates electricity, the mechanical energy is converted into *electrical energy*. This energy can be transmitted by wires and made to do useful work such as driving tram cars where electrical energy is reconverted into *mechanical energy*; lighting lamps in houses, where electrical energy is reconverted into *light energy*; and in this way various other transformations, may also take place but whatever are the transformations, the guiding principle remains that the total energy of the whole system will be constant.

28. The Principle of Conservation of Energy is obeyed by a Swinging Pendulum :—In the undisturbed position the pendulum acts like a plumb line and hangs vertically. At this position, the centre of gravity of the pendulum which is practically the same as the centre of the spherical bob, lies at the lowest level which may be called its zero or ground level as shown by the point *B* in Fig. 19. The vertical position of the pendulum is its mean position; for, when the pendulum is made to oscillate by drawing it to a side and then let go, it swings about this position with almost equal amplitude on either side of it in each oscillation. When it moves to one side of the mean position, the centre of the bob rises and the bob gains potential energy. When it is at the extreme end position as shown by *C* or *D*, its whole energy is potential there being no kinetic energy.

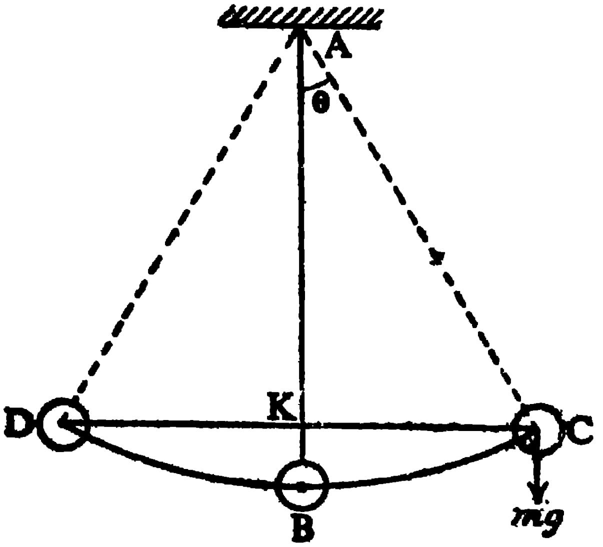


Fig. 19—A Swinging Pendulum

for the bob is at rest there momentarily. The vertical height BK, through which the bob rises when at the extreme position C or D, multiplied by the weight of the bob, gives the potential energy gained.

From the extreme position C or D, when the bob moves towards the mean position B, the potential energy is gradually transformed into kinetic energy till finally the whole of the potential energy is transformed into kinetic energy when the bob reaches the mean position B, its ground or zero level. At this position the whole energy being kinetic, it attains maximum velocity. At positions intermediate between B and C or D, the energy of the bob is partly potential and partly kinetic. On crossing the mean position by virtue of inertia and acquired velocity, when the bob begins to move to the other side the kinetic energy of the bob gradually reduces at a rate in which its potential energy increases till finally the whole of the kinetic energy is again transformed into potential energy at C or D. If there had been no friction due to the air or at the point of support no energy would have been lost by the pendulum and it would oscillate with the same amplitude for ever, once being set into motion. That is, for an ideal pendulum *oscillating in vacuum* the sum of the potential and kinetic energies at any instant should be constant.

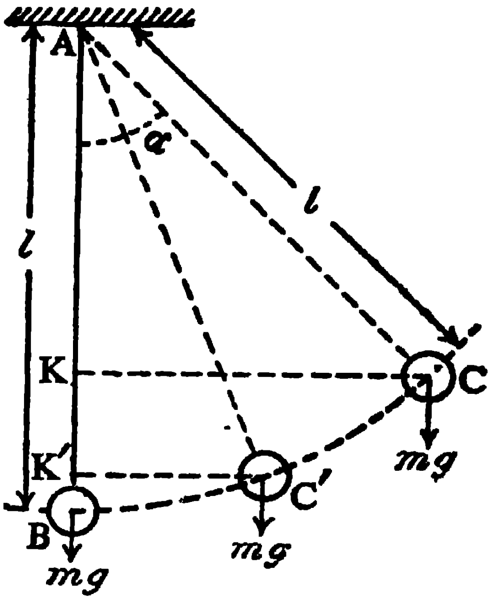


Fig. 20 – A Swinging Pendulum.

Mathematical Proof.—Let the position C denote the extreme end position for a pendulum and C' any subsequent position while moving towards the mean position B (Fig. 20). Draw CK and $C'K'$ perpendiculars on AB.

At C, the total energy (which is wholly potential) $= mg \times BK$. At C', $P.E. = mg \times BK'$, and $K.E. = \frac{1}{2}mv^2 = \frac{1}{2}m \times (2g \times KK') = mg \times KK' = mg \times (BK - BK')$.

$$
\begin{aligned}
\therefore \text{ Total energy at } C' &= P.E. + K.E. \\
&= (mg \times BK') + mg(BK - BK') \\
&= mg \times BK = \text{energy at start} \\
&= mg \times l(1 - \cos \alpha)
\end{aligned}
$$

where l = length of the pendulum, and α = amplitude.

29. Total Energy of a Falling Body is Constant:—The potential energy of a body of mass m at a height h (Fig. 21) above the ground $= mgh$.

When it falls through a distance x, its potential energy at the time $= mg\,(h-x)$.

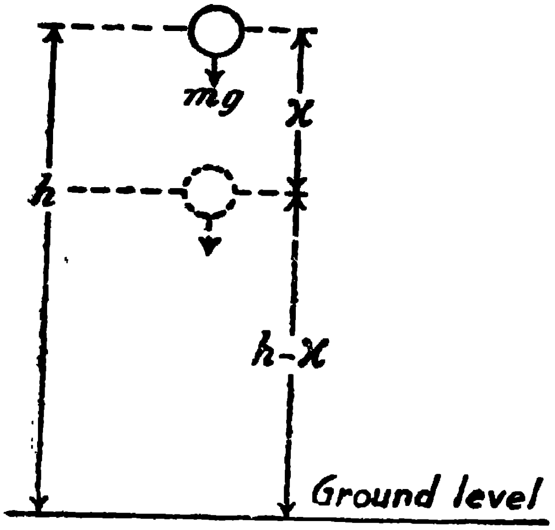


Fig. 21

Its kinetic energy at that instant $=\frac{1}{2}mv^2$ (where v is the velocity acquired during this interval) $=\frac{1}{2}m\times 2gx$ ($\because$ $v^2=2gx$) $=mgx$.

$\therefore$ At that instant, potential energy + kinetic energy

$=mg(h-x)+mgx=mgh=$ potential energy in the beginning.

Hence, neglecting the effects of air resistance, it is seen that the total amount of energy (kinetic + potential) of the body remains constant as it falls. When the body strikes the ground, it is brought to rest and loses its kinetic energy. Then the potential energy is also reduced to zero. The energy, however, is not destroyed. It is converted mainly into heat, the body and the ground being warmer as a result of the impact ; a small part of the energy is also converted into sound energy.

30. A Particle sliding down a Smooth inclined Plane obeys the Principle of Conservation of Energy throughout its Motion :— Consider a particle of mass m, say, which is allowed to slide down a *smooth* inclined plane AB having an inclination α to the ground level BC (Fig. 22). Suppose the particle starts from rest at a point whose height from the ground level is h. The $P.E.$ at this point is mgh and the $K.E.$ is zero, so that the total energy at start $=(mgh+0)$ $=mgh$.

Let v be the velocity acquired by the particle at any instant when the particle has slided down through a distance x along the inclined plane. The acceleration down the plane is given by, $g\cos(90^\circ-\alpha)=g\sin\alpha$. So, we have $v^2=2g\sin\alpha\times x$.

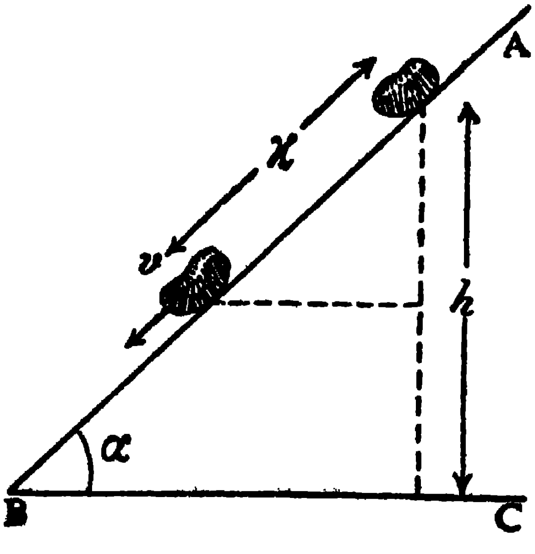


Fig. 22—A body sliding down an incline.

$\therefore$ $K.E.=\frac{1}{2}mv^2=mgx \sin \alpha$. But $x \sin \alpha$ is evidently the vertical height through which the particle has descended ; so its new height above the ground at this position is $(h-x \sin \alpha)$ and therefore, its $P.E.=mg\ (h-x \sin \alpha)$.

$$\therefore \quad K.E.+P.E.=mgx \sin \alpha+mg\ (h-x \sin \alpha)$$
$$=mgh.$$

This is independent of x and is equal to the initial total energy. So the total energy remains constant as the particle slides down the inclined plane and thus the principle of conservation of energy is obeyed by the sliding particle.

31. A Projectile obeys the Principle of Conservation of Energy throughout its Motion :—Let a particle of mass m be thrown from the ground (Fig. 23) with a velocity u at an angle α with the horizon. At start its total energy $=K.E.+P.E.$

$$=\tfrac{1}{2}mu^2+0=\tfrac{1}{2}mu^2.$$

Suppose, v is the velocity of the particle at an angle θ with the horizon, at the instant when it is at some vertical height h above the ground.

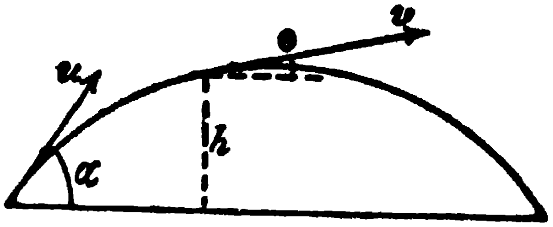


Fig. 23—A Projectile.

Since the only acceleration acting on the particle is due to gravity *i.e.* g vertically downwards, its horizontal velocity all along remains unchanged, and so,

$$v \cos \theta=u \cos \alpha \quad \cdots \quad \ldots \quad (a),$$

and considering the motion of the body vertically upwards, we have,

$$(v \sin \theta)^2=(u \sin \alpha)^2-2gh \quad \ldots \quad (b)$$

Squaring equation (a) and adding it to equation (b), we have,

$$v \ =u^2-2gh.$$

$$\therefore \quad K.E.=\tfrac{1}{2}mv^2=\tfrac{1}{2}m(u^2-2gh)=\tfrac{1}{2}mu^2-mgh.$$

At this position, the vertical height of the particle above the ground being h, its $P.E.=mgh$.

$\therefore$ Total energy $=P.E.+K.E.$ (neglecting air resistance to motion, etc.)

$$=mgh+(\tfrac{1}{2}mu^2-mgh)$$
$$=\tfrac{1}{2}mu^2=\text{initial energy, and is the same at all heights.}$$

32. Perpetual Motion :—The principle of conservation of energy indicates the impossibility of the existence of a "perpetual motion" machine, *i.e.* a machine which, when once set in motion, will continue its motion perpetually without the supply of an equivalent amount of energy from outside. Even when no useful work is done by the machine, the energy, supplied in the beginning, will be gradually used up in overcoming frictional and other resistances and machine will ultimately come to a stop.

33. The velocity of the Bob of a Pendulum at its Lowest Point :—When the bob of the pendulum, of length l cm., is set free from its extreme position C, it moves in an arc of a circle CBD, B being the lowest position (Fig. 24). From C draw a perpendicular CK on AB. At C the bob of the pendulum has potential energy $mg \times BK$, which represents the work done in raising the bob from B to C, *i.e.* vertically through BK. When the bob is released from the position C, it gradually loses its potential energy and gains kinetic energy. At the lowest point B, it loses all its potential energy $mg \times BK$, and the kinetic energy $\frac{1}{2}mv^2$, which it gains, is equal to this.

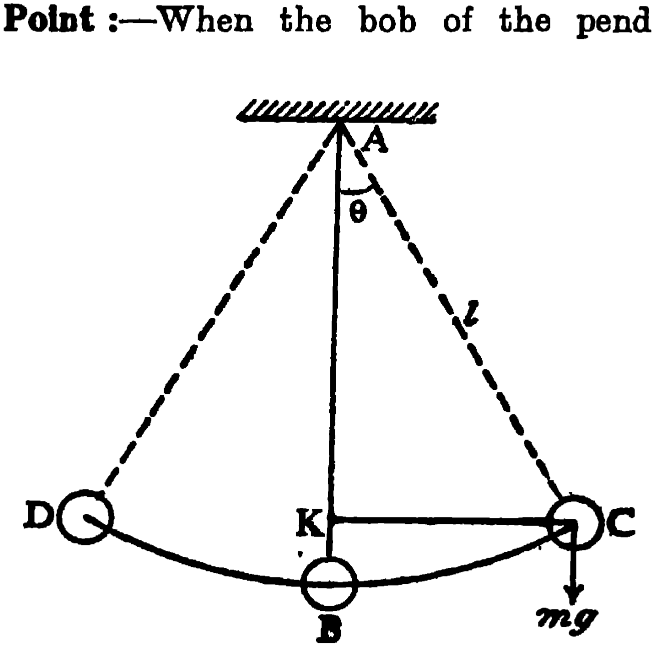


Fig. 24

$\therefore \quad mg.\,BK = \frac{1}{2}mv^2$, where v is the velocity of the bob at B;

or, $g(AB - AK) = \frac{1}{2}v^2$; or, $g\,(l - l\cos\theta) = \frac{1}{2}v^2$;

or, $v^2 = 2gl(1 - \cos\theta)$. $\therefore \quad v = \sqrt{2gl(1-\cos\theta)}$;

or, $v = \sqrt{2gl \times 2\sin^2\frac{\theta}{2}} = 2\sqrt{gl} \times \sin\frac{\theta}{2}$.

Example

The heavy bob of a pendulum is drawn aside so that the string makes an angle of 60° with the horizontal and then let go. Find the velocity with which the bob passes through its position of rest.

Ans. **: (Draw the diagram and proceed as described in the preceding article.)**

$\theta = (90° - 60°) = 30°$, $\therefore \quad v = \sqrt{2gl\left(1 - \frac{\sqrt{3}}{2}\right)}$; or, $v = \sqrt{0{\cdot}268 \times gl}$.

34. Other Forms of Energy :—As already stated, the mechanical energy which a body possesses may be due to either or both of the two forms, kinetic and potential. Besides mechanical energy there are also other forms of energy, *e.g.* heat, light, sound, electrical, magnetic, and chemical energy.

35. The Sun is the Ultimate Source of all Energy :—For the terrestrial people the sun is generally considered to be the ultimate source of all forms of energy. We get considerable amount of energy from solar radiation in the form of heat, light, etc. For example, the energy of the steam engine is derived from coal. Coal again is nothing but wood decomposed and subjected to great pressure and temperature inside the earth for thousands and thousands of years. The energy in the wood is due to the sun's action on trees and plants. When the coal burns, the stored-up potential chemical energy derived from solar radiation is given back as heat and light.

36. Further Examples of Transformation of Energy :—

(i) **Mechanical.**—(*a*) Kinetic to potential.—The bob of a pendulum moving from the normal position (maximum kinetic energy position) to the extreme position of swing. (*b*) Potential to kinetic.—A body falling from a raised position to the earth ; a pendulum returning from the extreme position of swing towards the normal position. (*c*) Kinetic to heat.—Heat produced by rubbing two stones ; a moving wheel stopped by applying brakes. (*d*) Kinetic to sound.—Sound produced when a reed vibrates. (*e*) Kinetic to electrical.—A dynamo.

(ii) **Heat.**—(*a*) Heat to mechanical.—Heat engines. (*b*) Heat to light.—White hot ball ; filament in an electric bulb. (*c*) Heat to sound.—Singing flame. (*d*) Heat to electrical —Thermo-electric phenomena. (*e*) Heat to chemical.—A magnesium ribbon burning in air produces magnesium oxide. (*f*) Heat to mechanical.—The pressure of a confined gas increases when the gas is heated.

(iii) **Light.**—(*a*) Light to electrical.—Photo-electric cell. (*b*) Light to chemical.—Photography.

(iv) **Sound.**—(*a*) Sound to mechanical.—Forced vibration and resonance. (*b*) Sound to electrical.—Telephone transmitter.

(v) **Magnetic.**—(*a*) Magnetic to heat.—Rapid magnetisation and demagnetisation repeated in a specimen of iron. (*b*) Magnetic to mechanical.—Deflection of a needle by magnet.

(vi) **Electrical**.—(*a*) Electrical to mechanical.—Electric motors ; tram cars. (*b*) Electric to heat.—Electric iron ; electric furnace. (*c*) Electric to light.—Electric lamps. (*d*) Electric to sound.—Calling bell ; Telephone. (*e*) Electrical to chemical.—Charging of batteries ; electro-plating. (*f*) Electric to magnetic.—Electro-magnet.

(vii) **Chemical**.—(*a*) Chemical to heat.—Burning of a fuel—petrol, kerosene, coal, etc. (*b*) Chemical to light.—Gas lighting. (*c*) Chemical to electrical.—Voltaic cells. (*d*) Chemical to mechanical. —Explosives.

37. Different Examples of Work done :—Work is measured by the product of the force and the distance through which the point of application of the force moves in the direction of the force.

(i) ***Work done in raising a load vertically upwards.***

If w represents the work done, $w = mgh$; where m is mass of the load and h the vertical height through which the load is raised.

(*ii*) ***Work done in taking a load up along an inclined plane.***

In this case, $w = mg \sin \alpha \times l$, where l is the length of the inclined plane [*vide* Art. 30] and α the inclination of the plane of the horizon $= mg \times l \sin \alpha = mgh$, where h is the height of the inclined plane. Thus, the work done in taking the load up the inclined plane is the same as that required to raise the load m vertically through a height h. Hence, the work done in raising a body to a height h against gravity is independent of the path along which the body is taken and depends only on the vertical height.

(*iii*) ***Work required to generate a velocity v in a body originally at rest.*** $W = P \times S$, where P is a force which generates an acceleration f in a body of mass m and S is the distance traversed by the body in time t. Here $P = mf$ and $S = \frac{1}{2}ft^2$. $\therefore$ $W = P \times S = mf \times \frac{1}{2}ft^2 = \frac{1}{2}m(f^2t^2) = \frac{1}{2}mv^2$, where $v\ (= ft)$ is the velocity acquired by the body after time t starting from rest.

Units of Force, Work and Power

C.G.S. SYSTEM

Quantity	Theoretical unit	Practical unit
Force	*Dyne*	(*i*) *Newton* $= 10^5$ dynes (*ii*) *Gram-weight* $= 981$ dynes (gravitational unit)
Work	*Erg*	(*i*) *Joule* $= 10^7$ ergs. (*ii*) *Kilowatt-hour* $= 36 \times 10^{12}$ ergs (*iii*) Gram-centimetre $= 981$ ergs
Power	One erg per sec. (No special name)	*Watt* $= 1$ Joule per sec. $= 10^7$ ergs per sec.

F.P.S. SYSTEM

Quantity	Theoretical unit	Practical unit
Force	*Poundal*	1 *Pound-weight* $= 32$ poundals (gravitational unit)
Work	*Foot-poundal*	1 *Foot-pound* $= 32$ foot-poundals (gravitational unit)
Power	One foot-poundal per second (No special name)	1 *Horse-Power* (H. P.) $= 550$ ft.-pounds per sec.

Examples

(1) *A rifle bullet of mass 14 gms. travelling at the rate of 36,000 cms. per sec. is just able to pierce a block of wood 21 cms. thick. Find the average force of the bullet while penetrating the wood.*

Ans. : Let F be the force in dynes : then $F\times s=\frac{1}{2}mv^2$;

$$\text{or,}\quad F=\frac{mv^2}{2s}=\frac{14\times(36,000)^2}{2\times21}=432\times10^6 \text{ dynes.}$$

(2) *A train of mass 100 tons is travelling at 30 miles per hour. Calculate the force necessary to bring it to rest (a) in a distance of 120 ft.; (b) in 10 seconds.*

Ans. : (*a*) Let F be the force applied which produces a negative acceleration f to bring the train to rest; then, $F=-mf$ (where m is the mass of the train) ; or $f=-F/m$. We have, $v^2=u^2+2fs$ $\therefore$ $0=u^2-2\frac{Fs}{m}$; or, $Fs=\frac{1}{2}mu^2$.

Here $m=100\times2240=224{,}000$ lbs. ; $u=30$ miles per hour $=44$ ft. per sec.

$$\therefore\quad F=\frac{mu^2}{2s}=\frac{224{,}000\times44^2}{2\times120}=1{,}806{,}933{\cdot}3 \text{ poundals}$$

$$=56{,}466{\cdot}6 \text{ lbs.-wt. (taking } g=32).$$

(*b*) We have, $v=u+ft$; or, $0=u-\frac{Ft}{m}$; or, $Ft=mu$;

$$\therefore\quad F=\frac{224{,}000\times44}{10}=985{,}600 \text{ poundals}=\frac{985{,}600}{32}=30{,}800 \text{ lbs.-wt.}$$

(3) *Find the energy stored in a train weighing 250 tons and travelling at the rate of 60 miles per hour. How much energy must be added to the train to increase its speed to 65 miles per hour ?*

Ans. : Mass $=250$ tons $=250\times2240$ lbs. Velocity $=60$ miles per hour $=88$ ft. per sec.

$\therefore$ The kinetic energy of the train $=\frac{1}{2}\times(250\times2240)\,88^2$ foot-poundals.

$$=2{,}168{,}320{,}000 \text{ ft.-poundals.}$$

Again, 65 miles per hour $=\frac{65\times1760\times3}{60\times60}=\frac{286}{3}$ ft. per sec.

$\therefore$ The K. E. of the train, when the speed is 65 miles per hour

$=\frac{1}{2}\times(250\times2{,}240)\left(\frac{286}{3}\right)^2=2{,}544{,}764{,}444{\cdot}4$ ft.-poundals.

$\therefore$ The energy to be added $=2{,}544{,}764{,}444{\cdot}4-2{,}168{,}320{,}000$

$=376\,444\,444{\cdot}4$ ft.-poundals.

(4) *If clouds are 1 mile above the earth and rainfall is sufficient to cover 1 square mile at sea-level, ½ inch deep, how much work was done in raising the water to the clouds ?*

Ans. : If w lbs. be the mass of rain water and h *ft.* the height of the clouds above the surface, of the earth, the work done in raising w lbs. of water through h ft.

$=w\times h$ foot-pounds. Here $h=1760\times3=5280$ ft.

The volume of rain water $=1$ square mile $\times\frac{1}{2}$ in. $=(5280)^2\times\frac{1}{2\times12}$ cu. ft.

The mass of 1 cubic foot of water $=62{\cdot}5$ lbs. $\therefore$ Mass of rain water $=(5280)^2\times(1/24)\times62{\cdot}5$ lbs. $\therefore$ The work done $=(5280)^2\times\frac{1}{24}\times\frac{125}{2}\times(5280)=(5280)^3\times\frac{1}{24}\times\frac{125}{2}=383{\cdot}328\times10^6$ foot-pounds.

Questions (Chapter II)

1. Show that if a piston is moved along a cylinder against a constant pressure, the work done in a stroke is equal to the product of the pressure and the volume swept out by the piston. Explain clearly the units in which the work will be given by this calculation.

[Pressure=force on unit area. Work done=force × distance=(pressure × area) × distance through which the piston moves=pressure × volume swept out. The work is expressed in ergs, if pressure is measured in dynes per sq. cm. and volume in c.c.]

2. What is the work done when a weight of 500 kilograms falls through a height of 50 metres and is then stopped? Assume the normal value of gravity. [*Ans.*: $24{,}525 \times 10^8$ ergs.]

3. How much power is required to pump water at the rate of 90 litres per minute to a height of 20 metres? [*Ans.*: 294·3 watts.]

4. Water is pumped up from a well through a height of 30 feet by means of a 5 horse-power motor. If the efficiency of the pump is 85%, find in gallons the quantity of water pumped up per minute. (1 gallon of water weighs 10 lbs.) [*Ans.*: 467·5 gallons].

5. An engine is employed to pump 6,000 gallons of water per minute from a well through an average height of 21 feet. Find the horse-power of the engine, if 45% of the power is wasted. [*Ans.*: 69·42]

6. What is the potential energy of the water which fills a cubical tank of each side 10 ft. and whose base is 20 ft. above the ground?

[*Ans.*: $1{\cdot}56 \times 10^6$ ft.-lbs.]

7. A railway train is going up-hill with a constant velocity. What is the source from which the energy of the train is supplied?

Describe the various transformations of energy that go on in this case.

[*Hints*: The energy of the train is derived primarily from the burning coal. This is utilised in running the train against friction and air resistance, and also raising the train up-hill against the force of gravity and thus work is done. The energy of the coal is derived from the sun. So the sun is the ultimate source of energy.]

8. A solid mass of 100 gms. is allowed to drop from a height of 10 metres. Calculate the amount of kinetic energy gained by the body, g being 980 cms. per sec.^{2} [*Ans.*: 98×10^6 ergs]

9. A shot travelling at the rate of 200 metres per second is just able to pierce plank 2 inches thick. What velocity is required to pierce a plank 6 inches thick? [*Ans.*: $200\sqrt{3}$ metres per sec.]

10. A mass of 10 lbs. falls 10 ft. from rest and is then brought to rest by penetrating 1 ft. into sand; find the average thrust of the sand on it. [*Ans.*: 110 lbs.-wt.]

11. Distinguish between poundal and pound-weight.

Prove that in the case of a body falling freely under gravity, the sum of the potential and kinetic energies is constant.

12. Explain the meaning of the 'Principle of Conservation of Energy'. Show that this principle is applicable at every stage of the journey of a particle falling freely under gravity from a height till it reaches the ground.

13. A bullet weighing 1 oz. is dropped from the top of a tower 60 ft. high and is brought to rest by penetrating 5 ft. into mud. Find the average thrust on the mud. [*Ans.*: 26 Poundals].

14. A pendulum consisting of a ten-gram bob at the end of a string thirty centimetres long oscillates through a semi-circle; find its velocity and kinetic energy when it passes its lowest point. Specify the units in which your answer is given. [*Ans.*: $v=242{\cdot}61$ cms. per sec.; K.E.$=294,300$ ergs.]

15. A body falls under gravity and strikes the ground. Explain how the phenomenon supplies an illustration of the transformation of energy. Does it also illustrate the principle of conservation of energy?

16. What are the practical units of power in the F.P.S. and C. G. S. systems? Write out the relations between these units.

17. A steel ball of 100 gms. drops through a height of 10 metres. What is its velocity when it reaches the ground? ($g=980$ cms. per sec.^{2}). [*Ans.*: 1,400 cms. per sec.]

CHAPTER III

FRICTION

38. **Friction**:—No solid surface is perfectly smooth. In other words, all solid surfaces are rough more or less. So when two contiguous solid surfaces (which are dry) are in contact and any attempt is made to move either over the surface of the other, it is always attended with a resistance which tends to oppose the motion. Such resistance to motion is called *friction*.

Friction arises on account of the adhesion, *i.e.* the mutual force of attraction between the molecules of the two contacting surfaces, and the interlocking of the irregularities present on the contacting surfaces.

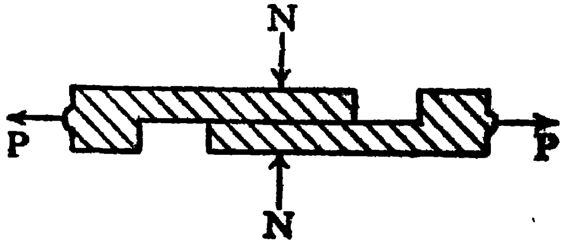


Fig. 25

Friction can be thought of as equivalent to a *force acting along the plane of contact between two surfaces* opposite in direction to any force attempting to produce a relative motion between the two surfaces. This will explain why a force is necessary to drag a book along a table, a rectangular box along the ground, and so on. Consider next a more general case when two plates are pressed together by normal forces N (Fig. 25). To overcome friction and to cause sliding between the two surfaces, a certain force P (its value depending on the value of N and the nature and condition of the two surfaces) acting along the common plane of contact, will be required.

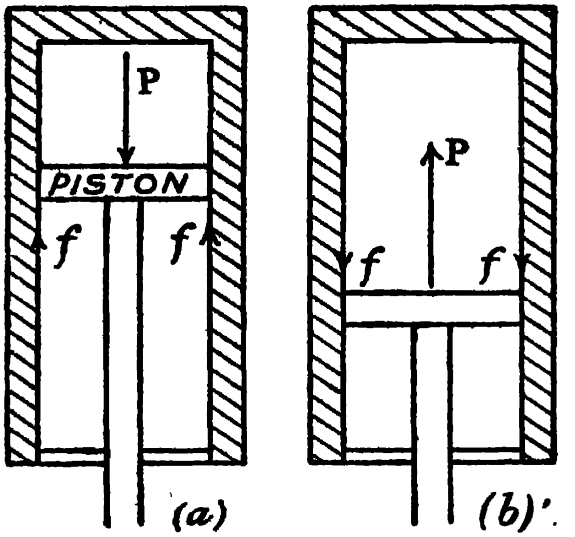


Fig. 26

Friction is perverse.—That is, it always opposes motion irrespective of the direction in which the motion may take place. In Fig. 26(a) a very closely fitting piston working in a cylinder of an engine is shown moving outwards under a force P, while in (b) it is moving inwards at the return stroke. In either case the motion of the piston will be opposed by frictional forces (f) operating along its surfaces of contact.

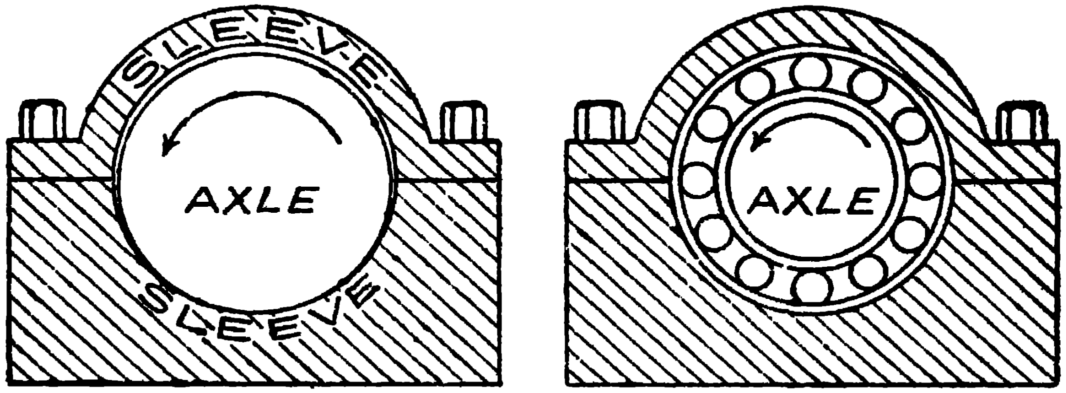
39. Different Kinds of Friction : (a) (i) Static its limiting value :—Frictional force is self-adjusting exert itself only up to a limiting maximum value. A[illegible] attempting to drag a surface over another gradually increases from zero, the frictional force opposing it also increases equivalently. The two contacting surfaces remain in static equilibrium upto a maximum value of the applied force. That is, up to this stage the frictional force which acts in opposition is equal to the applied force. When the applied force just exceeds this maximum value, the body on which the force is applied begins to slide. This maximum value of the applied force is a measure of the ***limiting value of static friction*** between the two surfaces, and is called the force of limiting friction—often also called the ***force of friction*** (F).

(ii) Kinetic or Sliding Friction.—It is found that the force, F, necessary to start sliding of one surface on another, is greater than that necessary to maintain sliding. That is, the force of sliding or kinetic friction is less than that of limiting friction.

N.B. ***It must be remembered that if two surfaces are separated by a film of liquid, such as a lubricant, the nature of the friction is altogether changed.***

(b) Rolling Friction.—This is also a kind of kinetic friction and occurs between two solid surfaces in contact but one of them rolling or tending to roll on the other, as in the case of a marble rolling on the floor, a football rolling on the turf, a rope passing over a rolling pulley, etc. It is a common experience that the force required to drag a rectangular box along the ground is much greater than that required to move it on rollers. **This means that rolling friction is much smaller than sliding friction.** That is the reason why vehicles are mounted on wheels instead of sleeve bearings. There are a number of different types of ball and roller bearings known collectively as **anti-friction bearings.** Basically, all these consist of the rolling elements (balls or rollers), the *race rings* on which are provided tracks for the rolling elements and in the majority of cases a separator for the rolling elements known as the cage.

Sleeve and Ball-bearings.—Fig. 27 (*a*) illustrates a sleeve type of bearing where it will be seen that the rotating *axle* slides on the bottom of the sleeve at low speeds. It however, tries to climb up the side of the sleeve at increased speeds.



(*a*) Sleeve-bearing (*b*) Ball-bearing
Fig. 27

Fig. 27(*b*) illustrates a ball type of bearing where it will be seen that the axle rotates on the balls *without sliding*. The groove, in which the balls themselves roll on account of reaction, is called the '*race*'.

(c) **Fluid Friction**.—Friction occurs when a liquid or gas is made to pass around a stationary body or the body made to move in a liquid or gas, *i.e.* friction occurs when there is relative motion between the two. It arises in the propulsion of a ship through water or automobiles, trains and aeroplanes through the air and so on. Take the case of a rain-drop falling through the air. Its speed depends upon its size and not upon its height above the ground. Starting with zero velocity, the velocity increases as the drop descends until the retarding force of friction equals the downward pull of gravity. When this condition of equilibrium is reached the body falls with a steady velocity, called its **terminal velocity**. For small particles like *fog drops*, this terminal velocity is low and the air-flow around them is *laminar* ; when, however, the particles are large, the terminal velocity may exceed the *critical velocity* and *turbulent flow* sets in around a moving body, which then determines the frictional resistance mostly. The considerations are important for aeroplanes which move in the air and for ships moving in the water.

40. The Role of Friction :—Friction is useful in many ways, though it is also wasteful in other ways.

Usefulness of Friction.—Friction is important in our daily life. If there were no friction, walking would have been impossible, nails and screws would not remain in the wood, fibres of a rope would not hold together, a ladder would not rest on the ground, locomotive engines would not draw a train on the rails, and so on. In designing automobiles and their parts, steps are taken to increase friction where it is needed. Brake linings in automobiles require special materials and tyres are given special thread designs for purposes of increasing the friction consistent with minimum wear and tear.

Wastefulness of Friction.—Friction is ordinarily looked upon as an evil. It is inevitably present, to an extent large or small, whenever there is motion of one body relative to another in contact. The effect of friction is to reduce the relative motion to certain extents and, to that extent, there is loss of mechanical energy of the moving member. So, in designing engines and all other moving machineries, precautions are taken to reduce friction in the bearings to the minimum. Ball and roller-bearings entail much lesser friction than sleeve-bearings and that is why these bearings are rapidly replacing the latter type in modern machineries. Lubrication of the surfaces in contact further decreases the frictional wastage of energy, as also the wear and tear.

41. Friction and Lubrication :—To reduce friction between two rubbing surfaces, a suitable oil or grease, called a *lubricant* is usually introduced in between. This prevents a direct contact between the surfaces and also minimises wear. The use of a lubricant for the above purpose is called *lubrication.*

We primarily owe to Osborne Reynolds and Hardy for our present knowledge of the theory of lubrication. Depending of the thickness of the layer of the intervening lubricant, two types of lubricated friction have been distinguished—**boundary friction** and **floatation friction**. When the protecting lubricant layer is very thin—of the order of molecular diameters—the frictional resistance depends on the roughness of the interfaces and also on the boundary conditions and the friction is known as the *boundary friction.* When, however, the layer is thick enough the friction depends only on the viscosity of the lubricant and is independent of the contact area and the relative velocity of the surfaces and the friction is known as *floatation friction.* In floatation, in the case of a bearing, the journal is lifted and floats on a film of oil. In starting large turbines or alternators, the shaft is first lifted by pumping oil under pressure into the bearings so as to make it oil-borne before it commences to rotate. This oil-borne condition is subsequently maintained by the help of a pump driven by the turbine itself which forces the lubricant to spread over the required surface (forced lubrication). A lubricant layer, even when very thin, requires a very great pressure to rupture it when once it establishes itself.

Liquids possessing the property of adherence to the solid surfaces and comparatively high viscosity will serve as the best lubricants. The chemical stability of a lubricant is another essentially important requirement from the practical standpoint of view.

42. Limiting Friction :—Let a rectangular block of wood *D* rest on a horizontal table *BC* (Fig. 28). The forces acting on the block are its weight *W*, acting vertically downwards and the reaction *R* of the table acting normally upwards at the surface of contact. In this case *R* is equal and opposite to *W*, there being no motion in the vertical direction. Now, suppose a small force *P* is applied to the block parallel to the surface *BC*. If the body is still at rest, *an equal force F opposite in direction* to *P*, must have been called into play to oppose the motion on account of friction arising from contact between the two bodies. As the applied force *P* is gradually increased, the opposing force of friction *F which is a self-adjusting one*, also increases at the same rate until a certain maximum value is reached. If the applied force be increased beyond this value, the block begins to move. The magnitude of this maximum force, when the block is just on the point of sliding is a measure of what is called the force of **limiting friction.**

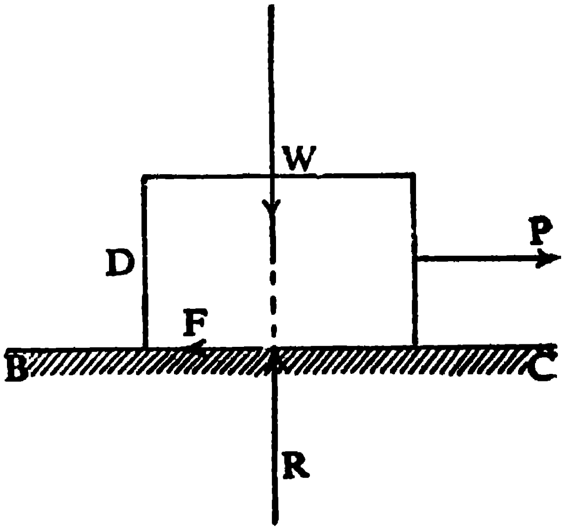


Fig. 28

When the block has once started to move, a *smaller* force would be sufficient to keep the block moving with a constant velocity ; this smaller force is called **Kinetic friction or Dynamic friction.** The same considerations also apply to rolling friction. But it should be remembered that rolling friction is even less than kinetic friction.

43. The Laws of Limiting Friction :—The following generalisations, known as the laws of friction, are due to A. T. Morin, a Frenchman, though some of these facts were previously established by A. Coulomb, another Frenchman who published the results of a large number of experiments on the subject in 1781.

(*i*) Friction always opposes motion.

(*ii*) The force of friction is proportional to the normal reaction between the two surfaces in contact.

(*iii*) It is independent of the extent of the areas of the surfaces in contact, but depends on the material, nature and condition of the surfaces in contact.

44. The co-efficient of Friction :—If the normal reaction acting across two solid surfaces in contact be equal to R and if F denotes the force of limiting friction, the ratio F/R, is found to be a constant and is called the co-efficient of *static friction* or *limiting friction* and more universally as **co-efficient of friction** and is generally denoted by μ, *i.e.* $\frac{F}{R}=\mu$. For any pair of surfaces in contact, the co-efficient μ is always less than unity.

45. The angle of friction :—In the case of limiting friction if the normal reaction and the frictional forces be compounded into a single force, which is sometimes referred to as *resultant* or *total reaction*, the angle which this resultant makes with the normal reaction is called the *angle of friction*.

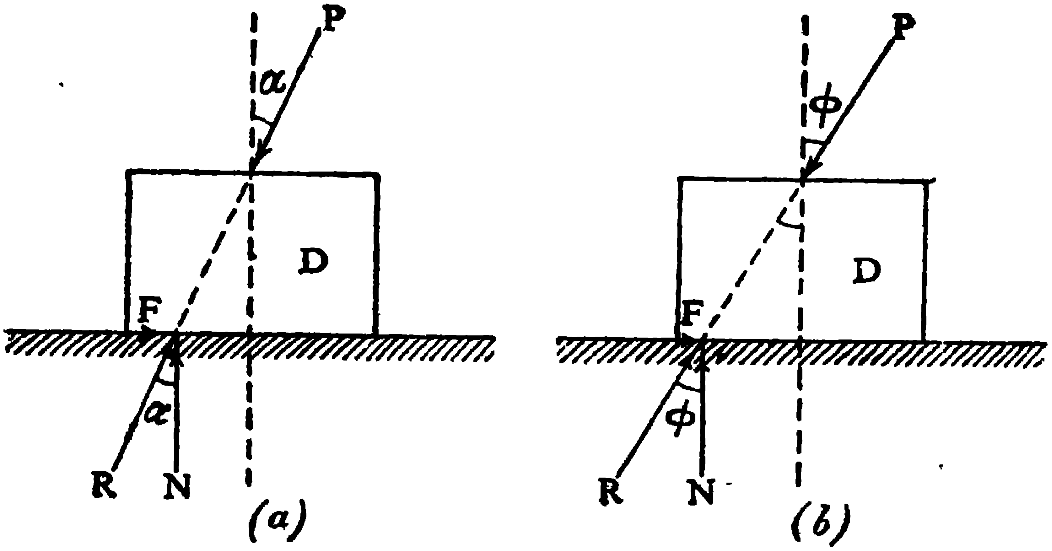


Fig. 29

Consider a small block D resting upon a horizontal plane surface and acted upon by a force P making an angle α with the vertical (Fig. 29, *a*) [P may be considered as the resultant of some applied force and the force of gravity on the block D]. As long as equilibrium exists, the reaction of the supporting surface is equivalent to the reactive force R which will be equal, opposite and collinear with the force P. R may be replaced by its two components, F and N, acting tangentially and normally respectively, to the surface of contact. The component F will represent the friction between the surfaces and the component N, the normal pressure so that $\frac{F}{N}=\tan\alpha$, for equilibrium. Suppose the sliding of the block impends when the force P makes an angle ϕ with the vertical (Fig. 29, *b*), then $\frac{F}{N}=\tan\phi$ ... (1)

Again, from the condition of sliding to begin, $\frac{F}{N}=\mu$ ... (2)

(where μ = co-efficicent of friction, or limiting friction)
From (1) and (2) $\tan\phi=\mu$ ... ... (3)

The limiting angle ϕ whose tangent is equal to the co-efficient of friction is the angle of friction or angle of static friction.

[**Note**—The above furnishes the idea of how friction affects the reaction exerted by a supporting surface acted on by a force. When motion impends, the total reaction *R* exerted by the supporting surface is inclined to the normal by the angle of static friction (ϕ) and acts so as to oppose the motion.

When motion is not impending the total reaction *R* inclines to the normal by whatever angle is necessary to maintain equilibrium. For an ideal surface ($\mu=0$), ϕ is also zero, *i.e.* the total reaction is normal to the supporting surface.]

46. Cone of Static Friction :—In the preceding consideration, the force *P* (Fig. 29) was supposed to be in the plane of the figure. We can, however, generalise it and say that if the force *P* remains confined within a cone generated by a line making the angle of static friction (ϕ) with the normal to the supporting surface, the block *D* will continue to be equilibrium whatever is the magnitude of the force *P*. This cone is called the *cone of static friction*.

47. Determination of the Co-efficient of Friction :—

(i) Horizontal Plane Method.—Place on a horizontal wooden table a rectangular block of wood (Fig. 30) to act as a slider. The contacting surfaces of both these pieces of wood should be as smooth as possible. The slider is attached to a light string which is passed over a light pulley fixed at the end of the table. A scale pan is attached to the end of the string passing over the pulley. The pulley should be so fixed that the position of the string above the table should be horizontal.

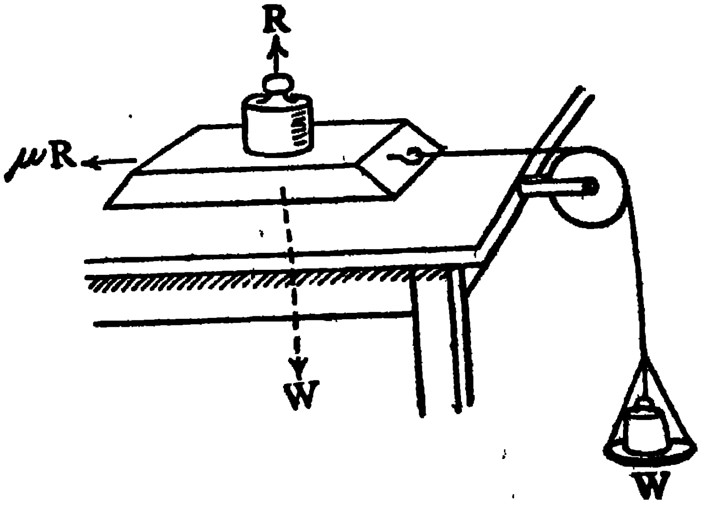


Fig. 30

Weigh the slider and put a known weight on it. Now put weights on the scale pan until the slider is just on the point of motion. Near about the slipping point, gently tap the table to ascertain the required weight to be placed on the scale pan. If W be the total weight of the slider and the weight placed on it and W' the weight of the scale pan including the weight w placed on it, the value of the limiting friction $= W'$, and that of the normal reaction $= W$. So we have $\mu = W'/W$.

Repeat the experiment several times with different weights on the slider and again on reversing the block.

The ratio W'/W for each set of experiment will be approximately the same. The mean value of the ratio is the value of μ.

(ii) Inclined Plane Method.—Place a rectangular slab of wood D on an inclined plane AB (Fig. 31) and gradually increase the inclination of the plane to θ, until D just begins to slide down the plane. Ascertain this by gentle tapping as in the last method. When this is the case, the friction F $(=\mu R)$ acts up the plane and balances the component $(= W \sin \theta)$ of the weight acting down the plane. The normal reaction R acts at right angle to the plane, AB. Resolving W in directions perpendicular and parallel to the plane, we have $W \cos \theta = R$ and $W \sin \theta = F$.

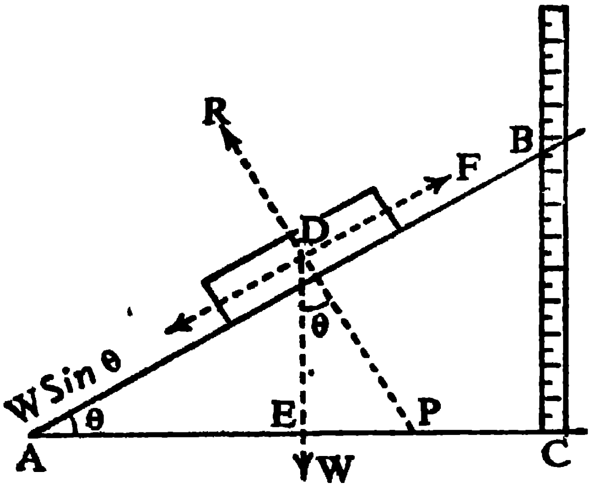


Fig. 31

Hence, $\frac{F}{R} = \frac{W \sin \theta}{W \cos \theta} = \tan \theta$. But $\frac{F}{R} =$

$\therefore$ $\mu = \tan \theta$; or, the co-efficient of friction is simply the tangent of the angle at which sliding begins. Again, $\tan \theta = \frac{\text{height}}{\text{base}}$ of the plane $= \frac{BC}{AC}$. Hence, the **co-efficient of friction** is obtained by dividing the height of the plane by the length of the base.

Repeat the experiment several times and calculate the mean value of μ.

48. The Angle of Repose :—In the case of an inclined plane the angle of incline θ, which the plane AB (Fig. 23) makes with the horizontal AC when a body D on it just begins to slide down, is called *the angle of repose.* It is proved above that the tangent of this angle is equal to the co-efficient of limiting friction. It is also equal to *the angle of friction.*

If the inclination of the inclined plane AB is greater than the angle of repose, the force component down the inclined plane is greater than that required to overcome the friction F and the difference between them produces an acceleration.

Co-efficients of Friction (μ) :—

Static Friction		Rolling Friction	
Wood on wood	...0·3 to 0·5	Rubber tyres on Concrete	...0·03
Metal on metal	.. 0·8 (average)	Ball-bearing on Steel	...0·002
Metal on wood	...0·2 to 0·6	Cast Iron on Rails	...0·004
Leather on wood	.. 0·3 to 0·5	Roller bearings	.. 0·002 to ·007
Leather on metal	...0·3 to 0·6		
Greased surfaces	...0·05		

49. Laws of Kinetic (or Sliding) Friction :—

(1) The frictional force is proportional to the normal reaction between the two rubbing surfaces. [The force necessary to maintain sliding is less than limiting friction, *i.e.* the frictional force here is less than limiting friction.]

(2) The frictional force is independent of the area of contact between the two surfaces, but depends on the material, nature and condition of the surfaces.

(3) The frictional force is independent of the velocity of sliding provided the velocity is low.

50. Co-efficient of Kinetic (or Sliding) Friction :—

If the normal reaction between a sliding body and the supporting surface be R, and F_k denotes the force (less than limiting friction) necessary to maintain a low steady velocity of sliding, once it has been started, then the ratio, F_k/R, is a constant for the given two surfaces and is known as the co-efficient (μ_k) of kinetic (or sliding) friction. That is, $\mu_k = F_k/R$.

The effect of kinetic friction on a body is to oppose the motion of the body with a constant force, $\mu_k R$. If the sliding body be of mass m and moving under a constant applied force P, the acceleration of the body $=(P-\mu_k R)/m$. If the surfaces are smooth ($\mu_k = 0$) the acceleration $= P/m$.

51. Acceleration of a body on an inclined plane :—Suppose a body of mass M and weight $W=Mg$, is placed on a plane whose inclination is θ (Fig. 32). The vertically downward force on M is $W=Mg$. The normal reaction R on the plane is $W\cos\theta$ and the force along the plane is $W\sin\theta$ acting downwards. If μ be the co-efficient of static friction and if $\tan\theta$ is less than μ, the body will not slide because $W\sin\theta$ is less than $\mu W\cos\theta$. But when $\tan\theta$ is greater than μ, *i.e.* $W\sin\theta>\mu W\cos\theta$, the sliding begins and then μ_k, the co-efficient of sliding friction, replaces μ in the opposing force. So the resultant force down the plane is $W(\sin\theta-\mu_k\cos\theta)$. This force produces an acceleration, f, down the plane given by the relation,

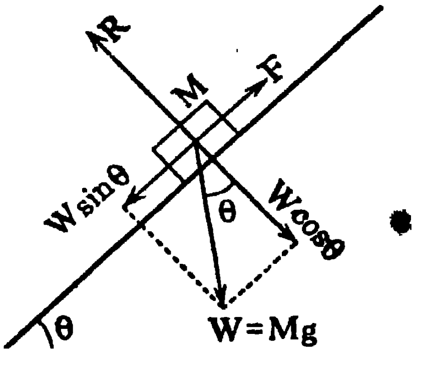


Fig. 32

$$f=\frac{W(\sin\theta-\mu_k\cos\theta)}{M}=g(\sin\theta-\mu_k\cos\theta).$$

52. Equilibrium of a body on an inclined plane :—Suppose a body of weight W is held in position on an inclined plane by means of a force F directed up the plane. If the angle of incline θ be greater than the angle of repose ϕ (say), the condition of equilibrium of the body can be easily deduced.

The force down the plane—$W\sin\theta$ and the normal reaction is $R=W\cos\theta$ (Fig. 32). If the body be just on the point of sliding down, the opposing force acting up the plane is $F+\mu R$ where $\mu=$ co-efficient of friction $=\tan\phi$.

$\therefore$ For equilibrium,

$$W\sin\theta=F+\mu R=F+\tan\phi\,.\,W\cos\theta\,;$$

$$\text{or,}\quad F=W\left[\sin\theta-\frac{\sin\phi}{\cos\phi}\cos\theta\right]=\frac{W}{\cos\phi}\sin(\theta-\phi)\ldots(1)$$

Again, if the body be just on the point of sliding up due to the action of F, μR will act down the incline and the equation representing the equilibrium condition will be, $F=W\sin\theta+\mu R$

$$=W\left[\sin\theta+\frac{\sin\phi}{\cos\phi}\cos\theta\right]=\frac{W\sin(\theta+\phi)}{\cos\phi}\qquad\ldots(2)$$

Thus, for the equilibrium of the body, the force acting up the plane must lie between $\frac{W}{\cos\phi}\sin(\theta+\phi)$ and $\frac{W}{\cos\phi}\sin(\theta-\phi)$.

53. Minimum force needed to move a body up an incline :— Let the force F applied at an angle α to the plane just cause the body to move up the incline (Fig. 33). Then $F\cos\alpha$ is the effective force directed up the inclined plane and $F\sin\alpha$ reduces the normal reaction.

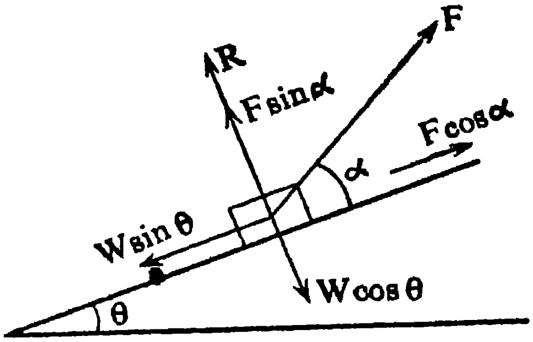


Fig. 33

Since the body is just on the point of sliding up,

$$F\cos\alpha = W\sin\theta + \mu R.$$

But $R = W\cos\theta - F\sin\alpha$

$$\therefore F\cos\alpha = W\sin\theta + \mu W\cos\theta - \mu F\sin\alpha.$$

Again since $\mu = \tan\phi$, ϕ being the angle of repose, we have,

$$F\cos\alpha + F\sin\alpha\tan\phi = W\sin\theta + W\cos\theta\tan\phi\ ;$$

$$\text{or, } \frac{F(\cos\alpha\cos\phi + \sin\alpha\sin\phi)}{\cos\phi} = W\frac{(\sin\theta\cos\phi + \cos\theta\sin\phi)}{\cos\phi}\ ;$$

$$\text{or, } F = W\frac{\sin(\theta+\phi)}{\cos(\alpha-\phi)}$$

$\therefore$ F is minimum when $\cos(\alpha-\phi) = 1$;

or, when $\alpha = \phi$, *i.e. when the upward force acting on the body is inclined to the plane at an angle equal to the angle of repose.*

Examples

1. A body rests upon an inclined plane and just slides down the plane when the slope of the plane is 30°. Calculate the acceleration of the body down the plane when the slope is increased to 60°.

Ans. : Here the angle of repose is 30°.

$\therefore\ \mu = \tan 30°.$

$\therefore\ f = g(\sin\theta - \mu\cos\theta)$

$$= g(\sin 60° - \tan 30°\cos 60°) = g\left(\frac{\sqrt{3}}{2} - \frac{1}{\sqrt{3}}\cdot\frac{1}{2}\right).$$

$$= \frac{g}{2}\left(\frac{3-1}{\sqrt{3}}\right) = \frac{g}{\sqrt{3}} = \frac{981}{\sqrt{3}} = 566 \text{ cm./sec.}^2.$$

2. A body of mass 8 lbs., hanging freely over the edge of a rough table, is connected to a body of mass 5 lbs. resting on the table by means of a light string passing over a smooth pulley at the edge. This is pulled 0 5 ft. along the table in 0·25 sec. from rest. Calculate the co-efficient of friction.

Ans. : If T and F be the tension in the string and the force of friction in poundals respectively, the resultant force pulling the 5 lb.-mass is $(T-F)$ *poundals.*

$\therefore$ The acceleration is $f = \frac{T-F}{5}$ ft./sec.^{2}.

Since $S=\frac{1}{2}$ ft.², we have, $0{\cdot}5=\frac{1}{2}\times f\times(\frac{1}{4})^2$; or, $f=16$ ft./sec.².

$$\therefore\quad T-F=80 \text{ poundals.}$$

Considering the suspended mass, we get from the relation, $mg-T=mf$.

$$8\times32-T=8\times16\,;$$

$$\text{or,}\quad T=256-128=128 \text{ poundals.}$$

$$\therefore\quad F=T-80=128-80=48 \text{ poundals.}$$

$\therefore$ The co-efficient of friction, $\mu=\dfrac{F}{5\times32}=\dfrac{48}{5\times32}=\dfrac{3}{10}=0{\cdot}3.$

3. Calculate the acceleration of a block sliding down an inclined plane when the co-efficient of friction is 0.25. and inclination of the plane is 30°. If the mass of the block be 8 lbs., what are the forces needed just to prevent it (i) from sliding down, and (ii) from moving up ?

Ans. : From Art. 51, the required acceleration is

$$f=g\,(\sin\theta-\mu\cos\theta)$$

$$=32\ (\sin30^\circ-0{\cdot}25\times\cos30^\circ)=32\left[0{\cdot}5-{\cdot}25\times\frac{\sqrt{3}}{2}\right]$$

$$=32\times0{\cdot}2835=9{\cdot}072 \text{ ft./sec}^2.$$

(i) Again from Art. 52, the force needed to prevent sliding down is F, where

$$W\sin\theta=F+\mu R\,;$$

$$\text{or,}\quad F=W\sin\theta-\mu R.$$

Here the normal reaction is $R=W\cos\theta$.

$\therefore\quad F=W\,[\sin\theta-\mu\cos\theta]=8\times32\,[\sin30^\circ-0{\cdot}25.\cos30^\circ]$

$$=256\left[0{\cdot}5-0{\cdot}25\times\frac{\sqrt{3}}{2}\right]=256\times0{\cdot}2835=72{\cdot}56 \text{ poundals.}$$

(ii) From Art. 52, the force needed to prevent upward movement is F_1, where

$$F_1=W\sin\theta+\mu R$$
$$=W\sin\theta+\mu\,W\cos\theta$$
$$=W\,(\sin\theta+\mu\cos\theta).$$

$\therefore\quad F_1=8\times32\left(0{\cdot}5+0{\cdot}25\,\dfrac{\sqrt{3}}{2}\right)$

$$=256\times0{\cdot}7165=183{\cdot}424 \text{ poundals.}$$

CHAPTER IV

MACHINES

54. **Machine** :—A machine is a contrivance by which a force applied at some point of it is overcome by means of another force applied at some other point of it with alteration in direction or magnitude or both. It is used to be the practice to call the former force the *weight* and the latter force the *power*. But as the force to be overcome is not necessarily that of gravity, it is better to name it the **resistance** (or **load**), and since the term *power* is used in connection with the *rate of work*, it will be better to use the term **effort** in referring to the driving force in a machine. The points at which the effort and the resistance act are usually termed the **driving point** and the **working point** respectively.

55. (a) **Mechanical Advantage** :—The ratio, $\frac{\text{load}}{\text{effort}}$, is called the mechanical advantage of a machine. The term *force-ratio* is sometimes used instead of mechanical advantage. Ordinarily, a machine is so constructed that the mechanical advantage is greater than one. If in a machine, this ratio is less than *one*, it would be more accurate to call it *mechanical disadvantage*.

(b) **Velocity Ratio.**—

The ratio, $\frac{\text{displacement of driving point}}{\text{displacement of working point}}$, is called the velocity ratio of a machine. In some machines it is a constant while in some others it is not.

Thus in a simple wheel and axle (Fig. 34) the displacement, say a of the effort E will bear a constant ratio to the displacement say b, of the load W.

That is, its velocity ratio $=\frac{a}{b}$.

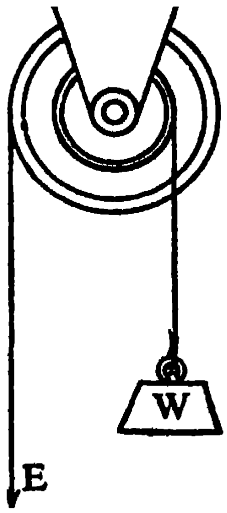


Fig. 34—A Simple Wheel and Axle.

In a toggle joint (Fig. 35) the ratio of the displacement of the effort E to that of the load W will be different for different positions of the moving parts of the machine. In such a machine in which the ratio is variable, the velocity ratio for any given position of its parts is the ratio of the displacement of the driving point to the displacement of the working point, when these displacements are indefinitely small.

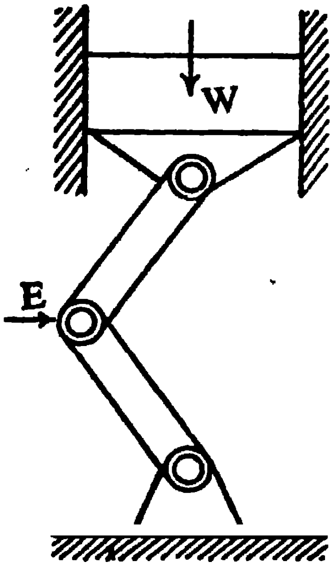


Fig. 35—
A Toggle Joint.

(c) **Efficiency of a Machine.**—In all machines some work is always wasted in overcoming *friction.* The result of it is that the work done by the effort in a given time, called *total work* or *work input* ($=E \times a$), is always greater than the work done on the resistance or load ($=W \times b$) called—*useful work* or *work output*. The difference of the latter from the former $=$ *lost work* $=(Ea-Wb)$.

The **Efficiency** is defined as the ratio, (*useful work*)/(*total work*). Efficiency evidently will always be less than unity. Often it is expressed as a *percentage* by multiplying with 100.

(d) **Mechanical Advantage = Efficiency × Velocity ratio.**—Let E be the effort and W the load. The *mechanical advantage* $=\frac{W}{E}$.

Suppose the displacement of the *driving point* is *a* and that of the *working point* is *b*.

Then, $\textit{efficiency}=\frac{\text{useful work}}{\text{total work}}=\frac{W\times b}{E\times a}=\frac{W/E}{a/b}=\frac{\text{Mech. advantage}}{\text{Velocity ratio}}$;

or, **Mechanical advantage = efficiency × velocity ratio.**

56. (a) **The Principle of work** :—In any *actual machine*, the useful work obtained in overcoming the resistance is always less than the total work done by the effort. This is because (*i*) work has to be done in lifting its parts which have weight, and (*ii*) because there is always some internal friction which has to be overcome. A *perfect* or *ideal* machine is one which has no weight and no internal friction. For if the useful work is equal to the total work and the *efficiency of the machine is unity*. So the **principle of work**, *viz. whatever be the machine provided there is no friction and that the weight of the machine is neglected, the work done by the effort is always equivalent to the work done against the load* (that is, $E\times a=W\times b$) is a universal principle relating to a machine. It is no new principle but is merely a modified form of the principle of conservation of energy.

(b) What is gained in Power is lost in Speed.—From the principle of work, $E \times a = W \times b$, assuming the machine to be an ideal one. If in a machine the effort E is less than the resistance W, the distance a through which the driving point moves will be greater in the same proportion than the distance b through which the working point moves in the same time. This is, in popular language, expressed as, "*What is gained in power* (effort) *is lost in speed.*" The meaning of the statement is that whenever mechanical advantage is gained, it is gained at a proportionate decrease of speed.

There is never any gain of work in a machine, though mechanical advantage is generally arranged for.

57. The Uses of a Machine :—

(1) A machine enables one to lift weights or to overcome resistances much greater than one could do unaided, as in the case of a pulley-system, a wheel and axle, a crow-bar, a simple screw-jack, etc.

(2) It enables one to convert a slow motion at some point into a more rapid motion at some other desired point, as in the case of a bicycle, a sewing machine, etc. An opposite effect may also be arranged in practice if necessary. Such changes of speed are brought about by belting, gearing, etc.

(3) It enables one to use a force acting at a point to act at a more convenient point, as in the case of a poker stirring up a fire or to use a force acting at a point in a more convenient manner, as in the case of lifting of a mortar-bucket to the top floor by means of a rope passing over a pulley fixed at the top of the building, the other end of the rope being pulled down by an agent remaining on the ground.

(4) It enables one to convert a rotary motion into a linear motion or *vice versa*, as in the case of a *rack and pinion*, etc.

(5) It enables one to convert a reciprocating (to-and-fro) motion into a rotary motion or *vice versa*, as by a crank used in the heat engine.

58. Types of simple machines :—

The following six simple machines represent the type of principles used in making practical machines :

(1) Pulley, (2) Inclined plane, (3) Lever, (4) Wheel and axle, (5) Screw and (6) Wedge.

59. The Pulley :—A pulley is a simple machine which consists of a grooved wheel, called the **sheave**, over which a string passes. The wheel is capable of turning freely about an axle passing through its centre. The axle is fixed to a framework called the **block**. The pulley is termed **fixed** or **movable** according as its block is fixed or movable.

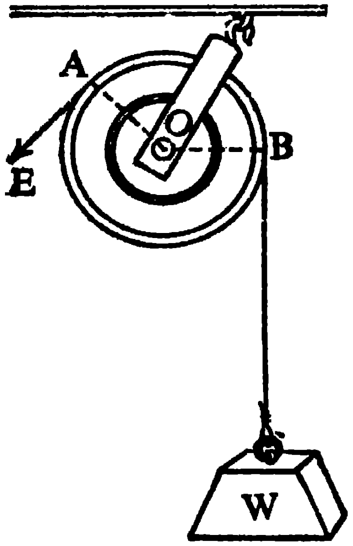


Fig. 36

(a) The Single Fixed Pulley.—In this (Fig. 36) the load W is attached to one end of the string and the effort E is at the other end. With a perfectly smooth pulley and a weightless string, the tension of the string will be the **same** throughout. Hence the distance through which the load is raised is equal to the distance through which the effort descends. For equilibrium, the moments of E and W about O, the centre of the wheel, must be equal and opposite; or $E \times AO = W \times OB$ but $AO = OB$, being radii of the same circle. Hence $E = W$.

$\therefore$ The mechanical advantage $= W/E = 1$.

In practice, pulleys are not perfectly frictionless, and W is always less than E, that is, the mechanical advantage is always less than 1 but in spite of this, the arrangement is useful as the operator can use the weight of his body for raising the load. It is generally used for raising weights, drawing curtains, pulling *punkhas*, etc.

(b) The Movable Pulley.—Here one end of the string passes round the pulley A and is attached to a fixed support (Fig. 37), and the effort E is applied at the other end. The load W to be raised is attached to the movable pulley. For a frictionless pulley the tension of the string is the same at any point of it and is equal to E. When the strings are vertical, the total upward force is $2E$ and neglecting the weight of the pulley, the downward force is W. So, for equilibrium, $W = 2E$. So the mechanical advantage $= W/E = 2$, *i.e.* a given effort can raise twice its weight.

If the weight of the pulley is w and cannot be neglected, we have $W + w = 2E$; or the mechanical advantage $= W/E = 2 - w/E$.

When the direction of the force is to be changed another fixed pulley (B in

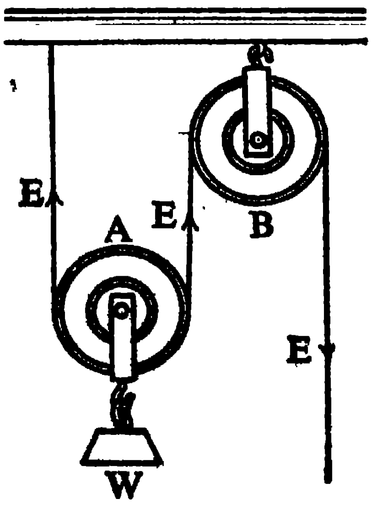


Fig. 37

Fig. 37) should be used, which does not, in any way, affect the mechanical advantage.

(c) **Weston's Differential Pulley.**—In Weston's differential type of pulley an endless chain replaces the string. The depressions in the grooves of the pulleys prevent the slip—the links of the chain fitting into these depressions. The arrangement is represented in Fig. 38, wherein the two pulleys of the upper block move together as a single pulley round a common axis just like a wheel and axle. The effort E is applied on the chain in the manner shown in the figure. The excess portion of the chain hangs loosely from the inner of the two upper pulleys. If a load W hangs from the lower pulley and T be the tension on the chain, supposing the machine to be an ideal one, we have for the equilibrium condition of the load,

$$W = 2T.$$

Again considering the moments of the forces round the common axis O of the upper pulley we get,

$$E \times R + T \times r = T \times R \;;$$

or, $ER = T(R-r)$ when R and r are respectively the radii of the larger and the smaller pulleys of the upper block.

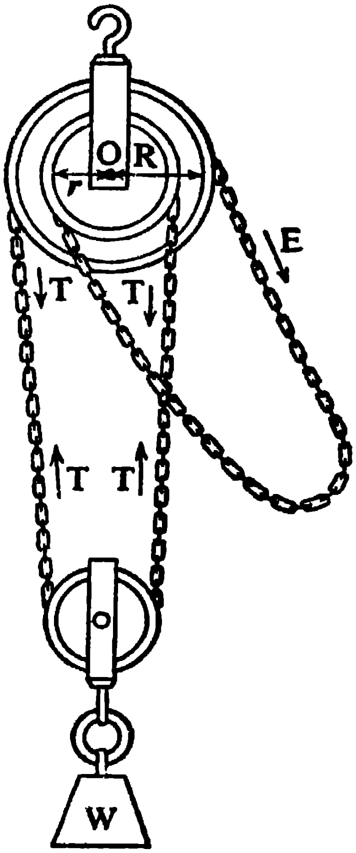


Fig. 38

$$\therefore\ E = T\,\frac{R-r}{R} = \frac{W}{2} \times \frac{R-r}{R}.$$

Hence the mechanical advantage of the system is given by, $\frac{W}{E} = \frac{2R}{R-r}$. Here since the machine is supposed to be ideal, its efficiency is unity or 100 per cent. So the velocity ratio is also equal to $2R/(R-r)$.

(d) **Combination of Pulleys.**—A combination of pulleys is very often used in order to secure a mechanical advantage greater than two. Different systems having different mechanical advantages are used for different purposes. The most important combinations, which are in general use, are given here.

(i) Separate String system of Pulleys.—(*First system of Pulleys*) :

In this system of pulleys there is a fixed beam RR' (Fig. 39) and a number of strings, one end of each of which is attached to the beam. Each of these strings supports a movable pulley, the other end of the string being attached to the next higher pulley. In this way the pulleys, p_1, p_2, and p_3, are attached to the successive strings. The number of pulleys and strings can be increased according to the need. In addition to these movable pulleys there is an auxiliary fixed pulley, A, over which the free end of the last string passes and hangs vertically downwards so that the effort may be applied as a downward vertical force. The load is always attached to the first pulley, p_1, which is the lowermost member of the system.

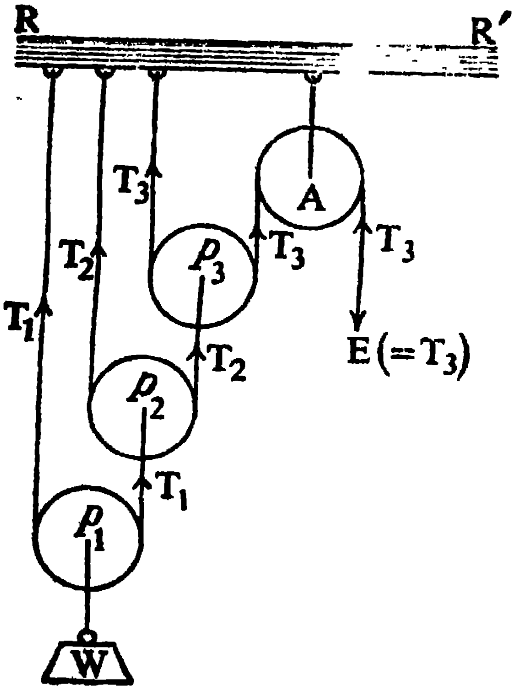


Fig. 39

Let the tension of the strings passing round the pulleys, p_1, p_2 and p_3 be T_1, T_2 and T_3 and let W and E respectively represent the load and the effort, the latter being just sufficient to maintain equilibrium.

For simplicity of calculation, neglect the weights of the pulleys. For equilibrium of p_1, $2T_1 = W$, or $T_1 = W/2$, and for equilibrium of p_2, $2T_2 = T_1$, or $T_2 = \frac{T_1}{2} = \frac{W}{4} = \frac{W}{2^2}$. Similarly, $T_3 = \frac{W}{2^3}$.

In the case of a three-pulley system, the effort $E = T_3 = \frac{W}{2^3}$.

For a n-pulley system, the effort will be evidently, $E = W/2^n$.

$\therefore$ The mechanical advantage of the system $= \frac{\text{Load}}{\text{Effort}} = W/E = 2^n$.

If the weights of the pulleys be taken into account and if each pulley has an weight w, then it can be shown that, $2^nE = W + w(2^n - 1)$ (see any book on Statics). Thus the mechanical advantage increases as the weight of the pulley decreases.

(ii) Pulley Block.—This system (often known as the ***Second System of Pulleys***) consists of two blocks each containing 2 or 3 pulleys, the upper one being fixed to a support and the lower one movable to which the load W is attached (Fig. 40). The string is attached to the upper, or to the lower block, and is then passed round a movable and a fixed pulley in turn, finally passing over a fixed pulley, the effort E being applied at the free end. It should be noted that, when the string is attached to the upper block, the number of wheels in the two blocks must be the same, but when it is attached to the lower block, the number of wheels in it will be one less than that in the upper one.

The tension everywhere round the string is the same and is equal to the effort E, the pulleys being assumed frictionless. If the number of portions of the string in the lower block be n, the total upward force on it is nE and this must be equal to the load W supported. Thus, we have, $nE = W + w$, where w is the weight of the lower block.

Hence the mechanical advantage

$$=\frac{W}{E}=n-\frac{w}{E}.$$

(iii) Third system of Pulleys.—

In this system of pulleys, the load, W, to be lifted is attached to a bar, PQ, to which is tied one end of each of the strings passing over the pulleys (Fig. 41). The uppermost pulley is fixed to a rigid support RR'. The other end of the string passing over the uppermost pulley is attached to the next lower movable pulley. The string passing over the second pulley supports the third pulley and so on. The effort is applied to the free end of the last string passing over the last pulley.

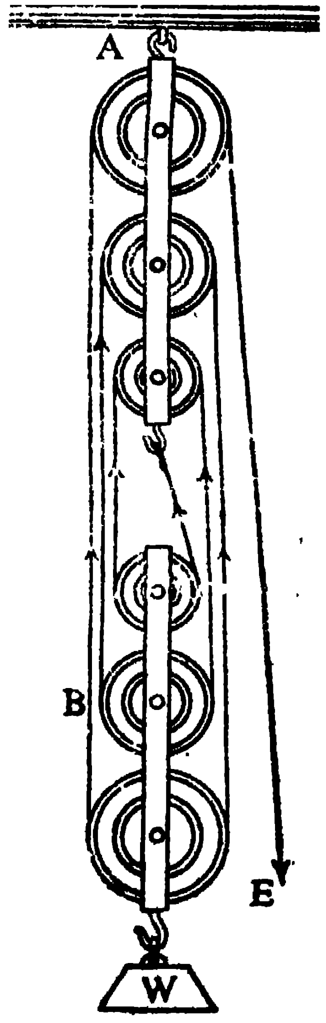


Fig. 40

Let T_1, T_2, T_3 be the tensions on the third, second and the first pulley respectively. The effort E is evidently equal to T_1.

Neglecting the weights of the pulleys, we have, for equilibrium of the pulleys taken in order from the lowest one,

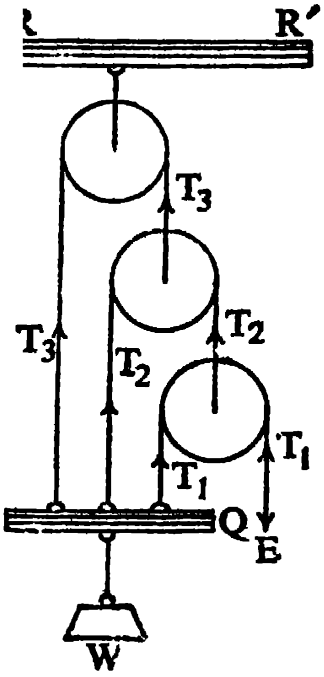


Fig. 41

$$T_2 = 2T_1 = 2E\ ;$$
$$T_3 = 2T_2 = 2^2E.$$

Now since PQ from which W is suspended is also in equilibrium,

$$W = T_1 + T_2 + T_3 = E(1+2+2^2) = E(2^3-1).$$

For n pulleys in the system,

$$W = T_1 + T_2 + T_3 + \ldots + T_n$$
$$= E(1+2+2^2+\ldots+2^{n-1}) = (2^n-1)E.$$

Hence the mech. advantage $= \frac{W}{E} = 2^n - 1.$

If the weight, w, of each of the pulley (movable) be taken into account, it can be shown that

$$W = (2^n-1)E + w(2^n-n-1).$$

Example

Find the mechanical advantages of the third system of pulleys consisting of 4 pulleys each of weight $\frac{1}{64}$th that of the weight.

***Ans.*: Here $W=(2^n-1)E+w(2^n-n-1)$ and $w=\frac{W}{64}$, n being 4.**

$$W = (2^4-1)E + \frac{W}{64}(2^4-4-1) = 15E + \frac{11}{64}W$$

$$\therefore \quad \frac{53}{64}W = 15E.$$

Hence the mechanical advantage $= \frac{W}{E} = \frac{64\times 15}{63} = 18{\cdot}1.$

60. The Inclined Plane :—It is also a simple machine and has been in use in India also from very ancient times. In building temples of great height heavy stones and other materials used were raised by the ancient Indians utilising the principle of this machine.

An inclined plane is a smooth rigid flat surface inclined at an angle to the horizontal. It is used to facilitate the raising of a heavy body to a certain height by the application of a force (or effort) which is less than the weight of the body.

Let AB be a plane inclined at an angle θ to the horizontal line AC and BC the height of the plane (Fig. 31).

The body D is acted upon by three forces, (*i*) W, its weight acting vertically downwards along DE, (*ii*) F, the force or effort, and *iii*) R, the reaction of the plane. Let the normal at D to the inclined plane meet AC in P. Then $\angle PDE = 90^\circ - \angle ADE = \angle A = \theta$.

Case I.—Let the force F act upwards along the plane (*vide* Fig. 31).

In order that the body may be in equilibrium, the sums of the resolved parts of the forces parallel and perpendicular to the plane AB are separately equal to zero. Resolving W parallel and perpendicular to AB, we have $W \sin\theta$ along DA and $W \cos\theta$ perpendicular to DA.

Hence, $W \sin\theta - F = 0$; $W \cos\theta - R = 0$;

$\therefore$ $F = W \sin\theta$; $R = W \cos\theta$.

The mechanical advantage, $\frac{W}{F} = \frac{1}{\sin\theta} = \frac{\text{length}}{\text{height}}$ of the plane.

When $\theta = 30^\circ$, the mechanical advantage $= 1 \div \frac{1}{2} = 2$, that is, a body of weight W can be supported by a force, $F = W/2$ acting up the plane.

Case II.—Let the force F act horizontally, *i.e.* parallel to the base AC (Fig. 42).

The vertical and horizontal components of R are $R \cos\theta$ along ED and $R \sin\theta$ along FD.

$R \cos\theta = W$; and $R \sin\theta = F$.

The mechanical advantage, $\frac{W}{F} = \frac{R\cos\theta}{R\sin\theta} = \cot\theta$

$= \frac{\text{base}}{\text{height}}$ of the plane.

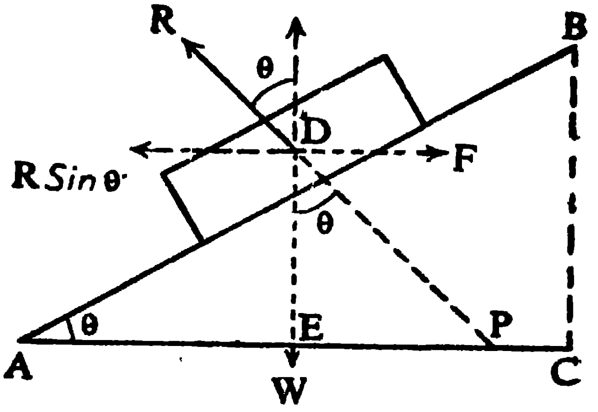


Fig. 42

Example

A mass of 16 lbs. is placed on a smooth inclined plane, whose height is 3 feet and base is 4 ft. Calculate reaction on the plane when the mass is at rest due to an applied horizontal force. Evaluate this force also.

Ans.: For the inclined plane $\cos\theta = \frac{4}{5}$ and $\sin\theta = \frac{3}{5}$.

From case II above, $R \cos\theta = W$; or, $R \times \frac{4}{5} = 16$; or, $R = 20$ lbs.-wt.

and $F = R \sin\theta = 20 \times \frac{3}{5} = 12$ lbs.-wt.

61. The Lever :—The knowledge of the principle of the lever is as old as Archimedes. A lever is a simple machine and consists of a rigid bar (straight or bent) having one point fixed about which the rest of the lever can turn. This fixed point is called the **fulcrum.** The forces exerted on or by the lever may be parallel or inclined to one another. As in all machines, the driving force is called the **effort** (or power), and the working force, the **weight** (or resistance or load) and let them be denoted by E and W respectively. The perpendicular distances between the fulcrum and the lines of action of the effort and the weight are called the **arms** of the lever. The ratio of the arm 'a' of the effort to the arm 'b' of the weight, in the position of equilibrium, is often called the **leverage,** *i.e.* leverage$=a/b$. The

$$\text{mechanical advantage}=\frac{\text{Weight}}{\text{Effort}}=\frac{W}{E}.$$

The principle of the lever is practically the principle of moments which may be stated as *"If a lever is in equilibrium, the sum of the moments tending to turn it clockwise round any point is equal to the sum of the moments tending to turn it anti-clockwise round that point."*

So for a lever, if it be in equilibrium, *clockwise moment round the fulcrum= contra-clockwise moment round the same point.*

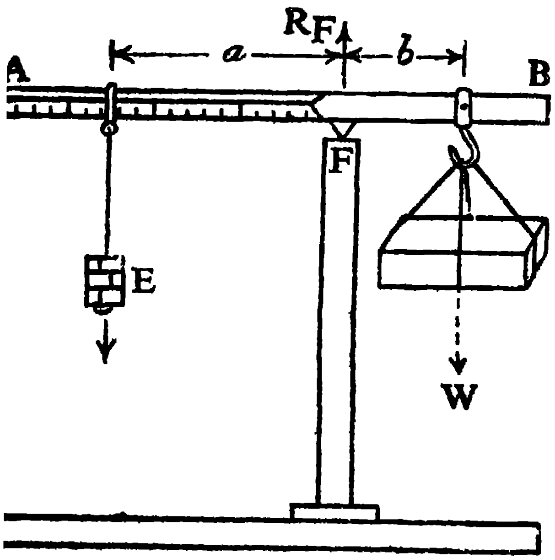


Fig. 43

Experiment.—Let a metre-stick AB balance on the sharp edge of a wedge-shaped piece of wood (Fig. 43) and let a load W says, 200 gms., be suspended by a string from a point 20 cms. from the fulcrum F. Now find, by experiment, a point on the other side of F such that an effort E, say, a weight of 100 gms. applied at the point will just support the load W. This point will be found to be at 40 cms. from the fulcrum. It is seen at once that the product of (200×20) is equal to the product of (100×40). If instead of an effort of 100 gms. an effort of 400 gms., is taken, the point of balance will be found now at 10 cms. from the fulcrum. Again it is seen that the product (200×20) is equal to the product of (400×10). The driving moment in this example, *i.e.* the moment of E about F is anti-clockwise and the working moment, *i.e.* the moment of W about F is clockwise.

62. The Straight Levers :—When a lever is straight and the effort and the weight act perpendicularly to the lever, the following three distinct classes of levers are found in practice (Fig. 44).

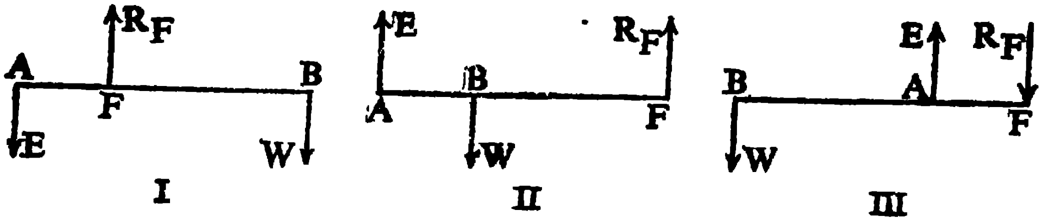


Fig. 44

(I) $E \times AF = W \times BF$, (II) $E \times AF = W \times BF$, (III) $E \times AF = W \times BF$

or, $E = \frac{BF}{AF} \times W$; or, $E = \frac{BF}{AF} \times W$; or, $E = \frac{BF}{AF} \times W$;

(I) R_F (reaction at fulcrum) $= E + W$. (II) $R_F = W - E$. (III) $R_F = E - W$.

Evidently, in class I and II types of levers, if F be taken very near to W, the ratio $\frac{BF}{AF}$ can be made very small *i.e.*, a small effort in E can be used to overcome a large resistance W, *i.e.* there is mechanical advantage in these cases. In class III type of levers, a large effort E overcomes a small resistance W, which shows a mechanical disadvantage. This arrangement gives W a large movement for a small movement of the effort E, a fact which is just opposite to what happens in the other two types of levers. The practical use of class III type of levers lies often in *convenience* ; for, in practice, it may not always be possible to find a convenient point to apply the effort relatively at greater distance referred to the fulcrum than that of the load.

Example

A lever is 8 ft. long with its fulcrum at a distance of 2 ft. from one end. Calculate the effort to be applied to raise a load of 240 lbs. placed at the end of the shorter arm, the efficiency of the lever being 80 per cent.

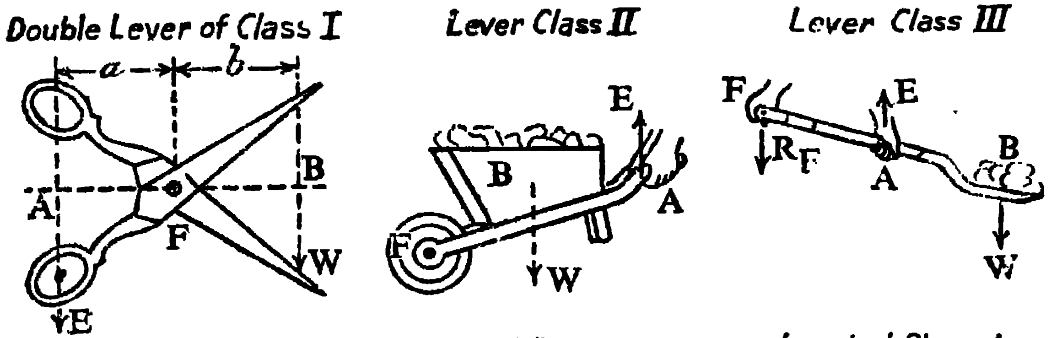
Ans. : Here mech. advantage $= 240/E$ and velocity ratio $= \frac{6}{2} = 3$.

$\therefore$ Efficiency $= M.A./$vel. ratio $= \frac{240}{3E} = 80\% = 0{\cdot}8$.

$\therefore$ $E = 100$ lbs.

63. Common Applications of the Principle of Levers :—

The lever principles, as described above, are used in daily life in various ways. Levers may be simple or double. Three common appliances representative of the three classes of straight levers are shown below in Fig. 45.


Double Lever of Class I Lever Class II Lever Class III

A pair of Scissors Wheel Barrow Loaded Shovel

Fig. 45

64. Examples of the Three Classes of Levers :—

Class I.—A common balance, pump handle of a tube well, a spade used in digging earth, a crow-bar used in moving a weight at one end, etc. A pair of scissors and a pair of pincers are examples of ***double levers*** of this class.

Class II.—A cork squeezer, a crow-bar with one end in contact with the ground, etc. A pair of ordinary nut-crackers is an example of ***double lever*** of this class.

Class III.—The human fore-arm (when a load is placed on the palm and the elbow is used as fulcrum, the tension exerted by the muscles in between acts as effort), the upper and lower jaws of the mouth, a pair of forceps used in a weight box, and a pair of coal-tongs are examples of ***double levers*** of this class.

Example

A pair of nut-crackers is 5 inches long and when a nut is placed $\frac{7}{8}$ in. from the hinge, a force of $3\frac{1}{2}$ lbs. wt. applied at the end will crack the nut. What weight simply placed on the top of the nut would crack it ?

(Utk. U. 1952)

***Ans.* : It is a lever belonging to class II.**

Here $E=3{\cdot}5$ lbs.-wt. ; effort arm $=5''$; load arm $=\frac{7}{8}''$.

$\therefore\ 3{\cdot}5\times5=W\times\frac{7}{8}$; or, $W=20$ lbs.-wt.

Hence the weight to be placed on the top $=20$ lbs.-wt.

65. The Wheel and Axle :—It is a simple machine, and may also be looked upon as a modification of the lever. It consists of two cylinders of different diameters capable of turning about a common fixed axis, the larger of which is called the *wheel* and the smaller the *axle* [Fig. 46 (*a*)]. The load W to be raised is attached to a rope coiled round the axle and the effort E is applied to a rope coiled round the wheel in the opposite direction, so that when the rope round the wheel is uncoiled, the rope round the axle is coiled up and thereby the weight is raised. Fig. 46 (*b*) shows a section where OB is the radius (r) of the axle and OA the radius R of the wheel. Taking moments about O, on the axis, $E\times OA=W\times OB$.

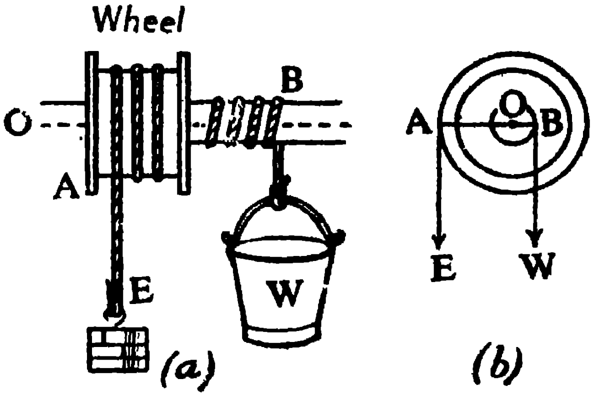


Fig. 46—
The Wheel and Axle.

$$\text{The mechanical advantage}=\frac{W}{E}=\frac{OA}{OB}=\frac{\text{Radius of wheel }(R)}{\text{Radius of axle }(r)}$$

The **windlass** by which water is drawn from a well is of the same class as the wheel and axle, the crank-handle of which serving the purpose of the wheel. The **capstan** (Fig. 47) used on board a ship for raising an anchor is also of this class. In it the length of the lever arm takes the place of the radius of the wheel, the radius of the barrel corresponding to the radius of the axle.

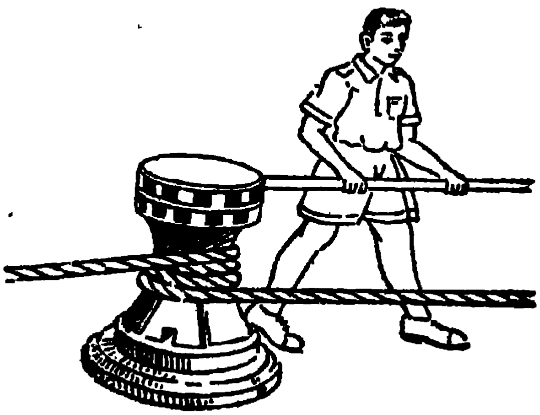


Fig. 47—The Capstan.

66. Screw :—An accurately cut screw has many important applications in modern industrial machines. The screw gauge and the spherometer which are two very common laboratory instruments also work on the principle of the screw and the nut.

A screw can be considered as an inclined plane wrapped round a cylinder. The connection between the inclined plane and the screw is shown in Fig. 48.

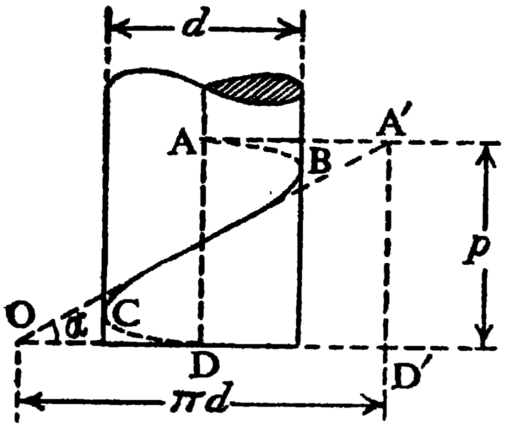


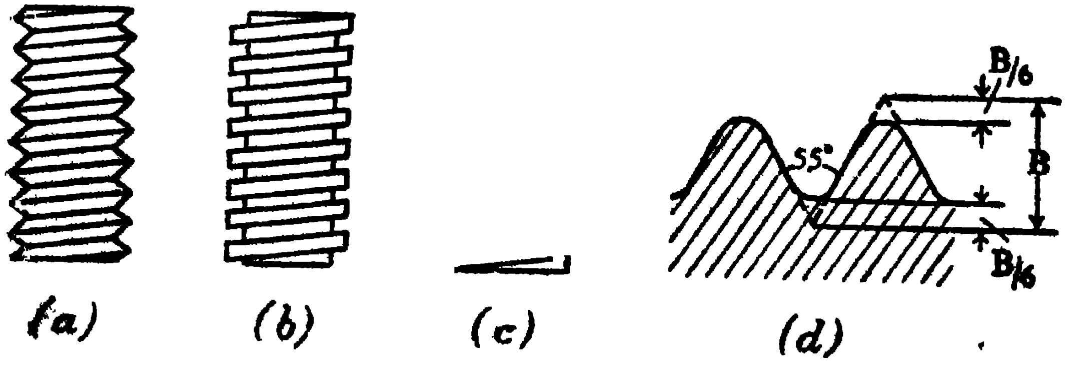
Fig 48

It shows a solid cylinder having one turn *ABCD* of a helix marked on its surface. The right-angled triangle *A'OD'* is the development of that part of the surface of the cylinder which is below the helix, p = pitch of the helix, α = inclination of the helix, d = diameter of the helix.

Then, $\tan \alpha = p/\pi d$.

Actual screws are of metal or of wood and differ from the above ideal screw in that they always have a protuberant thread (forming the helix) cut on the cylinder. This enables the screw to work in a nut which is a hollow collar on the inside surface of which a similar screw is cut, the threads of the screw fitting in the grooves of the nut. The screw is rotated in the nut or the nut on the screw by a force applied on a wheel or lever attached to the rotor. On account of the rubbing between the rotor and the stator some friction is inevitable and so the useful work obtained in an actual screw is less than the work that should be got out from an ideal screw. So the mechanical advantage of an actual screw is less than the velocity ratio and the efficiency of the screw is always less than unity.

Threads of screws are generally triangular or square in section as shown in Figs. 49 (*a*) and (*b*), respectively. Screws are conventionally represented as in Fig. 49 (*c*). A screw and a nut form a relative



(*a*) (*b*) (*c*) (*d*)

Fig. 49

pair. The Whitworth *V*-thread in which the angle of the thread is 55°, shown in Fig. 49(*d*), is perhaps the most used thread in engineering.

Pitch (p) of screw thread.—The linear distance through which a screw moves when it is rotated once about its axis is called the pitch of the screw. It is the same as the axial distance between two consecutive threads of it.

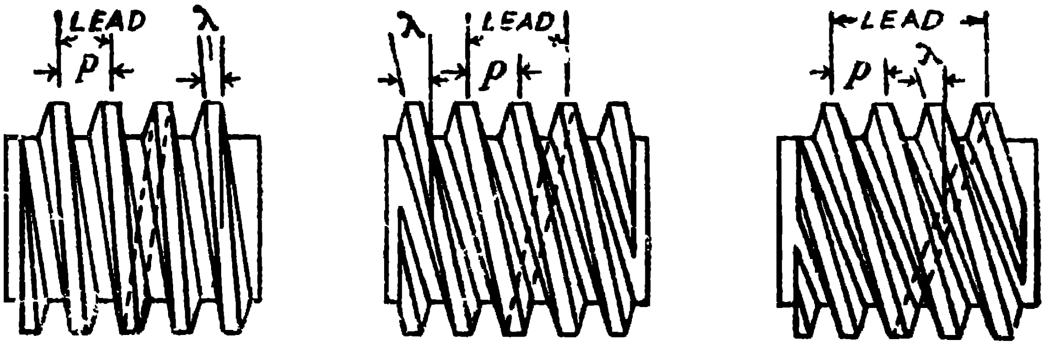


Fig. 50

Lead of screw thread.—It is the actual distance a nut on the thread would travel in making one complete rotation. When the screw is single threaded, the pitch and the lead are equal ; when double-threaded, the lead is twice the pitch. In general, when the screw is *n*-threaded, the lead is *n* times the pitch. Fig. 50 shows in the three diagrams from left to right, the lead of a single, double, and treble threaded screw.

Back-lash.— This error is present in almost all instruments with nut and screw. If due to wear, or any imperfection in manufacture, the screw is a loose fit in the nut, it may so happen that equal rotations of the screw-head in opposite directions may produce unequal linear movement of the screw, or any rotatory motion can be given to screw without causing any translatory movement of the nut when the latter should move ; then an error called **back-lash** exists in the instrument. When a screw and nut principle is utilised for measuring a small distance, as in the case of the spherometer, the screw should always be turned in the same direction to avoid *back-lash.*

67. Some common applications of the screw :—

A ***screw-jack*** (Fig. 51) used for lifting heavy loads like an automobile, a ***screw-press*** or ***letter-press*** used for compressing bound books

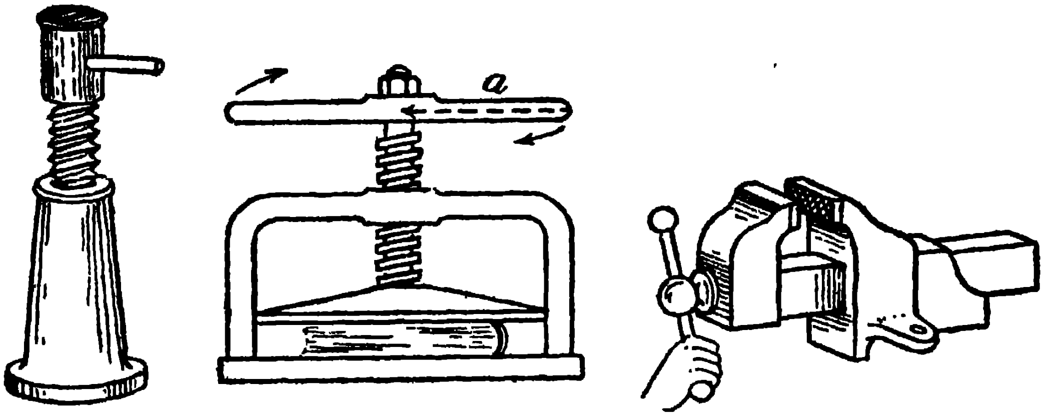


Fig. 51

Screw-jack Screw-press Vice

etc., a *vice* used in workshops for holding jobs with a strong grip, are common examples of a screw.

68. Velocity Ratio and Efficiency of a Screw-jack :—

Let a single-thread screw (Fig. 52 left) of pitch p working in a nut support a load of weight W and a force P be applied in the horizontal plane to the end D of a lever CD (length $= a$) fitted on the screw. In one complete turn of the lever arm, the distance travelled by the effort is $2\pi a$, while the load is moved up through a distance p.

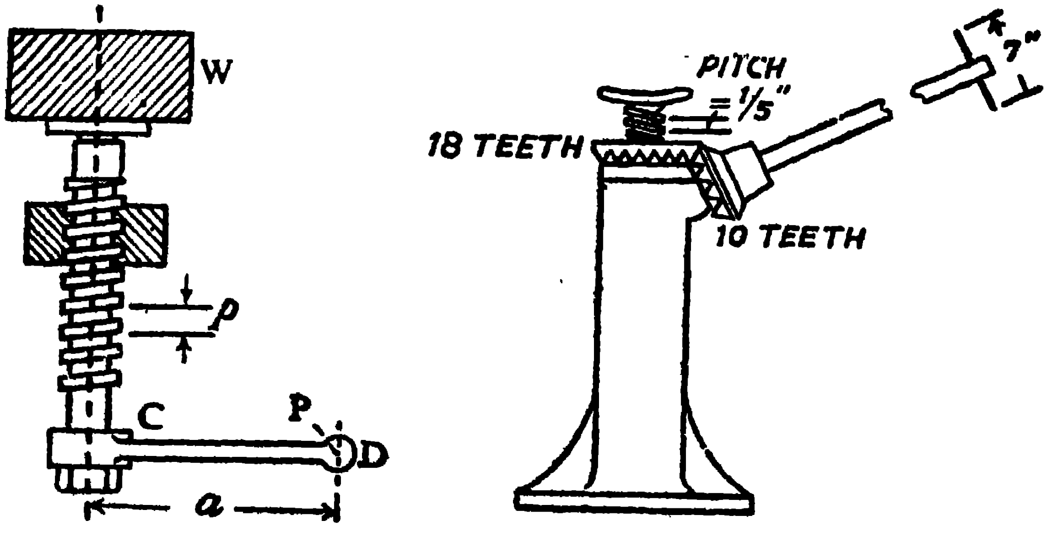


Fig. 52

The velocity ratio $(V.R.) = 2\pi a/p$. Since the mechanical advantage $(M.A.)$ is W/P,

$$\therefore \text{the efficiency} = \frac{\text{work got out}}{\text{work put in}} = (W \times p)/(P \times 2\pi a) = M.A. \times \frac{1}{V.R.}.$$

Examples

(*1*) *In an experiment, a screw-jack is arranged to be driven by a pulley as shown in Fig. 53. The lead of the screw (single-threaded) is ½ inch and the diameter of the pulley is 12 inches. Equal disc weights (P) of 4·5 lbs. are seen to raise a load W of 280 lbs. slowly and steadily. Find the efficiency of the jack.*

Ans. : In this case (single-thread screw), the lead is equal to the pitch of the screw.

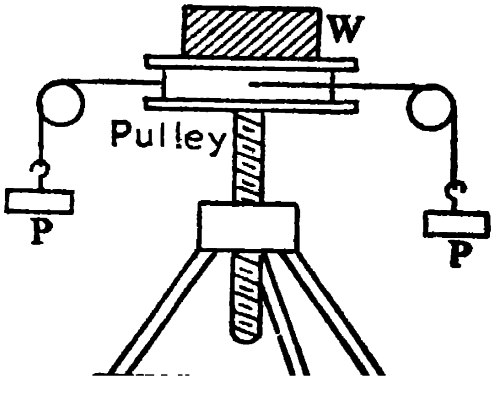


Fig. 53

$$V.R. = \frac{\text{distance moved by effort } P \text{ for one complete turn of screw}}{\text{distance moved up by load}}$$

$$= \frac{2\pi(12/2)}{1/2} = \frac{\text{circumference of pulley}}{\text{pitch of screw}} = 3{\cdot}14 \times 12 \times 2 = 75{\cdot}36.$$

$$M.A. = \frac{\text{load}}{\text{effort}} = W/P = \frac{280}{2 \times 4{\cdot}5} = 31{\cdot}11, \quad \therefore \text{ Efficiency} = \frac{\text{work got out}}{\text{work put in}}$$

$$= \frac{W \times \text{pitch of screw}}{P \times \text{circumference of pulley}} = 31{\cdot}11 \times \frac{1}{75{\cdot}36} = 0{\cdot}413, \text{ i.e., } 41{\cdot}3\%.$$

(*2*) *The length of each arm of a screw-press (Fig. 51) is 6 ins. and the pitch of the screw ¼ in. Forces of 14 lbs.-wt. are applied to each arm. Find the resistance overcome.*

Ans. : Work put in $= 2 \times 14 \times 2\pi \times 6/12$ ft.-lbs. Work got out $= W \times 1/(4 \times 12)$ ft.-lbs., where W = resistance in lbs.-wt. Neglecting friction. $\frac{28 \times 2\pi \times 6}{12} = \frac{W}{4 \times 12}$.

$W = 4224$ lbs.-wt.

69. Wedge :—A wedge is a simple machine consisting of a solid block of metal or wood shaped as an inclined plane. A small

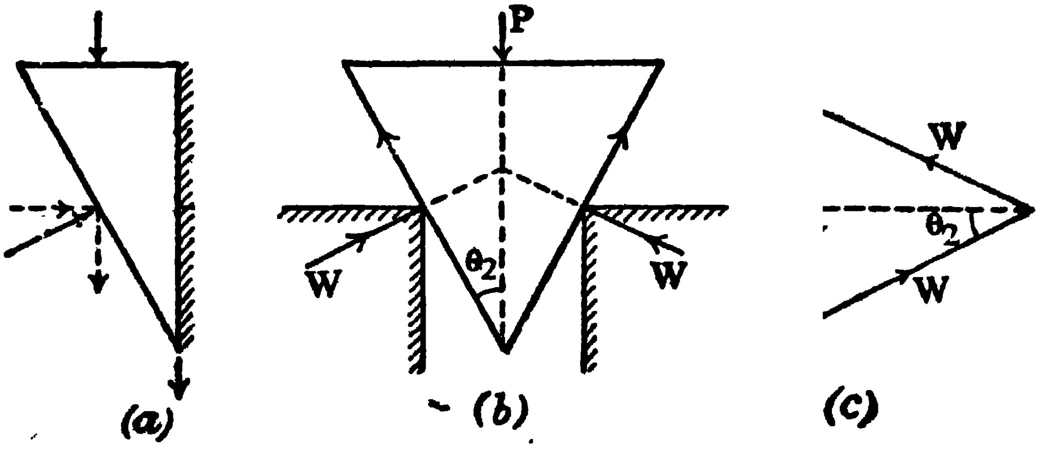


Fig. 54

driving force applied to the wedge results in a much larger splitting or separating force. It is commonly used for raising heavy bodies, for propping up a sinking wall, for widening a gap, for breaking strong cover joints, etc. [Fig. 54(*a*)].

A double wedge (semi-angle of the wedge $=\theta_2$) is shown in Fig. 54(*b*) being used for widening a gap. The separating forces generated produce equal reactions W, W at the edges of the gap. The forces P, W, W can be represented by a triangle shown in Fig. 54(*c*), neglecting friction.

$$\text{Here } P=2\,W\sin\theta_2\ldots(1)\ ;\ \text{and } M.A.=W/P=\frac{1}{2\sin\theta_2} \qquad (2)$$

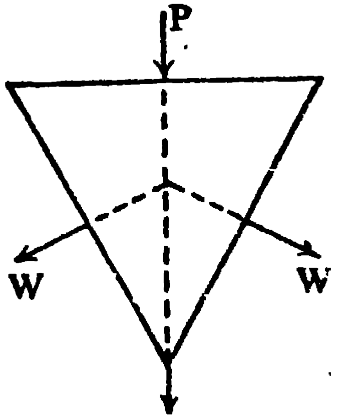


Fig. 55

The action of an axe, or a knife, or a nail may be treated as that of a combination of two wedges (Fig. 55).

Example

The angle of a wedge is 10°. Find the splitting force exerted by it when driven by a force of 15 lbs.-wt., and the mechanical advantage. Neglect friction.

Ans. : From equation (1) of the preceding article, $P=15=2\times W\sin 10^\circ/2$;

$$\text{or, } W=\frac{15}{2\sin 5^\circ}=\frac{15}{2\times 0{\cdot}0872}=86 \text{ lbs.-wt.}$$

$M.A.=W/P=86/15=5{\cdot}73$.

70. Magnification of Displacement by the Use of Levers : –

In machines and instruments it becomes often necessary either to magnify or to reduce the displacement of a moving element.

This is realised in practice usually by using a lever or levers. Fig. 56 represents an arrangement of double levers used to magnify a small displacement d_0 caused by an effort E at the end A_1 of a lever

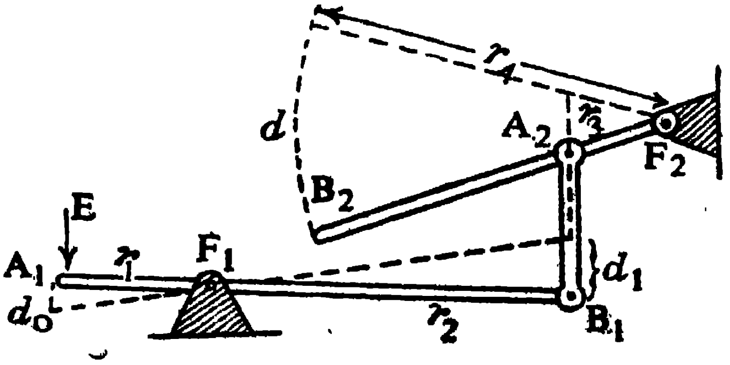


Fig. 56 – A Magnifying device of Levers.

A_1B_1 to a large displacement d (in two stages) at the free end B_2 of second lever. The fulcrum of the first lever is F_1 and that of the second

F_2, the working point B_1 of the first being rigidly connected by a stout rod to the driving point A_2 of the second lever. What happens, when the driving end A_1 of the first lever is given a finite displacement d_0 by the action of the effort E, is shown by the dotted lines.

$$\textbf{Overall magnification} = \frac{d}{d_0} = \frac{d}{d_1} \times \frac{d_1}{d_0}$$
$$= (r_4/r_3) \times (r_2/r_1).$$

71. Rack and Pinion :—A rack is a toothed wheel of infinite diameter. A rack and pinion in gear are shown in Fig. 57. When the rack is fixed, the pinion (with its attachments) rolls on it when rotated. When the pinion axle is fixed in position and the rack is movable, the latter with its attachments move as shown when the pinion is rotated.

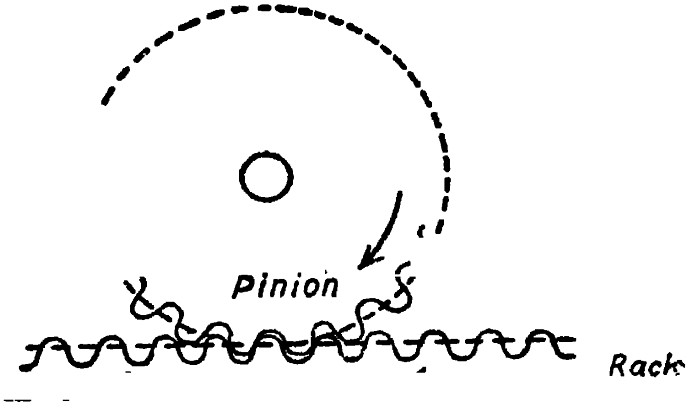


Fig. 57—Rack and Pinion.

72. Toothed Gearing or Gear wheels :—When energy is to be transferred in a rotating machinery from one part to another part, gear wheels are fitted provided the shafts of the two parts involved are quite close together. Here the rims of the wheels fitted to the shafts are provided with identical teeth which engage with each other in the fashion shown in Fig. 58. When the rotation of one shaft is transferred to another by the toothed gearing the angular speed varies inversely as the number of teeth in the corresponding gear wheel. If n_1 and n_2 be the number of teeth respectively on the driving and the driven gear, the mechanical advantage is given by, $M.A. = n_2/n_1$. When $n_2 > n_1$, the driven gear rotates at slower speed and when $n_1 > n_2$, the driven gear revolves faster than the driving gear.

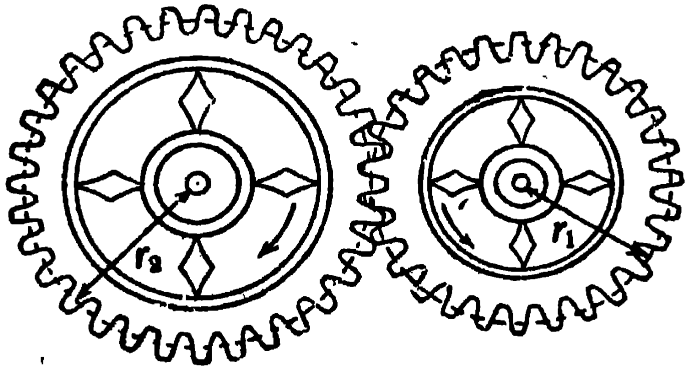


Fig. 58

If an imaginary circle be drawn with the axis of rotation as the centre and the distance from the axis to a point at half the height of any teeth as its radius, it is known as the ***pitch circle.*** If r_1 and r_2 be the radii of pitch circles of the driver and the driven gears,

then, $M.A. = \frac{n_2}{n_1} = \frac{r_2}{r_1}$.

Belt Driving.—When the revolving main shaft of the machine is at a considerable distance from the shaft of the part where the energy is to be transferred, each of these shafts are to be fitted with fly wheels with a broad rim. These wheels are to be connected with a belt of leather or rope running over the rims (Fig. 59). If r_1 and r_2 be the radii of the driver and the driven wheels, the mechanical advantages in this case is also r_2/r_1. The tensions on the two portions of the belt are not equal. Belt driving is advantageous for a machine running at a high speed. But in case the load on the driven shaft circuit is high the grip of the belting on the rim is not sufficient to prevent a slip between the two. So the required purpose is not served.

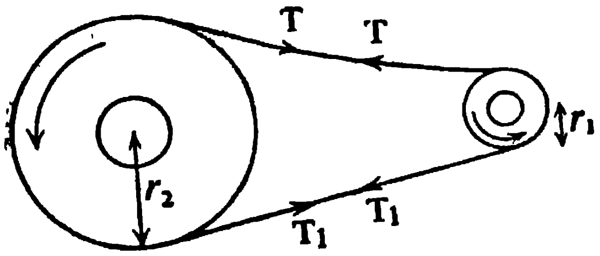


Fig. 59

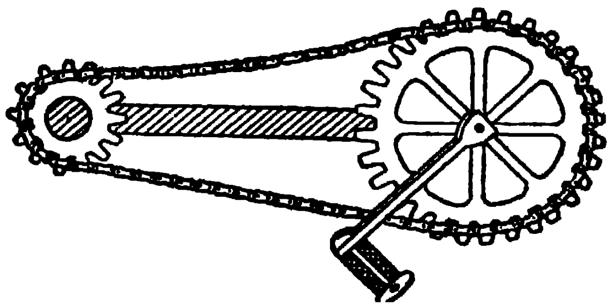


Fig. 60

Chain Driving.—To avoid the chance of slip between the wheel and the belting, the rims of the two fly wheels are cut into identical teeth and a chain exactly fitting in these teeth run over the two wheels. This ensures the best possible grip as we get in the case of the pedal of a bicycle connected to the axle of the rear wheel (Fig. 60). The high friction between the chain and the axle is disadvantageous at a comparatively high speed.

73. The Common Balance :—A common balance is an instrument of the utmost usefulness. It provides us with a ready means of measuring the *mass* of a body. We do not measure the *weight* of a body directly with it, though we ordinarily say that we do. A balance of this type is used by the grocer and this shows its importance in our daily life. A *sensitive* balance, of this type, usually referred to as a *physical* or *chemical* balance, is an indispensable necessity in the laboratory.

It consists of a horizontal rigid **beam** balanced at its centre on a knife-edge which rests on an agate plane fixed on the top of a vertical pillar (Fig. 61).

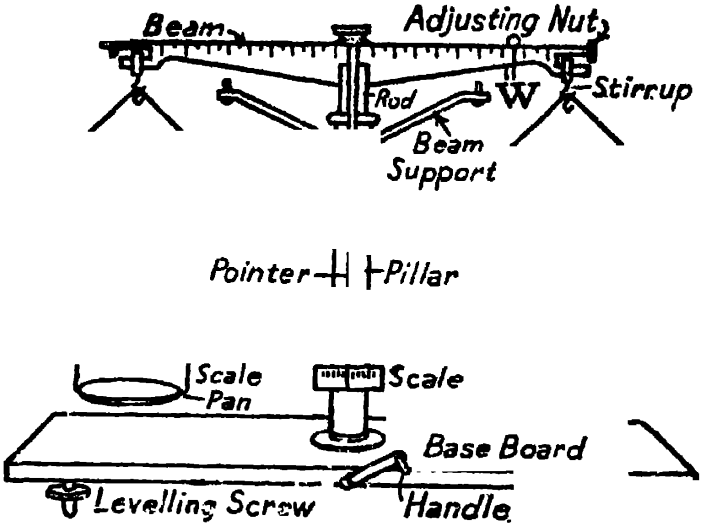


Fig. 61—A Physical Balance.

Two **scale pans** of equal weight are suspended from stirrups (or hangers) carried by knife-edges at the two extremities of the beam. The distances between the central knife-edge and those at the extremities are called the **arms** of balance, which should be equal. A long **pointer** attached to the centre of the beam moves over a **graduated** scale fixed on the pillar. For accuracy, the pointer should swing evenly to equal distance on each side of the middle mark of the scale. There is a **lever arrangement** (with a handle) by which the pillar, which supports the beam, can be raised or lowered and the beam arrested in order to preserve the sharpness of the knife-edges.

To use the balance, it is first of all levelled by levelling screws provided at the base board and then adjusted by means of two *screws* (adjusting nuts) at the two extremities of the beam until the pointer oscillates equally on both sides of the middle division. The *body* to be weighed is then placed on the *left-hand pan*, and *weights* from the weight box are added on the *right-hand pan*, until the pointer oscillates in the same way, as in the case of the unloaded beam ; it is then that the weights on the two sides are balanced. As the arms are equal, the two weights on the two pans are also equal. So the weight on the right-hand pan is equal to the weight of the body.

Principle of Measurement.—A common balance is an example of class I type of a lever in which the two arms, AF and BF, of a beam AB, F being the fulcrum, are equal (Fig. 62). Neglecting the weight of AB, and of the two scale-pans hung at A and B, if the beam remains in equilibrium at the horizontal position, when a weight W_1 is placed on the pan at A and a weight W_2 on the pan at B, we have, by taking moments

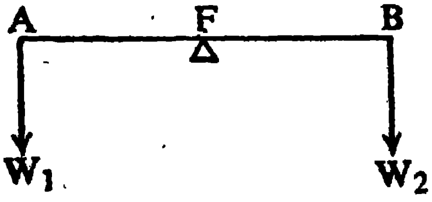


Fig. 62

about F, $W_1 \times AF = W_2 \times BF$ ... ... (1)
But $BF = AF$. $\therefore$ $W_1 = W_2$. That is, the weight on one pan is equal to the weight on the other at the position of balance when the beam is in the horizontal position. This is the principle of measurement by a common balance.

In practice, the weight of a given body is balanced by the combined weight of a number of standard masses of known values. Let m be the mass of the given body and m', the combined value of the standard masses required for balancing.

Then from (1), the two weights being equal, $mg = m'g$, where g is the acceleration due to gravity at the place, or $m = m'$. That is, an unkown mass is measured in terms of some standard masses supplied.

Note.—(*i*) *Weighing* by a balance means the determination of a known mass which has the same weight, as that of the unknown mass; and mass being proportional to weight, a common balance is used only to compare masses; for let W, W' be the weights of two bodies in poundals or dynes, as the case may be, and let their masses be m and m' respectively. Then we have, $W = mg$ and $W' = m'g$, where g is the acceleration due to gravity at the particular place. $\therefore$ $\frac{W}{W'} = \frac{mg}{m'g} = \frac{m}{m'}$.

Thus, the weights of two bodies at a given place are proportional to their masses.

(*ii*) The position of equilibrium for any two masses is unaltered by taking the balance to another place where the value of g is different when weighing is done by a common balance.

74. Theory of the Common Balance :—Suppose the beam AB having *equal* arms AF and BF turns round the fulcrum F, which, to diminish friction, is made of an agate or steel knife-edge resting on a smooth agate plate (Fig. 63). Let W_0 be the weight of the beam and

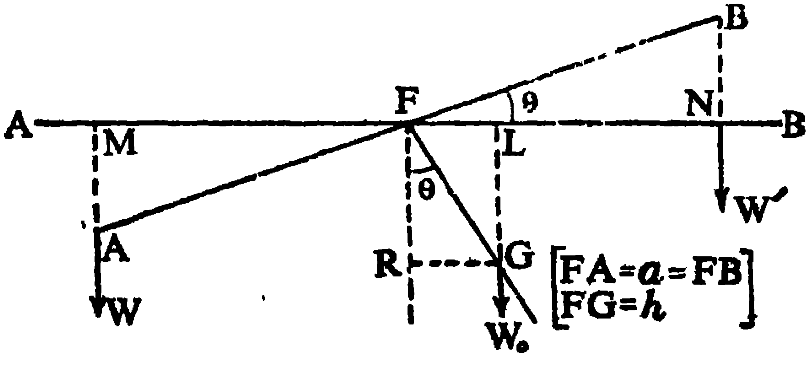


Fig. 63

pointer. Let us assume that the centre of gravity of the beam and pointer, through which W_0 acts, lies at G on the line FG which is perpendicular to the beam through F and is below F. Let two scale pans of equal weight S be hung at A and B. If now nearly equal weights W and W'

be placed on the pans at A and B respectively, and thereby the beam be tilted from the horizontal through θ, we have, by taking moments round F,

$$(W+S).\ FM = W_0.\ FL + (W'+S).\ FN.$$

$$= W_0 GR + (W'+S).\ FN \qquad \ldots \qquad \ldots \quad (1)$$

Also, $FM = FA \cos\theta = a\cos\theta = FN$, and $FL = GR = h\sin\theta$.

$$\therefore\ (W+S)\,a\cos\theta = W_0 h\sin\theta + (W'+S)\,a\cos\theta \qquad \ldots \quad (2)$$

$$\text{or, } \frac{\sin\theta}{\cos\theta} = \tan\theta = \frac{(W-W')a}{W_0 h} \qquad \ldots \qquad \ldots \quad (3)$$

75. Requisites of a good Balance :—A good balance must be (*a*) *true*, (*b*) *sensitive*, (*c*) *stable* and (*d*) *rigid.*

(a) True.—A balance is said to be true if the beam of the balance is horizontal when *equal weights* or *no weights* are in the pans. Equation (3) shows that $\theta = 0$, when either $W = W'$, or $W = 0 = W'$. Therefore, it follows from the assumptions made in arriving at equation (3) that a balance will be true provided the arms are (*i*) exactly of equal length, (*ii*) of exactly equal weight, *i.e.* the C.G. lies on the perpendicular to the beam at its middle, and (*iii*) the pans are equal in weight.

(b) Sensitive.—A balance is said to be sensitive, if for a small difference between W and W', the angle of tilt θ is large. Equation (3) shows that for a given difference between W and W', $\tan\theta$ or θ will be large if a is large, and W_0 and h small. Therefore, the conditions of sensitivity are that the beam should be (*i*) long, (*ii*) light, and (*iii*) its centre of gravity as near the fulcrum as possible.

(c) Stable.—A balance is said to be stable, if it quickly returns to its position of rest after being deflected, with equal weights in the pans. Equation (2) shows that, when $(W+S) = (W'+S)$, the only restoring couple (*i.e.* the couple which tends to restore the beam to its position of rest) arises from the weight of the beam. Hence for stability $W_0 h$ should be large (of course the C.G. must be below the fulcrum F). That is, for a given value for W_0 consistent with the rigidity of the instrument, the condition for stability is that h should be large, *i.e.* the C.G. as much below the fulcrum as possible.

(d) Rigid.—A balance is said to be rigid, if it be sufficiently strong so as not to bend under the weights it is intended to carry.

Note.—For a balance to be sensitive, the C.G. of the beam should be as near the fulcrum as possible ; while to be stable, the C.G. should be as much below the fulcrum as possible. Evidently, great sensitiveness and quick-weighing are incompatible in the same balance. In practice, these opposite conditions, however, do not present much difficulty ; for, in balances requiring high sensitivity as in the laboratory balances, accuracy of weighing forms the main criterion and quickness of weighing can be sacrificed to some extent. On the other hand, in commercial balances as used by grocers, etc. when large masses are used speed in weighing is more looked for than high accuracy. A compromise between the two opposite conditions is adopted when it is desired to combine the qualities of sensitiveness and quick weighing in the same balance to a moderate extent. This is done by making *h* neither too small, nor too large.

76. Test of Accuracy of a Balance :—Let *a* and *b* be the lengths of the arms of the balance, *S* and *S'* the weights of the scale pans.

Now, if the beam is horizontal with empty pans, we have, by taking moments about the fulcrum, $S \times a = S' \times b$............(1), provided the C.G. of the beam lies on the perpendicular to the beam through the fulcrum.

Again, the beam will be horizontal, if the equal weights *W*, *W* are placed on the pans. We have then, $(W+S).a=(W+S')b$ ... (2)

From (1) and (2) we get, $W \times a = W \times b$, or $a=b$, *i.e.* the arms must be of equal length ; and since $S.b=S'.b$, we have $S=S'$, *i.e.* the scale pans must be of equal weight. So, to test the accuracy of the balance, first see if the beam is horizontal when the scale pans are empty. Then put a body on one of the scale pans, and put weights on the other pan to balance it. Next interchange the body and the weights on the two pans. If the beam of the balance is still horizontal, the balance is true, otherwise it is not.

77. Weighing by the Method of Oscillation :—The operation of weighing by a sensitive balance takes a very long time before the beam comes to rest. It is, however, unnecessary to wait till the pointer comes to rest, for we can calculate the position which the pointer would occupy if the balance comes to rest. This can be done by observing the readings of the scale corresponding to the turning points of the pointer while the balance is swinging. The position so determined it called the **"resting point"** (written, R.P.) for a particular adjustment of weights and load, or for empty pans. It is more accurate and much quicker to perform the weighing by this method, which is called the **method of oscillation.** *This method is suitable when the weight to* be taken is small.

Procedure.—Imagine the scale divisions, over which the pointer moves, to be numbered from left to right. In an actual balance the scale divisions are numbered in the way shown in Fig. 64. In some types of balance, the rest position of the pointer, is maked zero and the graduations both on the left and on the right of zero are numbered in the increasing order as 5, 10, 15, etc. In Fig. 64 the graduation on the extreme left is zero and those on the right of it are in the increasing order 1, 2, 3, etc. Slowly raise and lower the beam two or three times so that the pointer swings over about ¾ of the whole scale. When after two or three oscillations the motion becomes regular, take a reading (say, 3) of the turning point of the pointer, by avoiding parallax, as it swings to the left, then read the extreme position (say, 10) of the subsequent swing to the right. Again read the next swing to the left (say, 4).

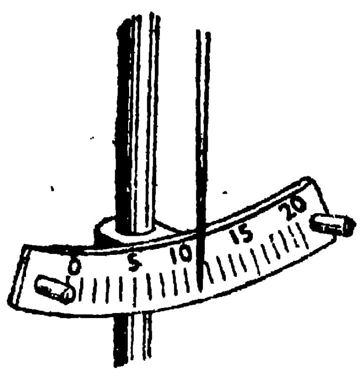


Fig. 64

Thus three readings, one to the right and two to the left, have been taken from which the R.P. for empty pans can be calculated in the following way. Take the mean of the two left-hand readings, *i.e.* the first and the third readings. Then the mean of this mean and the right-hand reading, *i.e.* the second reading, will give the mean R.P. for empty pans.

For greater accuracy five consecutive readings (two to the right and three to the left) should be taken. The R.P. for the empty pans is found as above a number of times and therefrom the mean R.P. (say, x) for the empty pans is obtained.

The reason for taking an odd number of observations is that the arc over which the pointer swings continually becomes less and less due to friction and air resistance, and thus if only two observations, say, one to the left and the next to the right, or two to the left and two to the right, are taken, the position of rest obtained by taking the arithmetic mean of these two will be too far to the left. The mean of any odd number of observations, obtained as above, will represent the true position of rest more accurately.

Next place the body to be weighed on the left-hand pan and try to get its weight (w) by adding wts. on the right-hand pan until the pointer oscillates within the scale. Let the mean R. P. for the loaded pans be y determined as in the case of the empty pans.

Next find out the mean R.P. (z) when the wt. w on the right-hand pan is increased by 1 milligram or any such small weight.

Now, calculate out as shown below, what weight must be added to or subtracted from w in order to reduce the R.P. from y to x.

Calculation.—

$$\text{True wt.} = w + \frac{0{\cdot}001}{y-z} \times (y-x) \text{ gm., when } y > x.$$

$$\text{If } y \text{ is less than } x, \text{ true wt.} = w - \frac{0{\cdot}001}{y-z} \times (x-y) \text{ gm.}$$

Note.—As the sensibility of a balance varies with the load, it should be calculated everytime a body is weighed. *The sensibility of a balance is defined as the change of the resting point due to a change of some definite weight, usually one milligram, in one of the pans.*

78. The Method of Double Weighing:—The true weight of a body can be determined with the help of a false balance by any one of the following two methods of **Double Weighing**—

(i) **The Method of Substitution** (*Borda's Method*).—Place the body to be weighed on the left-hand pan and counterpoise it by placing sand (or any other convenient substance) on the right-hand pan. Then remove the body and replace it by known weights to balance the sand. Since the weight of the body and the *weights* both balance the sand under exactly the same conditions, they must be equal.

(ii) **Gauss's Method.**—Let a and b be the lengths of the arms. Place a body of true weight W in the left-hand pan, and its apparent weight be W_1.

Then, taking moments of the force on each side,

$$W \times a = W_1 \times b \qquad \ldots \quad \ldots \quad \ldots \qquad (1)$$

Now put the body in the right-hand pan, and let W_2 be its apparent weight, then $W_2 \times a = W \times b$ $\qquad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (2)$

From (1) and (2), $\frac{W}{W_2} = \frac{W_1}{W}$; or, $W^2 = W_1 \times W_2$;

$$\text{or, } W = \sqrt{W_1 \times W_2} \qquad \ldots \quad \ldots \quad \ldots \qquad (3)$$

Thus, *the true weight is the geometrical mean of the two apparent weights.*

Ratio of the arms.—

From eqn. (1), $\frac{a}{b} = \frac{W_1}{W}$, and from eq. (2), $\frac{a}{b} = \frac{W}{W_2}$,

$$\therefore \frac{a^2}{b^2} = \frac{W_1}{W} \times \frac{W}{W_2} \text{ ; or, } \frac{a}{b} = \sqrt{\frac{W_1}{W_2}} \qquad \ldots \qquad (4)$$

79. A False Balance:—By using a false balance with unequal arms, a tradesman will defraud himself if he weighs out a substance (to be given to a customer), in equal quantities, by using alternately each of the scale pans. Let W be the true weight of the quantity of a substance which appears to weigh W_1 and W_2 successively by the two scale pans of a balance of which a and b are the length of the arms. Here, the customer gets $(W_1 + W_2)$ instead of $(W + W)$, *i.e.* $2W$; and we have, $W_1 + W_2 - 2W = W\frac{a}{b} + W\frac{b}{a} - 2W$ (from eqns. 1 and 2 of Art. 78).

$$= W\left(\frac{a^2+b^2-2ab}{ab}\right) = W\frac{(a-b)^2}{ab}.$$

The right-hand side of the equation is always positive whatever be the values of a and b, and so $(W_1 + W_2)$ is greater than $2W$. Thus the tradesman defrauds himself by the amount $W\frac{(a-b)^2}{ab}$.

Hence, at the time of purchasing a substance, *a customer should always insist on having half of that substance weighed on one pan and the other half on the other if he doubts the balance.*

Example

An object is placed on one scale pan, and it is balanced by 30 lbs. The object is then put into the other scale pan and now it takes 21 lbs. to balance it. When both scale pans are empty the scales balance. What is the matter with the balance, and what is the true weight of the object ?

Ans. : Two different weights are required to balance the same object when placed on different pans of the balance, because the arms of the balance are unequal [*vide* Art. 78 ii)]. The true wt. $= \sqrt{20 \times 21} = 20{\cdot}494$ lbs.

80. Buoyancy Correction in Weighing :—While weighing a body by a balance, the body to be weighed generally possesses a different density (and hence a different volume) from that of the standard weights. So a correction for the buoyancy of air is necessary to get the accurate value of the weight of the body.

Let ρ_1 = density of the material of the weights (generally brass) ;
ρ_2 = density of the material of the body ;
ρ_a = density of air ; W = true weight and W_1 = observed weight of the body.

The volume of the body $= \frac{W}{\rho_2}$ and the weight of the corresponding displaced air $= \left(\frac{W}{\rho_2}\right)\rho_a$ = upthrust due to air on W. Similarly, the upthrust on the standard weights (observed) $= \left(\frac{W_1}{\rho_1} \times \rho_a\right)$

For equilibrium, $W - \left(\frac{W}{\rho_2}\right)\rho_a = W_1 - \left(\frac{W_1}{\rho_1}\right)\rho_a$;

or $W\left(1 - \frac{\rho_a}{\rho_2}\right) = W_1\left(1 - \frac{\rho_a}{\rho_1}\right)$. $\therefore$ $W = W_1\left[1 - \rho_a\left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right)\right]$.

Example

A small piece of iron of density 7·8 gm./c.c., when weighed on a balance with standard weights of brass, weighs 10 gm. If the density of brass and air be respectively 8·5 gm./c.c. and 0·0013 gm./c.c. calculate the true weight of the iron piece.

Ans. : True weight $W=W_1\left[1-\rho_a\left(\frac{1}{\rho_1}-\frac{1}{\rho_2}\right)\right]$

$$=10\left[1-0{\cdot}0013\left(\frac{1}{8.5}-\frac{1}{7{\cdot}8}\right)\right]=9\,9998\text{ gm.}$$

Questions (Chapter III & IV)

1. Define the terms, *velocity ratio, mechanical advantage* and *efficiency*, as applied to machines.

2. Show that once a body is just ready to slide down an inclined plane, the tangent of the angle of inclination of the plane is equal to the co-efficient of friction.

3. What is the acceleration of a block sliding down a 30° slope, when the co-efficient of friction is 0·25 ? ($g=32{\cdot}2$ ft /sec.) [*Ans.* : 9·13 ft /sec.^{2}]

What h.p. is exerted in pulling a 200 lb. log up a 30° slope at the rate of 12 ft./sec. (co-eff. of friction=0·3) ? [*Ans.* : 3·3 h.p.]

4. A body starting from rest slides down an inclined plane whose slope is 30° (co-eff. of friction=0·2). What is its speed after sliding 76 ft. ? [*Ans.* : 40 ft./sec.]

5. What is the mechanical advantage of an inclined plane used as a machine, when $\theta=30°$ and the force acts (*i*) horizontally, and (*ii*) along the plane ? [*Ans.* : $\sqrt{3}:2$]

6. (*a*) State what you mean by 'limiting friction', and 'angle of friction.' Explain the laws of limiting friction and describe experiments to verify them.

(*b*) State the laws of friction establish the relation between the angle of repose and the co-efficient of static friction.

7. (*a*) Define 'machine' and 'mechanical advantage'.

(*b*) Justify the statement : 'What is gained in power is lost in speed' by considering two important machines.

8. Give a neat diagram and a brief description to show the working of the second system of pulleys, and deduce the mechanical advantage.

9. What are levers ? Give examples of different classes of levers.

10. A uniform beam weighing 72 lbs. and 12 ft. long is supported on two props at its ends. Where must a mass of 108 lbs. be placed so that the thrust on one prop may be twice as that on the other ? [*Ans.* : At $2\frac{2}{3}$ ft. from one end.]

11. Describe a lever of Class III. Calculate the mechanical advantage and show that the principle of work is satisfied there.

Give a brief description of the second system of pulley ; and deduce the mechanical advantage.

12. Deduce the M.A. of a *wheel and axle* from the general principle of conservation of energy.

13. A screw jack has a pitch of 0·05 inch. What weight will it lift (neglecting friction) when a force of 20 lbs. is applied at a point on the arm 18 inches from the axis ? [*Ans.* : 45216 lbs.]

14. Describe a jack-screw and state one of its practical applications with which you are familiar. Neglecting weight and friction of the machine, find out an expression for the mechanical advantage.

A screw-jack having a pitch of 0·25 inch is turned with a force of 50 lbs.-wt. applied at the end of a handle 3 ft. from the axis of rotation of the screw. Calculate the load which the jack will be able to raise. [*Ans.* : 45216 lbs.-wt.]

15. What are the requisites of a good balance? You are given an inaccurate balance; explain how it can be used to obtain accurate results.

The only fault in a balance being the inequality in weights of the scale pans, what is the real weight of a body, which balances 10 lbs. when placed in one scale pan and 12 lbs. when placed in the other? [*Ans.*: 11 lbs.]

16. What are the requisites of a good balance? A balance with unequal arms is used for weighing. The apparent weights of the same body placed in the two pans are 158·0 and 158·25 gms. respectively. Find the ratio of the balance arms.

[*Ans.*: $\sqrt{633} : \sqrt{632}$]

17. Explain with a neat sketch the principle and construction of a physical balance. Why is the method of double weighing adopted in the case of an inaccurate balance?

18. What are the requisites of a good balance? Explain clearly how you would proceed to determine the true weight of a body using a balance having unequal arms.

19. How would you determine whether the arms of a balance are of equal length, and how would you eliminate errors due to such an irregularity?

20. A body is placed on the pan of a balance, whose arms are unequal and is found to weigh W_1 gm. It is then removed to the other pan and weighed W_2 gm. Show that the actual weight is $\sqrt{W_1 \times W_2}$ gm.

21. Explain why a balance which is sensitive cannot be stable.

22. The turning points of a balance were observed to be successively 13,8,11. With the body on the left pan and 24·82 gm. on the right pan, the turning points were 14, 9, 12. On adding 10 more milligrams to the weights, the turning points became 10, 3, 8. Calculate the correct weight of the body. [*Ans.*: 24·822 gms.]

CHAPTER V

MOMENT OF INERTIA

81. Rectilinear Motion and Energy due to such Motion :– Whenever a body moves in a straight line, its motion is said to be *translatory* or rectilinear. In the case of a translatory motion, the force acting on the moving body is, from Newton's Second Law of motion, proportional to the rate of change of momentum. Since a body of mass m moving with a velocity v is attended with a momentum mv, the force P acting on the body may be written as

$$P=K\frac{d}{dt}(mv) \qquad \ldots \qquad \ldots \qquad \ldots \quad (1)$$

where K is the constant of proportionality which evidently reduces to unity when we define the unit of force either in the C.G.S. or in the F.P.S. system.

$\therefore$ Eqn. (1) reduces to $P=\frac{d}{dt}(mv)$ ... ... (2)

Since the mass of the body may be supposed to be fairly constant (except when its velocity approaches the velocity of light), eqn. (2) can be put in the form,

$$P=m\frac{dv}{dt}=mf \qquad \ldots \qquad \ldots \qquad \ldots \quad (3)$$

where f = acceleration of the body in the direction of the applied force, P.

The energy, that is associated with a body on account of its motion only, is kinetic in nature. This kinetic energy of the body is measured by the work done by the body against the opposing forces before coming to rest. If v be velocity of the body when opposing forces begin to act, the kinetic energy of the body is given by $E_k=\frac{1}{2}mv^2$ (4)

82. Rotational Motion and Energy of Rotation :—Whenever a body moves about a fixed straight line, the motion is said to be rotatory. The fixed straight line about which the rotation take place is called the ***axis of rotation*** or the ***gyrational axis***.

To consider rotational motion in detail it is better to start with a rigid body, which is supposed to be built up of particles that always maintain invariable positions with respect to one another. So when the body rotates round the fixed axis the constituent particles describe so many circles round this straight line. Since the distance of those particles from the axis are different and as they may possess different masses, their linear velocities as well as their kinetic energies will be different. To evaluate the total kinetic energy of rotation of the rigid body we shall have to calculate the kinetic energy of each of the constituent particles and then sum them up together.

Let *PQR* be a rigid body rotating uniformly round the fixed axis *OO′* (Fig. 65) with a constant angular velocity $\omega = d\theta/dt$ where $d\theta$ is the change in angular displacement in time dt. Let m_1, m_2, m_3, etc. be the masses of the particles forming the body and r_1, r_2, r_3 etc., the corresponding distances from the axis of rotation *OO′*. All the particles have the same angular velocity but their linear velocities are different. When the particle m_1 (or m_2, m_3 etc.) describes an angle θ, the corresponding linear displacement is given by the arc $r_1\theta$.

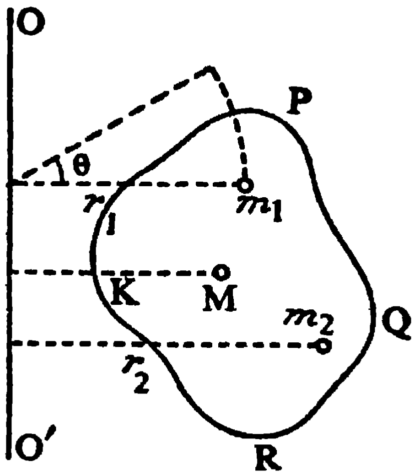


Fig. 65

∴ The linear velocity $= \frac{d}{dt}(r_1\theta)$

$$= r_1\frac{d\theta}{dt} = r_1\omega.$$

Similarly, for other particles the linear velocities will be $r_2\omega$, $r_3\omega$, etc.

∴ K.E. of particle $m_1 = \frac{1}{2}m_1(\omega r_1)^2$ and the kinetic energies of the other particles will be $\frac{1}{2}m_2(\omega r_2)^2$, $\frac{1}{2}m_3(r\omega_3)^2$ etc.

Therefore, the total K.E. of the rotating body

$$\begin{aligned} &= \tfrac{1}{2}m_1(\omega r_1)^2 + \tfrac{1}{2}m_2(\omega r_2)^2 + \tfrac{1}{2}m_3(\omega r_3)^2 + \ldots \\ &= \tfrac{1}{2}[m_1r_1^2 + m_2r_2^2 + m_3r_3^2 + \ldots]\omega^2 \\ &= \tfrac{1}{2}(\Sigma mr^2)\omega^2 = \tfrac{1}{2}I\omega^2 \quad \ldots \quad \ldots \quad (4) \end{aligned}$$

where I stands for Σmr^2 and is known as the **moment of inertia** of the rotating rigid body about the axis referred to above.. When the body under consideration is of discontinuous structure and consists of separate masses m_1, m_2 etc., I is obtained by direct summation as expressed by Σmr^2. But when it is of continuous structure, I must be obtained by the integral $\int r^2 dm$, where dm is an element of mass at a distance r from the axis. The units in which the moment of inertia is generally expressed are gm.-cm.^{2} and lb.-ft.^{2} and its dimensional formula is ML^2.

83. Moment of Inertia and Radius of Gyration :—We know that the kinetic energy of a body of mass M moving in a straight line with a velocity v is given by $E_T = \frac{1}{2}Mv^2$. But from eqn. (4) we get that the kinetic energy of the same body when subjected to rotational motion with a constant angular velocity ω as $E_r = \frac{1}{2}I\omega^2$ where I is the moment of inertia of the body about the gyrational axis. A simple comparative study of the two expressions of kinetic energy leads us to conclude that the moment of inertia, I, plays the same part in rotational motion as the mass M of the body plays in translatory motion. Since M is sometimes termed as the inertia of translatory motion, I may as well be termed as the inertia of rotational motion or simply the *rotational inertia. It is numerically equal to twice the kinetic energy of the body when rotating with unit angular velocity.*

If M be the total mass of the rigid body PQR (Fig. 65), we can write $I=\Sigma mr^2=MK^2$ where K is called the radius of gyration of the body about the axis OO'. The physical meaning of this supposition is that *if the distributed mass of the rigid body be supposed to be concentrated at a point at a distance K units of length from the axis of gyration, the moment of inertia under the circumstances will be the same as that of the body with the distributed mass.* The radius of gyration may thus be very well termed as the *equivalent radius* of the rotating body or simply the *swing radius.*

84. Angular Acceleration :—We know that the angular velocity $\omega\left(=\frac{d\theta}{dt}\right)$ is related to the linear velocity by the relation $\omega=v/r$.

$$\therefore\quad v=r\omega \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

Again, if we consider the rate of change of this angular velocity, which is the angular acceleration, it will be represented by $\frac{d\omega}{dt}=\frac{d^2\theta}{dt^2}$.

But linear acceleration, $f=\frac{dv}{dt}$. $\therefore\ f=\frac{d}{dt}(r\omega)=r\frac{d\omega}{d}$.

So the angular acceleration, $\frac{d\omega}{dt}$, is equal to the linear acceleration divided by the distance from the axis of rotation.

85. Torque and Uniform Angular Acceleration :—In rectilinear motion, any type of displacement is caused by a force whose magnitude is given by the product of the mass and the linear acceleration $(P=mf)$. In rotational motion, the angular displacement of any type is caused by a torque which is the moment of a force about the axis of rotation.

To find an expression for this torque in terms of a uniform angular acceleration, let us consider a rigid body rotating with a uniform angular acceleration about an axis and let a particle of mass m of the body be at a distance r_1 from this axis. If $\frac{d\omega}{dt}$ be the uniform angular acceleration, the linear acceleration of the body $=f_1=r_1\frac{d\omega}{dt}$.

Hence the force P_1 acting on m_1 is $m_1 r_1 \frac{d\omega}{dt}$, and moment of P_1 about the axis $= P_1 \times r_1 = m_1 r_1^2 \frac{d\omega}{dt}$. Similarly for another particle of mass, m_2, at a distance r_2 from the axis of rotation, the moment of the force $= P_2 r_2 = m_2 r_2^2 \frac{d\omega}{dt}$.

So is the case with all the constituent particles of the body.

∴ The total moment for the whole body = the required torque = $(m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2 + \cdots) \frac{d\omega}{dt} = \frac{d\omega}{dt} \Sigma mr^2 = I \times \frac{d\omega}{dt}$, where $I = \Sigma mr^2$ = the moment of inertia of the body about the axis.

Thus torque is also expressible in the form of the product of I and the angular acceleration. So the part played by mass in translational motion is performed by moment of inertia in rotational motion.

From the above expression of torque, we can, alternatively, define *moment of inertia of a body as the ratio of the mechanical moment to the angular acceleration of the body.*

Example

A wheel of moment of inertia 3000 gm.-cm.² rotates at a speed of 50 revs./sec., when a retarding torque is applied to it as a result of which its speed falls to 10 revs./sec. in to 2 minutes. Calculate the torque applied.

Ans.: Here $\omega_1 = 2\pi \times 50$ and $\omega_2 = 2\pi \times 10$.

∴ The angular acceleration $= \frac{d\omega}{dt} = \frac{\omega_1 - \omega_2}{t} = \frac{2\pi(50-10)}{2 \times 60} = \frac{2\pi}{3}$.

∴ The retarding torque $= I\frac{d\omega}{dt} = 3000 \times \frac{2\pi}{3} = 6284$ dyne-cm.

86. Angular Momentum or Moment of Momentum:—Let us consider a constituent particle of mass, m_1, of a rotating rigid body. Let its distance from the axis of rotation be r_1 and its angular velocity be ω. Then the linear velocity v of m_1 will be given by, $v = r_1 \omega$.

∴ Its momentum $= m_1 v = m_1 r_1 \omega$. The product of this linear momentum and the corresponding distance r_1 from the axis of rotation is generally known as the *moment of momentum.*

∴ The moment of momentum of the particle is $m_1 r_1^2 \omega$. When the moments of momentum of all the constituent particles are summed up, we get the moment of momentum of the rigid body. Thus the moment of momentum L is given by

$$L = \omega[m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2 + \ldots] = \omega \Sigma mr^2 = I\omega \qquad \ldots (5)$$

Thus from eqn. (5) we also see that I plays the same part in rotational motion as is played by M in translational motion. In case of linear momentum we know that is given by Mv. In case of moment of momentum, v is replaced by angular velocity ω and M by I. So $L = I\omega$ is more commonly called the *angular momentum.*

As the linear momentum is conserved in a system when no external force acts on it, so is the angular momentum conserved in absence of any external torque. This is the principle of *conservation of angular momentum.*

Example

An electron of mass 9×10^{-28} gm. revolves round a nucleus in an orbit of radius $0{\cdot}5\times10^{-8}$ cm. If the angular momentum of the electron in the orbit be 10^{-27} erg.-sec., calculate the linear velocity with which it travels in the orbit.

Ans.: Since the electron may be supposed to be a point-mass (or particle), its angular momentum is given by $m\omega r^2 = mvr$.

$$\therefore \quad mvr = 10^{-27} \text{ erg.-sec.}$$

$$\text{Hence} \quad v = \frac{10^{-27}}{9\times10^{-28}\times0{\cdot}5\times10^{-8}} = \frac{10^{9}}{4{\cdot}5}$$
$$= 2{\cdot}2\times10^{8} \text{ cm./sec.}$$

87. Kinetic Energy of a rolling Ball :—A rolling ball has two kinds of motion—one of which is rotational in character (rotating about an axis passing through the centre of the ball) and the other is translatory arising on account of the forward motion of the rolling ball. So the kinetic energy will also be partly rotational and partly translational.

Rotational $K.E. = \frac{1}{2}I\omega^2$, where I is moment of inertia of the body about the axis of rotation through the centre and ω is the angular speed of rotation. Translational $K.E. = \frac{1}{2}Mv^2$.

Since $I = Mk^2$ (where k = radius of gyration) and $v = \omega r$ (r = radius of the ball), the total kinetic energy of the rolling ball is given by, $E_k = \frac{1}{2} Mk^2\omega^2 + \frac{1}{2} M\omega^2 r^2 = \frac{1}{2} M\omega^2(k^2 + r^2)$ ... (6)

The $K.E.$ of a *rolling wheel* can be similarly calculated.

88. Work due to rotation —Suppose a body capable of rotation about an axis perpendicular to the plane of the figure and passing through the point O. Let the body be acted on by a force F applied at the point A at a distance l from O and the point of application displaced to B through a small distance $AB = ds$, where AB will form an arc of a circle of radius l. The corresponding angular displacement of the body will be $d\phi$. If the applied force makes an angle θ with AB, the effective component of the force along AB will be $F \cos\theta$. Hence the work done due to this small displacement will be,

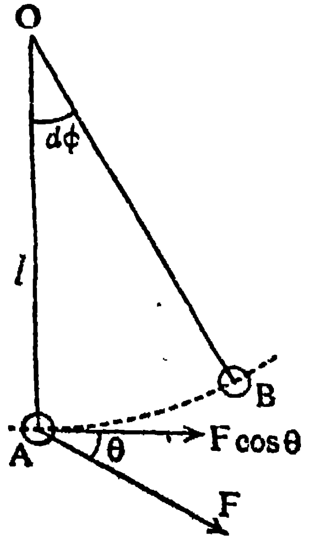


Fig. 66

$$dW = F\cos\theta\times ds = F\cos\theta\times l\,d\phi.$$

But $F \cos \theta \times l$ = moment of the effective force about O
= torque applied (C).

$$\therefore \quad dW = C d\phi \qquad \ldots \qquad \ldots \qquad \ldots \quad (1)$$

∴ The total work done due to movement of the body from its rest position to a final angular displacement ϕ is given by,

$$W = \int_0^{\Phi} C d\phi = C\phi.$$

This expression is true only when the applied torque is constant.

89. Analogy between Rotational and Translational Motions:—Rectilinear and the rotational motions are analogous in many respects. The mass and the linear velocity in rectilinear motion correspond to the moment of inertia and the angular velocity in rotational motion so that while mv and $\frac{1}{2} mv^2$ represent the linear momentum and kinetic energy in the former type of motion, $I\omega$ and $\frac{1}{2} I\omega^2$ represent the corresponding quantities in rotational motion.

A uniform linear acceleration (f) produced in a body (of mass m) by an applied force, P, is given by $f = \frac{dv}{dt} = \frac{P}{m}$.

Similarly, a uniform angular acceleration $\left(\frac{d\omega}{dt}\right)$ generated in a rotating body under the action of a torque C is given by,

$\frac{d\omega}{dt} = \frac{C}{I}$, ($I$ = the moment of inertia of the body).

It must, however, be remembered that while the mass of a body remains constant, its moment of inertia changes depending on the position of the axis of rotation.

Equations of Rotational motion.—The equations of motion as have been deduced in Art. 4 in the case of rectilinear motion have also their analogous forms in the rotational motion.

As the uniform speed in rotational motion is given by,

$$\omega = \frac{d\theta}{dt},$$

we have by integration, $\int d\theta = \omega \int dt$;

or, $\theta = \omega t + c$,

When $t = 0$, $\theta = 0$. ∴ $c = 0$.

(So, the angular displacement, $\theta = \omega t$) $\qquad \ldots \qquad \ldots \quad (1)$

Again, in the case of uniformly accelerated motion, let the angular acceleration be $a = \frac{d\omega}{dt}$.

Integrating we get,

$$\int d\omega = a \int dt;$$

or, $\omega = at + c_1$.

When $t = 0, \omega = \omega_0$, the initial angular velocity ; hence $c_1 = \omega_0$.

$$\therefore \quad \omega = \omega_0 + at \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

Now ω being the angular velocity in the case of an uniformly acceleration rotational motion at any arbitrary instant t, the angle described in infinitesimal time dt is $d\theta = \omega dt$;

or, $d\theta = \omega_0 dt + at.dt$.

Integrating, $\theta = \omega_0 t + \frac{1}{2}at^2 + c_2$.

When $t = 0$, $\theta = 0$, $\therefore c_2 = 0$.

$$\therefore \quad \theta = \omega_0 t + \tfrac{1}{2}at^2 \quad \ldots \quad \ldots \quad (3)$$

From (2), $\omega^2 = \omega_0{}^2 + 2a\omega_0 t + a^2 t^2$

$$= \omega_0{}^2 + 2a(\omega_0 t + \tfrac{1}{2}at^2)$$

$$= \omega_0{}^2 + 2a\theta \quad \ldots \quad \ldots \quad \ldots \quad (4)$$

Thus equations, (1), (2), (3) and (4) in rotational motion correspond to equation, $s = vt$, $v = u + ft$, $s = ut + \frac{1}{2}ft^2$, and $v^2 = u^2 + 2fs$ in rectilinear motion.

A table showing the analogy between the two types of motion is given below :

Rectilinear motion	Rotational motion
Displacement, S	Displacement, θ
Velocity, $v = ds/dt$	Velocity, $\omega = d\theta/dt$
Acceleration, $f = \frac{dv}{dt} = \frac{d^2s}{dt^2}$	Acceleration, $a = \frac{d\omega}{dt} = d^2\theta/dt^2$
Mass, M.	Moment of inertia, I.
Force, $M\frac{d^2s}{dt^2}$	Torque, $I\, d^2\theta/dt^2$
Kinetic energy, $\frac{1}{2} Mv^2$	Kinetic energy, $\frac{1}{2}I\omega^2$
Linear momentum, Mv	Angular momentum, $I\omega$

90. Theorems of Perpendicular Axes and Parallel Axes :— To calculate the moments of inertia of different bodies, we generally take the help of two well-known theorems, namely,

(a) Theorem of Perpendicular Axes, and
(b) Theorem of Parallel Axes.

(a) (i) Theorem of Perpendicular Axes (for a laminar body).—

Let OX, OY be two mutually perpendicular axes on the plate or lamina PQR (Fig. 67). The two axes meet at O. OZ is an axis perpendicular to the plane of the lamina and passing through O. The theorem states that *the moment of inertia of the lamina about an axis perpendicular to the plane of the lamina is equal to the sum of its moments of inertia about the two mutually perpendicular axes drawn through the same point and lying in the plane of the lamina.*

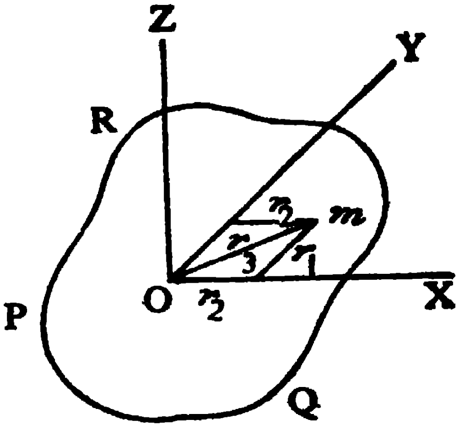


Fig. 67

So, mathematically, it can be written as,

$$I_z = I_x + I_y$$

where I_x, I_y, I_z, represent the moments of inertia of the lamina about the axes OX, OY and OZ respectively.

To establish the theorem let us imagine a particle of mass m of the lamina located as shown in the figure. Let r_1, r_2 and r_3 be the distances of m from the respective axes OX, OY, and OZ. Then, from geometry,

$$r_3^2 = r_1^2 + r_2^2$$

By definition, $I_x = \Sigma m r_1^2$, $I_y = \Sigma m r_2^2$ and $I_z = \Sigma m r_3^2$.

$$\therefore \quad I_z = \Sigma m r_3^2 = \Sigma m(r_1^2 + r_2^2)$$
$$= \Sigma m r_1^2 + \Sigma m r_2^2$$
$$= I_x + I_y.$$

(ii) Theorem of Perpendicular Axes (for three dimensional bodies).—

This theorem states that *the sum of the moments of inertia of a three dimensional body about any three mutually perpendicular axes drawn through the same point will be equal to twice the moment of inertia of the body about that point.*

Let a particle of mass m of the body be placed at P having co-ordinates x, y, z with reference to the perpendicular axes OX, OY, OZ passing through the point O (Fig. 68). Draw PM, PQ and PR perpendicular to OX, OY and OZ respectively.

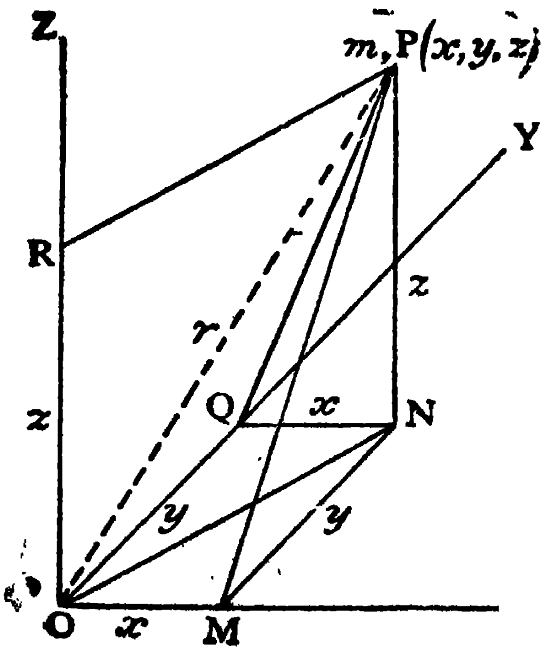


Fig. 68

Then, $I_x = \Sigma m.\ PM^2 = \Sigma m(y^2 + z^2)$

$I_y = \Sigma m.\ PQ^2 = \Sigma m(x^2 + z^2)$

and $I_z = \Sigma m.PR^2 = \Sigma m.\ ON^2 = \Sigma m(x^2+y^2)$

(where N is the projection of P on the XY plane).

$\therefore$ Summing up, we get,

$$I_x + I_y + I_z = \Sigma m(2x^2 + 2y^2 + 2z^2)$$
$$= 2\Sigma m(x^2+y^2+z^2) = 2\Sigma mr^2 = 2I_o,$$

where r = distance of P from O and

I_o = moment of inertia of the body about O.

This proves the theorem.

(b) Theorem of Parallel Axes (for laminar bodies).—This theorem states that if I be the moment of inertia of a lamina SQB (Fig. 69) of mass M about an axis AB and if I_g be that about a parallel axis, MN, passing through the centre of mass, G of the lamina, both axes lying in the plane of the lamina, then

$$I = I_g + Mr^2$$

where r is the perpendicular distance between the two axes. To prove the theorem, let us imagine a particle of the lamina of mass m at the point P of the body. Let x and $(r+x)$ be the perpendicular distances of P from the axes MN and AB respectively.

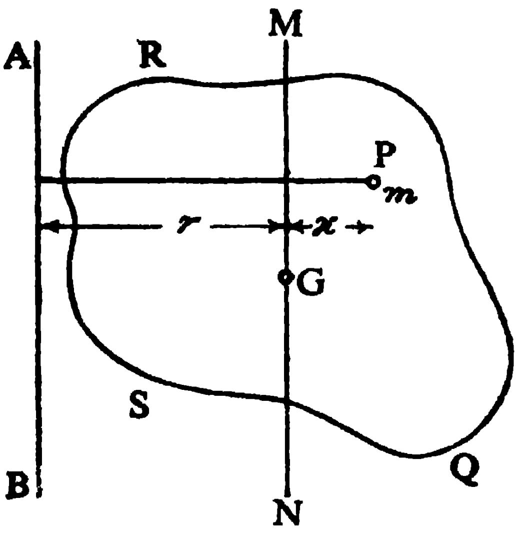


Fig. 69

Then, the moment of inertia of the body about AB is given by,

$$I = \Sigma m(x+r)^2 = \Sigma m(r^2 + 2rx + x^2)$$
$$= \Sigma mr^2 + \Sigma mx^2 + 2\Sigma mrx$$
$$= r^2\Sigma m + I_g + 2r\ \Sigma mx, \text{ since } r \text{ is here constant}$$
$$= Mr^2 + I_g + 2r \times \text{moment of the mass about the axis through the centre of mass } (G).$$

For the whole body the last term evidently vanishes.

$\therefore\ I = I_g + Mr^2.$

Thus the parallel axes theorem for a laminar body is established. It can be shown that the theorem can be extended even to the cases of three dimensional bodies.

91. Calculation of Moment of Inertia and Radius of Gyration :—

(a) Uniform thin rod.—

(i) About an axis passing through its centre and perpendicular to its length.

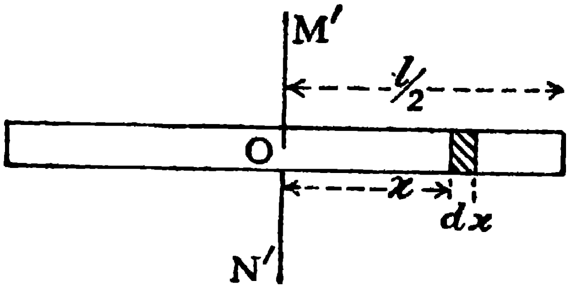


Fig. 70

Let M and l be respectively the mass and length of the rod. Consider a short element of length dx at a distance x from the axis $M'N'$ (Fig. 70) which passes perpendicular through the centre of mass O of the rod. Because of uniformity, the mass of dx is $\frac{Mdx}{l}$. As dx is very small every portion of dx may be supposed to be at the same distance x from $M'N'$. Therefore the moment of inertia of the element dx about $M'N'$ is given by $\frac{Mdx}{l} \cdot x^2$, and that of one half of the rod about $M'N'$ is

$$\int_0^{l/2} \frac{Mr^2}{l} dx = \frac{M}{l}\left[\frac{x^3}{3}\right]_0^{l/2} = \frac{Ml^2}{24}.$$

$\therefore$ For the whole rod the moment of inertia about $M'N'$ is given by

$$I = 2\int_0^{l/2} \frac{Mr^2}{l} dx = \frac{Ml^2}{12}.$$

Again $I = MK^2$.

$\therefore$ $K^2 = \frac{l^2}{12}$. Hence the radius of gyration, $K = \frac{l}{2\sqrt{3}}$.

(ii) About an axis through one of its ends and perpendicular to its length—

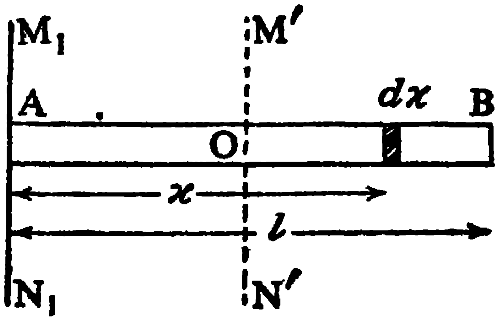


Fig. 71

Here M_1N_1 is the axis at the end A of the uniform thin rod AB (Fig. 71). At a distance x from A, take an element of length dx. Its mass is $\frac{M}{l} dx$, where M = total mass of the rod and l is the length of the rod.

Moment of inertia of the element 'dx' about M_1N_1

$$= \left(\frac{M}{l} dx\right) x^2.$$

Integrating for the whole rod, the moment of inertia of the rod about M_1N_1 is,

$$I=\int_0^l \frac{M}{l}x^2dx=\frac{M}{l}\cdot\frac{l^3}{3}=\frac{Ml^2}{3}.$$

Hence the radius of gyration is $K=\sqrt{\frac{l^2}{3}}=\frac{l}{\sqrt{3}}$.

[This case can also be deduced from case (*i*) by applying the theorem of parallel axis.

Since $I=I_g+Mr^2$

and here $I_g=Ml^2/12$ and r, the distance between the two axes $M'N'$ and M_1N_1 is $l/2$, we can write,

$$I=\frac{Ml^2}{12}+M\left(\frac{l}{2}\right)^2=Ml^2[\tfrac{1}{12}+\tfrac{1}{4}]=Ml^2/3.$$

(b) Rectangular uniform Lamina [*Thin rectangular plate*].—

(i) About an axis through its centre of mass and parallel to one of its sides.—

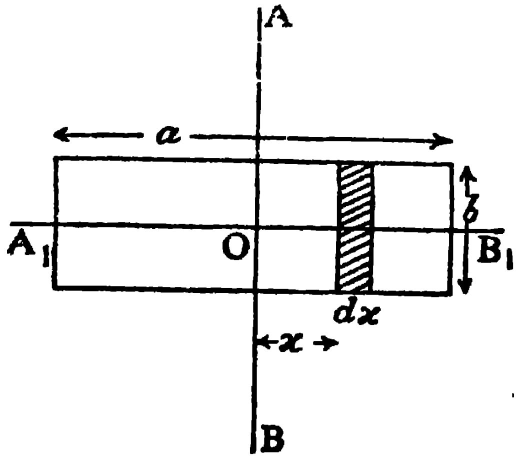


Fig. 72

Let the rectangular lamina of mass M have a length 'a' and a breadth 'b' (Fig. 72). Suppose I_1 be the moment of inertia of the lamina about AB which passes through the centre of mass O and is parallel to the side b. Consider a small slice of length dx at a distance x from AB. Since the lamina is uniform, the mass of the slice $=\frac{M}{a}dx$. Considering both halves of the lamina, its moment of inertia about

$$AB=I_1=2\int_0^{a/2}\frac{Mdx}{a}.x^2=\frac{Ma^2}{12}.$$

Similarly, the moment of inertia about A_1B_1 which is parallel to $a=I_2=\frac{Mb^2}{12}$

The radius of gyration is $K=\frac{a}{2\sqrt{3}}$ or $K=\frac{b}{2\sqrt{3}}$, as the case It ca of th)e.

9i About an axis through its centre of mass and perpendi- Gyrato the plane of the lamina.—

Moment of inertia of a lamina about an axis (Z-axis) passing through its centre of mass O and perpendicular to its plane is, according to the perpendicular axes theorem, $I=I_z=I_x+I_y$ where I_x and I_y are respectively the moments of inertia about the two axes XX' and YY' (Fig. 73) which lie in the plane of the lamina, passing through the centre of mass, O, and are parallel respectively to sides a and b. I_z is evidently equal to the moment of inertia about the axis ZZ' through O and perp. to the plane of the lamina.

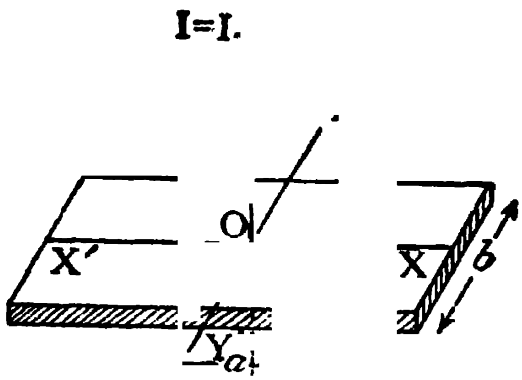


Fig. 73

From the previous consideration, b (i) above, we know,

$$I_x=\frac{Mb^2}{12} \text{ and } I_y=\frac{Ma^2}{12}. \quad \therefore \quad I=I_x+I_y=M\left(\frac{a^2+b^2}{12}\right).$$

The radius of gyration, $K=\sqrt{\dfrac{a^2+b^2}{12}}=\frac{1}{2}\sqrt{\dfrac{a^2+b^2}{3}}$.

(c) Circular Disc.—

(i) About an axis passing through its centre and perpendicular to its plane.—

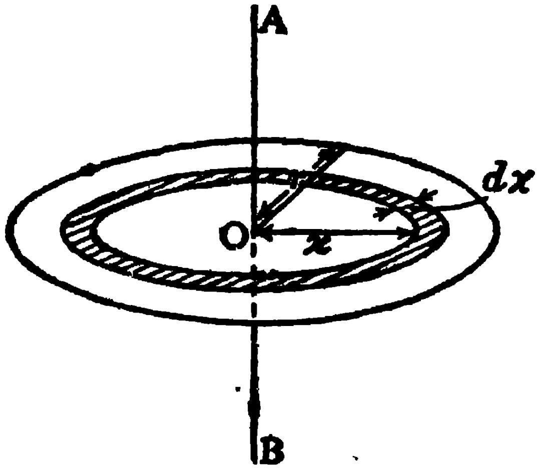


Fig. 74

Let the disc have a radius r and mass M. Consider an elementary annulus at a distance x from the centre O of the disc (Fig. 74). Let dx be the thickness of this annulus. The mass

$$\text{of this ring}=\frac{M}{\pi r^2}\times 2\pi x dx$$

$$=\frac{2M}{r^2}x.dx$$

$\therefore$ The moment of inertia of this ring about the axis AB through the centre O and perp. to the plane of the disc is given by $\frac{2M}{r^2}xdx.x^2$, because the annulus being of infinitesimally small thickness every point on it is at a distance x from O. $\therefore$ The moment of inertia of the whole disc about AB is given by,

$$I=\int_0^r \frac{2M}{r^2}.\ x^3dx=\frac{2M}{r^2}\left(\frac{x^4}{4}\right)_0^r=\frac{Mr^2}{2}.$$

The radius of gyration, $K=\sqrt{\frac{r^2}{2}}=\frac{r}{\sqrt{2}}$.

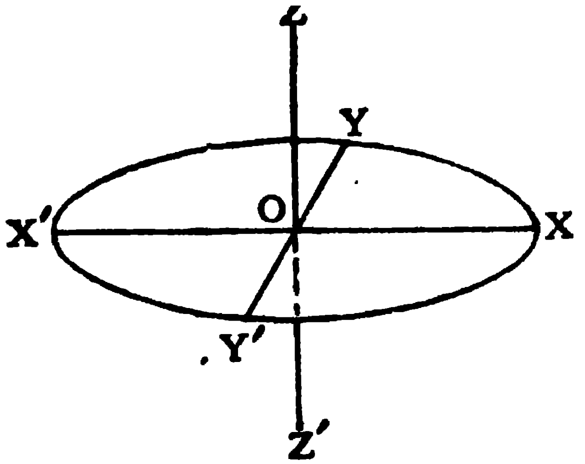


Fig. 75

(ii) About a diameter.—Let O be the centre of the circular disc and ZZ' the axis through O and perpendicular to the plane of the disc (Fig. 75 Let YY' and XX' be any two mutually perpendicular diameters of the disc. Since XX', YY' and ZZ' are mutually perpendicular, we can apply the theorem of perpendicular axes (in case of a laminar body) here. That is,

$$I_{zz'}=I_{xx'}+I_{yy'},$$

where the subscripts of the moment of inertia, I, refer to the corresponding axes.

From the last case we get, $I_{zz'}=\frac{1}{2}Mr^2$, and from symmetry of the circular disc, $I_{xx'}=I_{yy'}=I$, where I is the moment of inertia about any arbitrary diameter.

$$\therefore \tfrac{1}{2}Mr^2=I_{xx'}+I_{yy'}=2I; \text{ or, } I=\frac{Mr^2}{4}.$$

Hence the radius of gyration, $K=\frac{r}{2}$.

(iii) About a tangent.—Let $A'B'$ be the tangent at the point T of the disc. Let AB be the diameter parallel to $A'B'$ (Fig. 76). Then since the moment of inertia about AB is

$$I_1=\frac{Mr^2}{4},$$

we can get the moment of inertia I about the tangent $A'B'$ by applying the theorem of parallel axes.

$$\therefore I=I_1+Mr^2$$

$$=\frac{Mr^2}{4}+Mr^2=\tfrac{5}{4}Mr^2.$$

$\therefore$ The radius of gyration,

$$K=\frac{\sqrt{5}}{2}r.$$

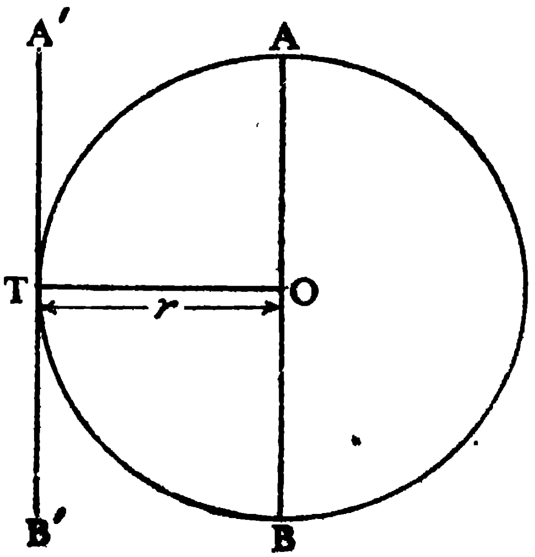


Fig. 76

(d) Flat Ring or Annular Disc.—

(i) About an axis passing through its centre and perpendicular to its plane.—

Let r_1, r_2, and M be the inner radius, outer radius and mass of the ring respectively. O is the centre of the ring OZ the axis through O perpendicular to the plane of the ring (Fig. 77). Consider an elementary concentric ring of radius x and thickness dx (shaded ring in the figure). Supposing the ring to be uniform, the mass of this elementary ring is $\frac{M}{\pi(r_2^2 - r_1^2)} \times 2\pi x dx = \frac{2Mxdx}{r_2^2 - r_1^2}$.

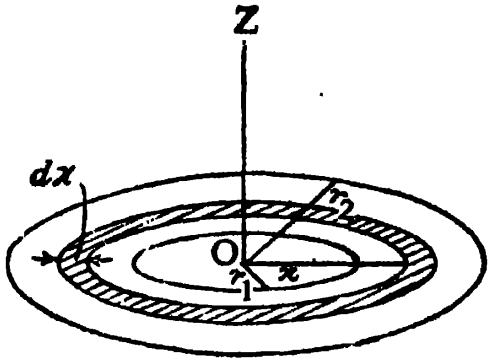


Fig. 77

Since dx is small, all points of this element are at a distance x from O.

∴ The moment of inertia of the elementary ring about OZ is

$$\frac{2M}{r_2^2 - r_1^2} \cdot xdx \times x^2.$$

The moment of inertia of the flat ring under consideration

$$= \frac{2M}{r_2^2 - r_1^2} \int_{r_1}^{r_2} x^3 dx = \frac{2M}{r_2^2 - r_1^2} \cdot \frac{r_2^4 - r_1^4}{4} = \frac{M(r_2^2 + r_1^2)}{2}$$

∴ The radius of gyration, $K = \sqrt{\frac{r_2^2 + r_1^2}{2}}$.

(ii) About a diameter.— Let the moment of inertia about any diameter be I. Then considering two mutually perpendiculars diameter XX' and YY' in the plane of the ring (Fig. 78), we get from symmetry,

$$I_{xx'} = I_{yy'} = I.$$

The moment of inertia about axis OZ perpendicular to the plane is $I_{zz'} = \frac{M(r_2^2 + r_1^2)}{2}$.

[from Case $d(i)$]

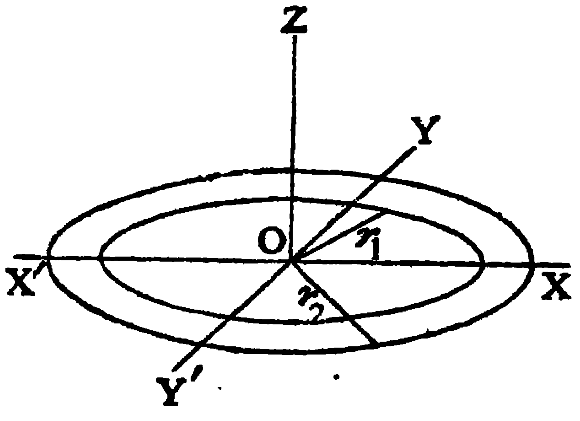


Fig. 78

From the perpendicular axes theorem, we get,

$$I_{zz'} = \frac{M(r_2^2 + r_1^2)}{2} = I_{xx'} + I_{yy'} = 2I.$$

$$\therefore \quad I = \frac{M(r_2^2+r_1^2)}{4}.$$

The radius of gyration, $K = \frac{\sqrt{r_2^2+r_1^2}}{2}$.

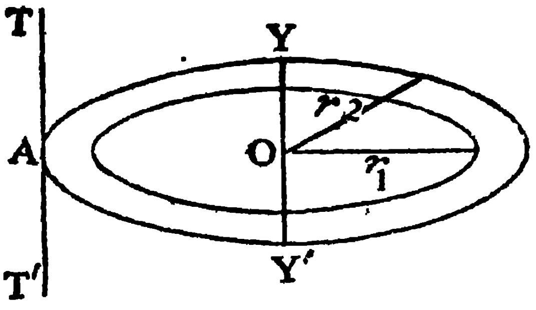


Fig. 79

(iii) About a tangent.— Let TT' be any tangent at the point A, YY' any diameter parallel to TT' (Fig. 79). Then by the theorem of parallel axes, the moment of inertia of the ring about TT' will be given by, $I = I_{yy'} + Mr_2^2$

$$= \frac{M(r_2^2+r_1^2)}{4} + Mr_2^2$$

$$= \frac{M}{4}(5r_2^2+r_1^2).$$

The radius of gyration, $K = \frac{1}{2}\sqrt{5r_2^2+r_1^2}$.

(iv) Special case of a thin ring of negligible width.— Since here the width is small $r_1 = r_2 = r$. So the moment of inertia about an axis through centre and perpendicular to the plane of the ring is given by $I_1 = \frac{1}{2}M(2r^2) = Mr^2$ [*cf.* case (*i*) above]. The moment of inertia of the thin ring about a diameter is $I_2 = \frac{M}{4}(2r^2) = \frac{Mr}{2}$, [*cf.* case (*iii*) above]. Its moment of inertia about a tangent is given by, $I_3 = \frac{M}{4}(5r^2+r^2) = \frac{3M}{2}r^2$, [*cf.* case (*iii*) above].

(e) Right circular cylinder.—

(i) About an axis through its centre of mass and perpendicular to its length.—

Let r be the radius of the cylinder whose centre of mass is at O (Fig. 80). The length of the cylinder is l and M is its mass. ZZ' represents the axis passing through O and perpendicular to the length of the cylinder.

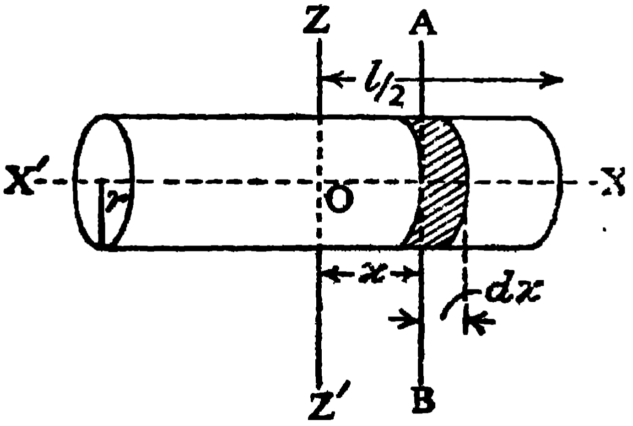


Fig. 80

Consider an elementary transverse section of thickness dx of the cylinder at a distance x from O. AB is a diametric axis of this element parallel to ZZ'. Let ρ be the density of the material of the cylinder.

Then moment of inertia of the elementary section about AB is given by $\frac{\pi r^2.dx\rho.r^2}{4}$ ($\because$ mass of the element is $\pi r^2 dx\rho$).

Its moment ol inertia about ZZ' is by parallel axes theorem $\frac{\pi r^4\rho.dx}{4}+\pi r^2 dx\rho x^2$.

Integrating this expression for the whole length of the cylinder, we get the moment of inertia of the cylinder about ZZ' as

$$I=\frac{\pi r^4\rho}{4}\int_{-l/2}^{+l/2}dx+\pi r^2\rho\int_{-l/2}^{+l/2}x^2dx$$

$$=\frac{\pi r^4\rho}{4}\Big[x\Big]_{-l/2}^{l/2}+\pi r^2\rho\left[\frac{x^3}{3}\right]_{-l/2}^{l/2}=\frac{\pi r^4\rho l}{4}+\frac{\pi r^2\rho l^3}{12}.$$

$$=\frac{Mr^2}{4}+\frac{Ml^2}{12}\ (\text{since } M=\pi r^2\times l\times\rho)=M\left(\frac{r^2}{4}+\frac{l^2}{12}\right).$$

The radius of gyration, $K=\sqrt{\frac{r^2}{4}+\frac{l^2}{12}}$.

(ii) **About its own axis.**— Let ZZ' be the axis of the cylinder of radius r (Fig. 81). Consider a co-axial cylindrical shell of radius, x, and of thickness dx. Its length is same as that of the original cylinder. If M be the mass of the cylinder, then from consideration of uniformity, mass of the elementary shell $=\frac{M}{\pi r^2}\times 2\pi x.\,dx$.

$\therefore$ Moment of inertia of the cylindrical shell about ZZ'

$$=\frac{2M}{r^2}xdx\times x^2=\frac{2M}{r^2}x^3.dx.$$

So the moment of inertia of the whole cylinder about ZZ' is given by

$$I=\frac{2M}{r^2}\int_0^r x^3.dx=\frac{2M}{r^2}\left[\frac{x^4}{4}\right]_0^r=Mr^2$$

The radius of gyration, $K=\frac{r}{\sqrt{2}}$.

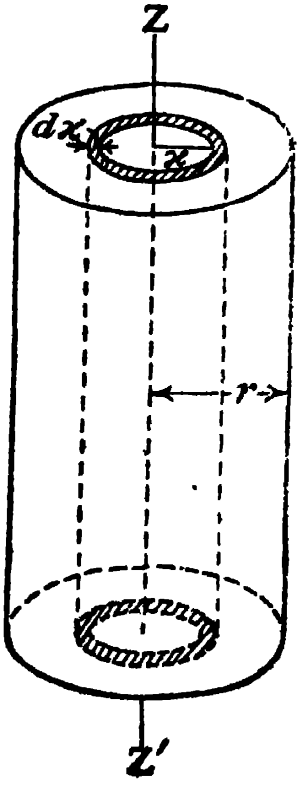


Fig. 81

Special cases of cylindrical shells (hollow cylinder).—As a cylindrical shell may be supposed to consist of a number of circular rings one above the other and as a solid cylinder is likewise comprised of a number of superimposed circular discs, the modifications that were introduced in the cases of circular rings over the discs should apply also in the case of hollow cylinders. Here if r_1 and r_2 be the inner and outer radii of cylinder, r^2 in cases (*i*) and (*ii*) above should be replaced by $(r_1^2+r_2^2)$. So the moment of inertia of the hollow cylinder about an axis through the centre of mass and perpendicular to its length given by, $I_1=M\left(\frac{l^2}{12}+\frac{r_1^2+r_2^2}{4}\right)$

and that about its own axis by,

$$I_2=\tfrac{1}{2}M(r_1^2+r_2^2).$$

[The complete calculations are left for the students as an exercise.]

(f) Solid Sphere.—

(i) About the centre.—

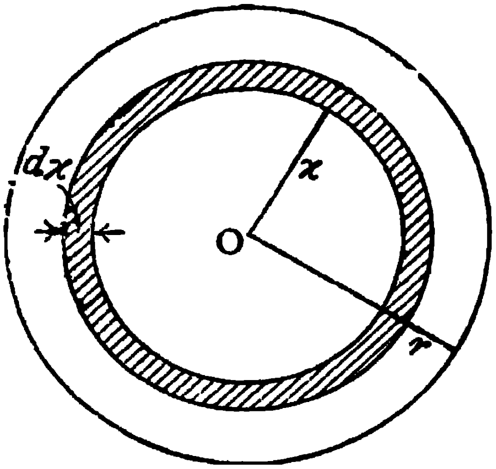


Fig. 82

Let the solid sphere of mass M be of radius r (Fig. 82). Consider a concentric spherical shell of radius x and of thickness dx. The mass of the shaded shell$=4\pi x^2.dx\rho$ where ρ is the density of the material of the sphere.

$\therefore$ The moment of inertia of the shell about the centre O is $4\pi x^2 dx\rho.x^2$.

Hence the moment of inertia of the whole solid sphere about the centre O is given by,

$$I_o=4\pi\rho\int x^4 dx=4\pi\rho\left[\frac{x^5}{5}\right]_0^r$$

$$=\frac{4\pi\rho}{5}r^5=\tfrac{4}{3}\pi r^3\rho.\frac{3r^2}{5}=\tfrac{2}{5}Mr^2$$

since $M=\frac{4}{3}\pi r^3\rho$. The radius of gyration, $K=\sqrt{\tfrac{2}{5}}.r$.

(ii) About a diameter.— Let O be centre of the sphere of mass M and of radius r (Fig. 83). If three mutually perpendicular diameters XX', YY' & ZZ' be drawn through the centre, the moments of inertia about these diameters will be identical from symmetry.

$\therefore$ $I_x=I_y=I_z=I$ (say), where subscripts indicate the respective axes about which the moment of inertia is considered.

Then according to the theorem of perpendicular axes in three

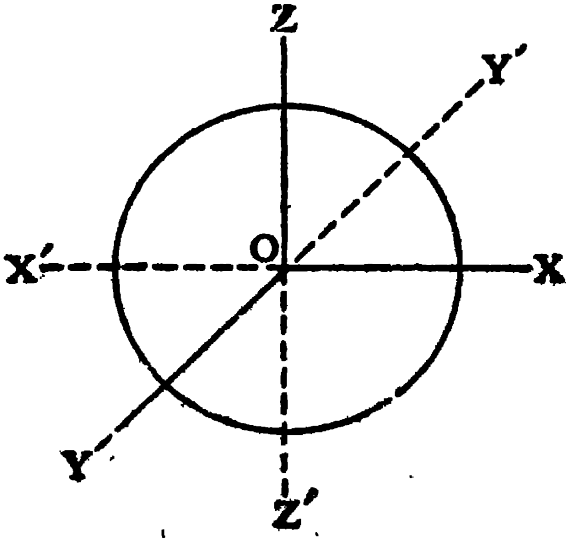


Fig. 83

dimensions, $2I_0 = I_x + I_y + I_z$ where I_0 is the moment of inertia about the centre. ∴ $3I = 2\times\frac{3}{5}Mr^2$, or, $I = \frac{2}{5}Mr^2$.

The radius of gyration, $K = r\sqrt{\frac{2}{5}}$.

(iii) **About a tangent.**—Applying the theorem of a parallel axes to the cases (*ii*) above, we get the moment of inertia of a solid sphere about a tangent as,

$I = \frac{2}{5}Mr^2 + Mr^2 = \frac{7}{5}Mr^2$.

(g) **Hollow Sphere.**—

(h) **About the centre.**—Let the follow sphere have its internal and external radii r_1 and r_2 respectively (Fig. 84). O is its centre. Its mass is $M = \frac{4}{3}\pi(r_2^3 - r_1^3)\rho$ where ρ is the density of the material of the shell. Consider a concentric elementary shell of radius x and of thickness dx. The mass of this elementary shell $4\pi x^2\,dx\rho$.

∴ Its moment of inertia about O is $4\pi x^4 dx\rho$. So the moment of inertia of hollow sphere about the centre is given by,

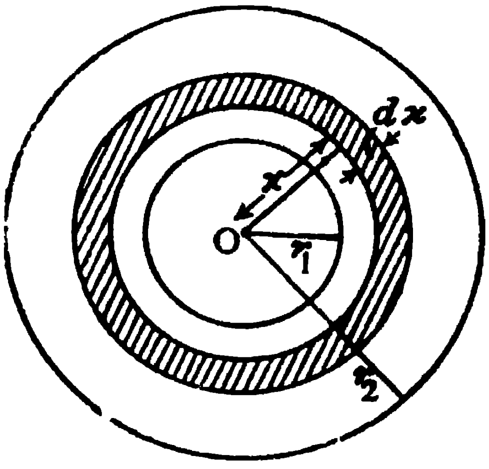


Fig. 84

$$I = 4\pi\rho\int_{r_1}^{r_2} x^4 dx = 4\pi\rho\frac{r_2^5 - r_1^5}{5}$$

$$= \frac{4}{3}\pi(r_2^3 - r_1^3)\rho.\frac{3}{5}.\frac{r_2^5 - r_1^5}{r_2^3 - r_1^3} = \frac{3}{5}M.\frac{r_2^5 - r_1^5}{r_2^3 - r_1^3}.$$

The radius of gyration, $K = \sqrt{\frac{3}{5}.\frac{r_2^5 - r_1^5}{r_2^3 - r_1^3}}$.

(ii) **About a diameter.**—

If at the point O, three mutually perpendicular diameters are drawn, the moment of inertia of the sphere about any one of them will be the same. Let it be I. Again its moment of inertia about the centre O is $I_0 = \frac{3}{5}M\frac{r_2^5 - r_1^5}{r_2^3 - r_1^3}$.

∴ From the perpendicular axes theorem (three dimensional) we get, $3I = 2I_0$; or, $I = \frac{2}{3}I_0 = \frac{2}{5}M\,\frac{r_2^5 - r_1^5}{r_2^3 - r_1^3}$.

The radius of gyration, $K = \sqrt{\frac{2}{5}.\frac{r_2^5 - r_1^5}{r_2^3 - r_1^3}}$.

Special case when the hollow sphere is of negligible thickness.—Here if M be the mass of the thin shell, then as points on the shell are at a distance r from the centre, the moment of inertia of the thin shell (spherical) about the centre is $I_0 = Mr^2$.

If I be the moment of inertia about any diameter then from the perpendicular axes theorem, $3I=2I_o$. $\therefore$ $I=\frac{2}{3}I_o=\frac{2}{3}Mr^2$.

92. Other important cases :—

(a) Moment of Inertia of a Right Circular Solid Cone about its Axis.—

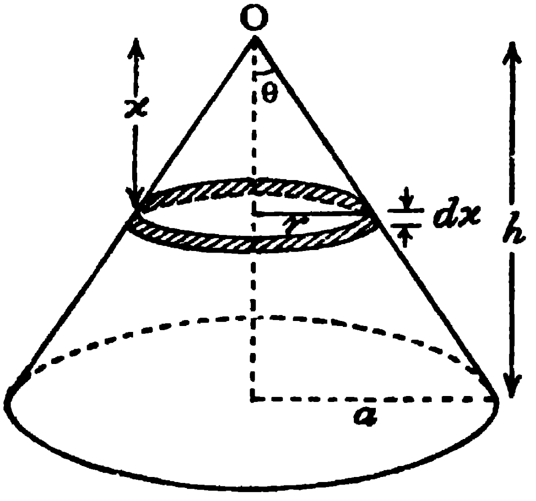


Fig. 85

Let O be the apex of a cone of height h and semi-vertical angle θ (Fig. 85). Let a be the radius of the base. If M be its mass and ρ the density of its meterial, then $M=\frac{1}{3}\pi a^2.h\rho$.

At a distance x from the apex O, consider a thin slice of the cone of thickness dx perpendicular to the axis. The radius of this slice is $r=x\tan\theta$.

So its mass is $m=\pi r^2.dx.\rho=\pi\rho\tan^2\theta.\,x^2.\,dx$. The moment of inertia of this slice (circular disc) about the axis of the cone $=\frac{1}{2}mr^2=\frac{1}{2}\pi\rho r^2\tan^2\theta.x^2.dx$ $=\frac{1}{2}\pi\rho\tan^4\theta.x^4dx$.

The moment of inertia of the whole cone about its axis,

$$I=\tfrac{1}{2}\pi\rho\tan^4\theta.\int_o^h x^4dx=\tfrac{1}{2}\pi\rho\tan^4\theta.\,\frac{h^5}{5}$$

$$=\tfrac{3}{10}.\,\tfrac{1}{3}.\,\pi\,(h\tan\theta)^2h\rho.\,(h\tan\theta)^2$$

$$=\tfrac{3}{10}\,Ma^2.\ (\text{since } h\tan\theta=a).$$

(b) Moment of Inertia of a Thin Hollow Right Circular Cone about its axis.—Let A be the apex of the hollow cone of height h and O be the centre of its base whose radius is a (Fig. 86). If l be the slant height of the cone and σ its mass per unit area, the total mass of the hollow cone is $\frac{1}{2}(2\pi a)\times l\sigma=\pi al\sigma=M$. Consider a slice at a distance x along the axis AO. The distance of the slice from A along the slant height is x_1. The thickness of the slice is evidently along the slant surface and sup[illegible] it to be dx_1. Its radius [illegible] denoted by r. From of [illegible]metry of the figure

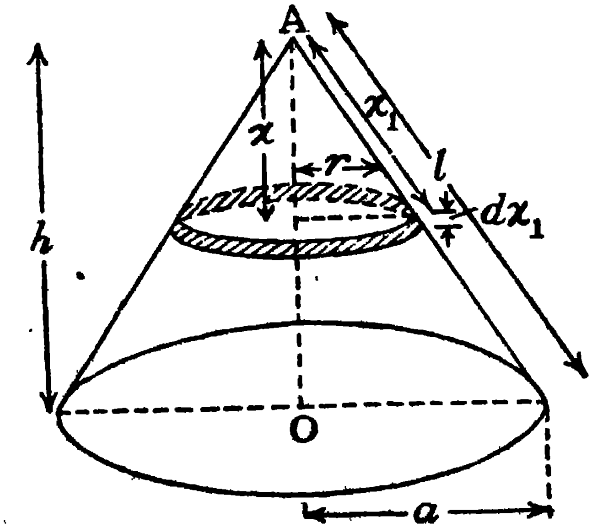


Fig. 86

we get, $\frac{r}{a}=\frac{x}{h}$; or, $r=\frac{a}{h}x$;

and $\frac{x_1}{l}=\frac{x}{h}$; or, $x_1=\frac{l}{h}x$. $\therefore$ $dx_1=\frac{l}{h}dx$.

Hence the mass of the slice $=2\pi r.dx_1.\sigma=2\pi\frac{al}{h^2}\sigma.xdx$.

$\therefore$ The moment of inertia of the slice about $AO=\frac{2\pi al\sigma}{h^2}x.r^2dx$ $=\frac{2\pi a^3l\sigma}{h^4}x^3dx$.

$\therefore$ The moment of inertia of the hollow cone about AO

$$=2(\pi al\sigma)\frac{a^2}{h^4}\int_0^h x^3dx=2.M\frac{a^2}{h^4}.\frac{h^4}{4}=\tfrac{1}{2}Ma^2.$$

(c) **Moment of Inertia of a triangular Lamina about one side of the Triangle.**—Let ABC be the triangular lamina under consideration (Fig. 87). It is required to find the moment of inertia of the lamina about BC. Suppose $BC=a$ and the perpendicular from A on BC be h, and $\sigma=$ mass per unit area of the lamina. Consider an elementary strip of thickness dx at a distance x from BC. Its mass $=$ length $\times$ thickness of the strip $\times\,\sigma$

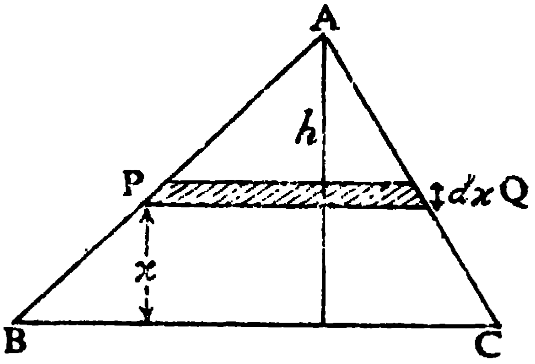


Fig. 87

$$=\frac{h-x}{h}a\times dx\times\sigma.$$

$\therefore$ Its moment of inertia about $BC=\frac{h-x}{h}adx\sigma\times x^2$.

$\therefore$ The moment of inertia of the lamina about BC.

$$I=\int_0^h\frac{h-x}{h}a\sigma.x^2dx=\frac{a}{h}\sigma\left(\frac{h^4}{3}-\frac{h^4}{4}\right)$$

$$=\tfrac{1}{12}a\sigma h^3=\frac{Mh^2}{6},$$

since the mass of the lamina is $M=\frac{1}{2}a\times h\times\sigma$.

Corollaries :—

(*i*) *Moment of inertia of a triangular lamina about an axis through its centre of mass and parallel to side BC.*

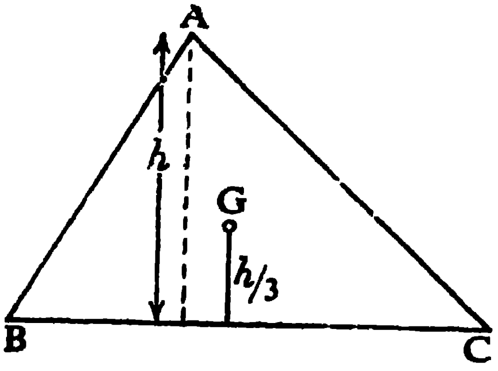


Fig. 88

Let G be the centre of mass of the triangular lamina (Fig. 88). Its distance from $BC=h/3$. Suppose I_0 be the moment of inertia of the lamina about an axis through G and parallel to BC, and I_a that about the side BC. Then by the parallel axes theorem

$$I_a=I_0+M\left(\frac{h}{3}\right)^2.$$

But $I_a=Mh^2/6$. $\therefore\quad I_0=\dfrac{Mh^2}{6}-\dfrac{Mh^2}{9}=\dfrac{Mh^2}{18}$.

(*ii*) *Moment of inertia of a triangular lamina about an axis through the vertex, A, and parallel to the side BC.*

Let G be the centre of mass of the lamina (Fig. 88). Its distance from $A=\dfrac{2h}{3}$. Let I be the required moment of inertia about the axis through A parallel to BC. Then by the parallel axis theorem,

$$I=I_0+M\left(\frac{2h}{3}\right)^2=\frac{Mh^2}{18}+\frac{4Mh^2}{9}=\tfrac{1}{2}Mh^2.$$

(d) Moment of Inertia of an Elliptic Lamina about one of its Principal Axes.—The equation of an ellipse referred to its principal axes is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \quad \cdots \quad (i)$$

where a and b refer to the two semi-axes.

We shall determine the moment of inertia of the elliptic lamina about the Y-axis.

Let the lamina be divided into thin strips parallel to the y-axis (Fig. 89) and one such strip of thickness dx be taken at a distance x from the centre O. Half length of this strip is evidently y. Hence its area is $2y.\,dx$. and mass is $2ydx\sigma$, where σ is the mass per unit area of the lamina. So its moment of inertia about YY' is $2y\sigma.x^2.dx$.

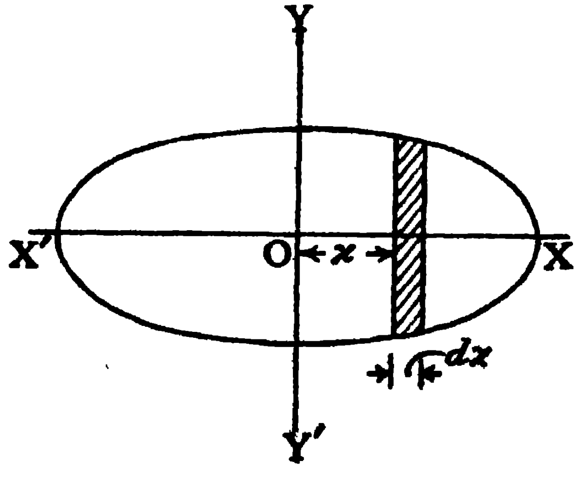


Fig. 89

Hence the moment of the whole elliptic lamina is,

$$I=2\sigma\int_{-a}^{+a} yx^2dx=2\sigma b\int_{-a}^{+a}\left(1-\frac{x^2}{a^2}\right)^{\frac{1}{2}}x^2\,dx \qquad \text{[From eqn. } \ldots \text{ (i)]}.$$

Put $x=a\sin\phi$, $\frac{x}{a}=\sin\phi$, and $dx=a\cos\phi\,d\phi$,

and the limits change to $\phi=-\pi/2$ when $x=-a$,
and to $\phi=\pi/2$ when $x=+a$.

$$\therefore\quad I=2\sigma b\int_{-\pi/2}^{+\pi/2} a^2\sin^2\phi\cos\phi\,.\,a\cos\phi\,.\,d\phi$$

$$=2a^3b\sigma\int_{-\pi/2}^{-\pi/2}\sin^2\phi\cos^2\phi\,.\,d\phi=2a^3b\sigma\,.\,\tfrac{1}{4}\,.\,{}^{\pi}/_{2}$$

$$=(\pi ab)\sigma.\frac{a^2}{4}=M\frac{a^2}{4},\ \text{where } \pi ab=\text{area of the ellipse.}$$

93. Routh's Rule :—Routh enunciated an important rule by means of which the moment of inertia in a few cases can be readily obtained. It is stated as follows—

"When the axis about which the moment of inertia is required is an axis of symmetry passing through the centre of mass of the body, the moment of inertia in a large number of cases is given by $I=\dfrac{\text{Mass}\times\text{sum of the squares of perpendicular semi-axes}}{3,\,4,\,\text{or }5}$.

where the denominator is to be taken 3, 4, or 5 according as the body is a rectangle, ellipse (including circle) or ellipsoid (including sphere).

Applications.—(1) Moment of inertia of a rectangular lamina of mass M and having sides a and b about an axis through the centre of mass and perpendicular to the plane of the lamina,

$$=M\frac{(a/2)^2+(b/2)^2}{3}=\frac{M}{12}(a^2+b^2).$$

(2) Moment of inertia of a circular disc of mass M and radius r about an axis passing through the centre and perpendicular to its plane

$$=M\times\frac{\left(\frac{D}{2}\right)^2+\left(\frac{D}{2}\right)^2}{4}\quad(\text{where } D=\text{diameter})$$

$$=M\frac{r^2+r^2}{4}=\frac{1}{2}Mr^2.$$

(3) Moment of inertia of a solid sphere of mass M and radius r about any diameter

$$=M\times\frac{\left(\frac{D}{2}\right)^2+\left(\frac{D}{2}\right)^2}{5}=M\frac{r^2+r^2}{5}=\frac{2}{5}Mr^2.$$

94. Time Period of Torsional oscillation of a body :— In rectilinear motion, the acceleration f, produced in a mass m by a force P is given by, $f=\frac{P}{m}$, according to Newton's Second Law of motion. Similarly, in rotational motion angular acceleration is given by $\frac{d^2\theta}{dt^2}=\frac{\text{turning force or torque}}{\text{moment of inertia of the body}}$.

If a piece of wire or a rod is held rigidly at one end and the other end is turned through small angle and let go, it is set to torsional oscillation (twisting and untwisting). Thus the twisting torque and the acceleration are oppositely directed. This torque may be regarded, within certain limits, as proportional to the angle through which the body has been turned. Thus, if at any instant, θ be the angular displacement of the body from its normal position, the torque$=\tau\theta$ where τ (torsional rigidity) is a constant depending on the length, radius and material of the wire or rod.

So the angular acceleration is given by,

$\frac{d^2\theta}{dt^2}=-\frac{\tau\theta}{I}$, where I is the moment of inertia of the body about the axis of turning.

Thus angular acceleration is proportional to the angular displacement and oppositely directed to it. So the oscillation is simple harmonic in nature (see Chapter VI).

$$\therefore\ \frac{d^2\theta}{dt^2}=-\frac{\tau\theta}{I}=-\omega^2\theta,\ \text{(where } \omega=\frac{2\pi}{T}=\text{angular velocity).}$$

$$\therefore\ \omega^2=\frac{\tau}{I};\ \text{or } \frac{T}{2\pi}=\sqrt{\frac{I}{\tau}};\ \text{or, } T=2\pi\sqrt{\frac{I}{\tau}}.$$

95. Experimental Determination of Moment of Inertia of a body :—To determine the moment of inertia of a body, first of all a torsion pendulum arrangement is to be set up as shown in Fig. 90. From the torsion head *H*, a cradle *C* is suspended by means of a wire, *w*, in an air-tight box *B*, provided with glass windows. The arrangement is so made that the torsion wire on prolongation passes through the centre of gravity of the suspended cradle. The torsion head is given a little twist so as to set the cradle into torsional oscillations. This must be carefully done so that pendulum oscillations are not set up. The time period of torsional oscillation of the cradle, T_0, is determined by means of a stop-watch from the observation of total time for nearly 50 oscillations. The oscillations may be observed by looking from above the chamber through the glass windows, *gg*, or by a lamp and scale arrangement with a small plane mirror attached to the cradle.'

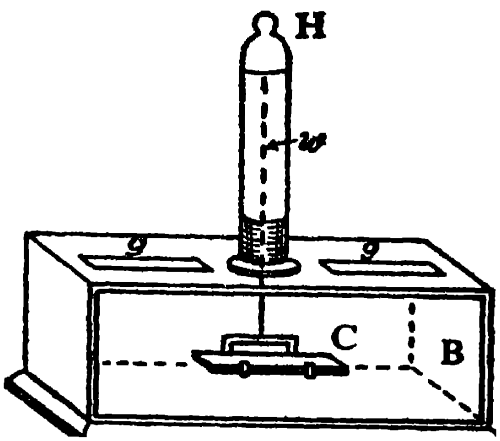


Fig. 90

An auxiliary rod, whose moment of inertia *I* can be known from its mass *M*, length *l*, breadth *b*, etc. [*e.g.* a rectangular bar whose axis of suspension is parallel to the thickness and passes through its centre of mass, and whose moment of inertia, $I=M\left(\frac{l^2+b^2}{12}\right)$] is taken and so placed on the cradle that its centre of mass lies on the torsional axis. Again the time period, *T*, of torsional oscillation of this combined system of cradle and auxiliary rod is determined in a similar manner taking care that no relative displacement takes place between the rod and cradle. The rod is now removed and replaced by the body whose moment of inertia is to be determined. It is also to be placed on the cradle so that the axis through which the moment of inertia is required coincides with the torsional axis. As before, the time period T_x of the new system (cradle and body) is also determined.

Let I_0, I and I_x be the moments of the inertia of the cradle, the rod and the body (experimental) respectively.

Then, $T_0=2\pi\sqrt{\frac{I_0}{\tau}}$, where τ = torsional rigidity of the wire, *w*.

$$T=2\pi\sqrt{\frac{I_0+I}{\tau}} \text{ and } T_x=2\pi\sqrt{\frac{I_0+I_x}{\tau}}$$

$$\therefore \quad T_0^2 = 4\pi^2\frac{I_0}{\tau}, \quad T^2 = 4\pi^2\frac{I_0+I}{\tau} \text{ and } T_x^2 = 4\pi^2\frac{I_0+I_x}{\tau}$$

$$\therefore \quad \frac{T_x^2 - T_0^2}{T^2 - T_0^2} = \frac{\frac{4\pi^2}{\tau}I_x}{\frac{4\pi^2}{\tau}I} = \frac{I_x}{I}$$

$$\therefore \quad I_x = \frac{T_x^2 - T_0^2}{T^2 - T_0^2} \times I.$$

All the terms on the right-hand side of the expression for I_x being known, I_x can be determined.

96. Determination of Moment of Inertia of a Flywheel :—A flywheel is nothing but a heavy wheel capable of rotation about an axle. In determining the moment of inertia of a flywheel it is to be mounted on ball-bearings and its axle is made horizontal at a suitable height from the ground (Fig. 91). A loop at the end of a fine cord is slipped over a tiny peg P fixed to the axle and the whole cord is uniformly wound on it. A known weight mg of suitable value is suspended from the free end of the string and is kept fixed at a particular height, x, from the ground.

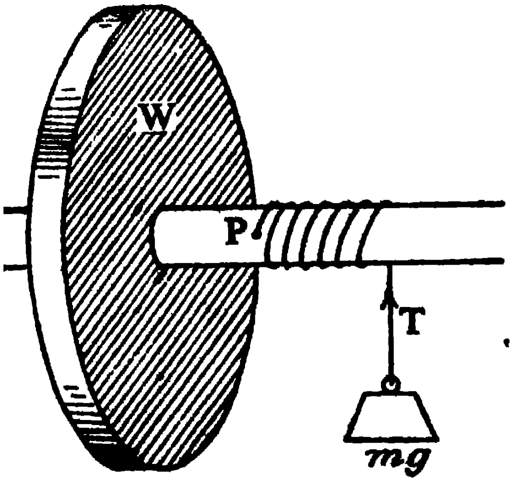


Fig. 91

Now as the weight is released it falls under its own weight thereby unwinding the string and as a result the wheel is set in rotation. The length of the cord is so chosen that as soon as the weight touches the ground the whole length of the cord would unwind and slip off the peg.

Let ω be the angular velocity of the wheel at the moment when the cord slips off the peg. Then at the instant when mg reaches the ground, the energy of the system is wholly kinetic—partly due to the movement of mg and partly due to the rotation of the wheel of moment of inertia I. If v be the velocity of the weight just at the moment it strikes the ground, the total kinetic energy of the system at that time

$$=\tfrac{1}{2}I\omega^2 \text{ (of the wheel)} + \tfrac{1}{2}mv^2 \text{ (of the weight)}$$

$$=\tfrac{1}{2}I\omega^2 + \tfrac{1}{2}m\omega^2 r^2 (r = \text{radius of axle}).$$

This energy must be equal to the potential energy mgx possessed by the weight at the start of its journey downwards (neglecting the effect of friction).

$$\therefore \quad mgx = \tfrac{1}{2}(I\omega^2 + m\omega^2 r^2) ;$$

$$\text{or,} \qquad I = \frac{2mgr - m\omega^2 r^2}{\omega^2} \qquad \ldots \qquad \ldots \qquad (1)$$

Hence from (1) I can be determined because g is known at a particular locality, m, x and r are directly measurable quantities and ω can also be found in the following way.

Starting from the time of detachment of the cord from the peg, the number of rotations undergone by the wheel before it comes to rest is counted and the corresponding time interval also noted. Let the number of rotations and the time interval be respectively n and t. As at the beginning and at the end of t, the angular velocities were respectively ω and 0, the average angular velocity of the wheel is evidently $\omega_a = \dfrac{\omega+0}{2} = \omega/2$.

Hence in t secs. the angle described is $\omega_a \times t$.

This is also equal to $2\pi n$;

$$\therefore \omega_a t = \frac{\omega}{2} t = 2\pi n ;$$

$$\text{or, } \omega = \frac{4\pi n}{t}.$$

Thus ω can be directly determined and hence I also, as stated above.

Note. If the friction at the bearing of the flywheel is not neglected, a small correction is to be applied. Let E be the energy used per revolution of the wheel in overcoming this friction and let n be the number of rotations of the wheel during the descent of the weight. Then the total energy needed to overcome friction $= nE$.

If the energy of the wheel $\frac{1}{2} I\omega^2$ is fully used in overcoming friction and it comes to rest after n_1 rotations then

$$n_1 E = \tfrac{1}{2} I\omega^2 \text{ ; or, } E = \frac{1}{n_1}\cdot \tfrac{1}{2} I\omega^2. \text{ Hence } nE = \frac{n}{n_1}\cdot\tfrac{1}{2} I\omega^2.$$

So the energy eqn. (1) takes from, $mgx = \frac{1}{2} m\omega^2 r^2 + \frac{1}{2} I\omega^2 \left(1 + \frac{n}{n_1}\right).$

$$\therefore \quad I = (2mgx - m\omega^2 r^2)/\omega^2\left(1 + \frac{n}{n_1}\right).$$

97. Determination of the Moment of Inertia of a Bar or a Cylindrical Rod by the Bifilar Suspension Method :—If a uniform bar or a cylindrical rod be horizontally suspended by two equal and parallel vertical threads equidistant from its C.G. the arrangement is known as a bifilar suspension. If such a system be slightly displaced from its equilibrium position, it will oscillate executing S.H.M. Timing the period of such oscillations and knowing the length of the vertical threads, the distance between the points of suspension and weight of the rod or bar, the moment of inertia of the same can be calculated.

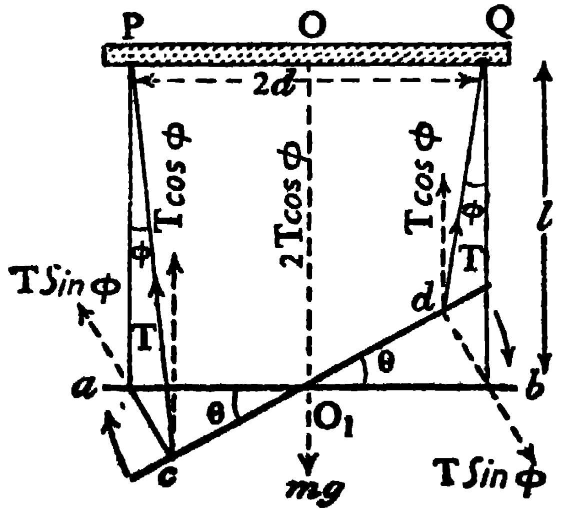


Fig. 92

Let ab be the bar suspended by parallel threads, each of length l, from the points P, Q where $PQ = 2d$ (Fig. 92), O_1 is the C.G. of the bar and OO_1 is the vertical line through it along which the weight of the bar mg acts downwards.

If ab be slightly displaced through a small angle θ about OO_1 in its own plane, the suspension threads will be displaced through an angle ϕ with respect to their original position. Now if T = tension along the threads, then in the displaced position it can be resolved into a vertical component $T \cos \phi$ and horizontal component $T \sin \phi$.

Considering both the threads together, we have.

(1) Sum of the two cosine components each acting vertically upwards $=2T\cos\phi$. This evidently acts through O_1 and balances mg.

$$\therefore\ 2T\cos\phi=mg.$$

$$\text{or, }\quad T=\frac{mg}{2}\ (\because \cos\phi=1,\ \phi \text{ being very small}) \quad \dots \quad \dots \quad (1)$$

(2) The two horizontally acting sine components being oppositely directed constitute a couple of arm $2d$. This couple tends to restore the rod to its rest position.

$$\begin{aligned}\therefore\ \text{Restoring couple} &= T\sin\phi\times 2d\\ &=2Td\,\phi\ (\because \phi \text{ is small})\\ &=mdg\,\phi\ [\text{ from (1) }].\end{aligned}$$

But as evident from the figure,

$$\phi=\frac{ac}{l},\ \text{and}\ \theta=\frac{ac}{d}.$$

$$\therefore\ \phi l=\theta d\ ;\ \text{or, }\phi=\frac{d}{l}\theta.$$

$$\text{Hence the restoring couple}=\frac{md^2g}{l}.\,\theta \quad \dots \quad \dots \quad (2)$$

The deflecting couple is $I\frac{d^2\theta}{dt^2}$ and is oppositely directed to the restoring couple.

$$\therefore\ I\frac{d^2\theta}{dt^2}=-\frac{md^2g\theta}{l}\ ;$$

$$\text{or, }\quad \frac{d^2\theta}{dt^2}=-\frac{md^2g}{I\times l}\,\theta \quad \dots \quad \dots \quad \dots \quad (3)$$

This is an equation of a S.H.M. whose time period is (vide Ch. VI),

$$T=2\pi\sqrt{\frac{Il}{md^2g}}\ ;\ \text{or, }\ I=\frac{T^2}{4\pi^2}\times\frac{mgd^2}{l} \quad \dots \quad (4)$$

Since in relation (4), T, m, g, d and l are determinable quantities, I can be also determined.

Table of Moments of Inertia

Body	Axis of Rotation	Moment of Inertia
1. Uniform thin rod (length $=l$)	(*a*) At end, perp. to length (*b*) At middle, perp. to length	(*a*) $Ml^2/3$ (*b*) $Ml^2/12$
2. Rectangular lamina, (sides *a* and *b*)	(*a*) Through C.G. and parallel to *b* (*b*) Through C.G. and perp. to the plane	(*a*) $Ma^2/12$ (*b*) $M\frac{(a^2+b^2)}{12}$
3. Rectangular parallelopiped (edges, *a*, *b*, *c*)	Through C.G and perp. to face *ab*	$M\frac{(a^2+b^2)}{12}$
4. Elliptical lamina (semi-axes *a*, *b*)	(*a*) About an axis (*b*) Through centre and perp. to plane	(*a*) $\frac{Mb^2}{4}$, or, $\frac{Ma^2}{4}$ (*b*) $M\frac{(a^2+b^2)}{12}$
6. Circular lamina (radius, *r*)	(*a*) Any diameter (*b*) Through centre and perp. to plane (*c*) About a tangent in its own plane (*d*) About a tangent perp. to its plane	(*a*) $\frac{Mr^2}{4}$ (*b*) $\frac{Mr^2}{2}$ (*c*) $\frac{5Mr^2}{4}$ (*d*) $\frac{3Mr^2}{2}$
6. Annulus or ring of outer and inner radii, *R* and *r*.	(*a*) Through centre and perp. to its plane (*b*) About a diameter	(*a*) $M\left(\frac{R^2+r^2}{2}\right)$ (*b*) $M\frac{(R^2+r^2)}{4}$
7. Right circular cylinder (length *l*, radius *r*)	(*a*) Axis of figure (*b*) Through centre and perp. to axis.	(*a*) $\frac{Mr^2}{2}$ (*b*) $M\left(\frac{l^2}{12}+\frac{r^2}{4}\right)$
8. Spherical shell (radius, *r*)	(*a*) About a diameter (*b*) About a tangent	(*a*) $2Mr^2/3$ (*b*) $5Mr^2/3$
9. Solid sphere (radius, *r*)	(*a*) About a diameter (*b*) About a tangent	(*a*) $2Mr^2/5$ (*b*) $7Mr^2/5$

Examples

1. *Two spheres of the same mass and same external radius are externally exactly similar in appearance. One of them is hollow while the other is solid. Explain how they can be identified. Indicate also how the internal radius of the hollow sphere can be determined.* (*C.U. 1961*)

Ans.: Let M be the mass of either sphere and r_2 the external radius of each of the spheres. The moment of inertia, I_S, of the solid sphere about a diameter $= \frac{2}{5} Mr_2^2$ and that of the hollow sphere is given by $I_H = \frac{2}{5}M\frac{r_2^5 - r_1^5}{r_2^3 - r_1^3}$ where r_1 is the internal radius of the hollow sphere.

Hence, $I_H - I_S = \frac{2}{5}M\left[\frac{r_2^5 - r_1^5}{r_2^3 - r_1^3} - r_2^2\right] = \frac{2}{5}M\left[\frac{r_2^2 r_1^3 - r_1^5}{r_2^3 - r_1^3}\right].$

Since r_2 is greater than r_1, right-hand side is positive. Hence I_H must be greater than I_S.

Now if the time periods of torsional oscillation of the solid and the hollow spheres are found out by successively suspending them at the end of the same torsion wire, the hollow sphere having greater value of the moment of inertia will have a greater time period since $T = 2\pi\sqrt{\frac{I}{\tau}}$. Thus the two spheres may be identified. [It may also be done by rolling them down an inclined plane.]

Once the spheres are identified, the internal radius of the hollow sphere can be easily determined from the values of the time periods obtained in the two cases and from the knowledge of M and r_2. If the time periods in the case of the solid and the hollow spheres be T_S and T_H, then $T_S = 2\pi\sqrt{\frac{I_S}{\tau}} = 2\pi\sqrt{\frac{2Mr_2^2}{5\tau}}$. From this expression knowing T_S, M and r_2, we get τ.

Again, $T_H = 2\pi\sqrt{\frac{I_H}{\tau}} = 2\pi\sqrt{\frac{2M(r_2^5 - r_1^5)}{5\tau(r_2^3 - r_1^3)}}.$

Here since τ has been previously determined and T_H, M and r_2 are known, r_1 can be easily obtained.

2. *A rod weighing 10 lbs. and of length 3 ft. revolves 50 times in a minute. Find its kinetic energy.*

Ans.: Kinetic energy $= \frac{1}{2}I\omega^2$ where I = moment of inertia and ω = angular velocity of rotation. Supposing the rod to be revolving round the axis passing through an end at right angles to the length,

$$I = \frac{Ml^2}{3} = \frac{10 \times 3^2}{3} = 30 \text{ lb.-ft}^2.$$

$$\omega = \frac{2\pi}{T} \text{ where } T = \text{time period} = \frac{60}{50} \text{ secs.}$$

$$\therefore \quad \omega^2 = \frac{4\pi^2 \times 50^2}{60^2} = \frac{4 \times (3{\cdot}14)^2 \times 50^2}{60^2} \text{ sec.}^{-2}$$

$$\therefore \quad \text{K.E.} = \frac{1}{2} \times 30 \times \frac{4 \times (3{\cdot}14)^2 \times 50^2}{60^2} \ \frac{\text{lb.-ft.}^2}{\text{sec.}^2}.$$

$$=410{\cdot}9\left(\text{lb.}\times\frac{\text{ft.}}{\text{sec.}^2}\right)\times\text{ft.}=410{\cdot}9\ \text{ft.-poundals.}$$

3. *A fly-wheel of mass 2·45 tons and of diameter 8 ft. makes 250 revolutions per minute. Find its kinetic energy.* (log $\pi=0{\cdot}4972$)

Ans.: Assuming the mass of the wheel to be concentrated at the rim, $I=Mr^2$ and $\omega=\frac{2\pi}{60/250}$.

$$\therefore\quad \text{K.E.}=\tfrac{1}{2}I\omega^2=\frac{2{\cdot}45\times2240\times4^2}{2}\times\frac{4\pi^2\times250^2}{60^2}$$

$$=3{\cdot}008\times10^7\ \text{ft.-poundals.}$$

4. *A sphere of mass 50 gms., diameter 2 cm, rolls without slipping, with a velocity of 5 cm. per sec. Calculate its total energy in ergs.* (P. U. 1943)

Ans: From the given data, mass of the sphere $(m)=50$ gm., its radius $(r)=1$ cm. and velocity, $v=5$ cm./sec.

Total energy of the body = K.E. due to translation + K.E. due to rotation $=\frac{1}{2}mv^2+\frac{1}{2}I\omega^2$.

Here $I=\frac{2}{5}mr^2$. and $\omega=\frac{v}{r}$.

$$\therefore\quad \text{Total energy}=\tfrac{1}{2}.mv^2+\tfrac{1}{2}.\tfrac{2}{5}.\ mr^2.\ \frac{v^2}{r^2}$$

$$=\tfrac{1}{2}mv^2(1+\tfrac{2}{5})=\tfrac{1}{2}\times50\times25\times\tfrac{7}{5}=875\ \text{ergs.}$$

5. *Four spheres each of diameter 2r and mass m are placed with their centres on the four corners of a square of side a. Calculate the moment of inertia of the system about one side of the square.*

Ans.: If we consider any side of the square as the axis, the two spheres at the ends, of this side will have the *axis* passing through their centres. Hence the moment of inertia of these two spheres about the side will be $2(\frac{2}{5}mr^2)=mr\frac{4}{5}^2$.

The other two sphere are at a distance a from the above axis chosen. The moment of inertia of each of them about that side of the square which is opposite to the chosen axis will also be $\frac{2}{5}Mr^2$. But as the axis chosen is parallel to that side, by the parallel axes theorem, the moment of inertia about the required axis for these two latter spheres will be $2(\frac{2}{5}mr^2+ma^2)=\frac{4}{5}mr^2+2ma^2$.

Hence the required moment of inertia $=\frac{4}{5}mr^2+\frac{4}{5}mr^2+2ma^2=\frac{2}{5}m(4r^2+5a^2)$.

6. *A flat thin uniform disc of radius a has a hole of radius b in it at a distance c from the centre of the disc* $[c<(a-b)]$. *If the disc were free to rotate about a smooth cylindrical rod of radius b passing through the hole, calculate the moment of inertia about this axis.*

Ans.: If M be the mass of the disc provided with a hole of radius b, supposing this hole to be at the centre of the disc, the moment of inertia about an axis through the centre and perpendicular to the plane of the disc is $I_0=\frac{M}{2}(a^2+b^2)$.

Now supposing the hole to be shifted through a distance c and the rotation axis to be passing through this hole, the moment of inertia.

$$I=I_0+Mc^2=\frac{M}{2}(a^2+b^2+2c^2).$$

7. *A fly-wheel of mass 500 kgm. and 2 metres in diameter, makes 500 revolutions per minute. Assuming the mass to be concentrated at the rim, calculate the angular velocity, the moment of inertia and the kinetic energy of the fly-wheel.*

(C.U 1965

Ans.: (*i*) *Angular velocity.*

The angle described per revolution is 2π radians.

∴ The angle described per min. $= 2\pi \times$ no. of rev. per min.

$$= 2\pi \times 500.$$

∴ The angular velocity = angle described per second

$$= \omega = \frac{2\pi \times 500}{60} = \frac{50\pi}{3} \text{ radians per sec.}$$

(*ii*) *Moment of Inertia.*

Since the mass is concentrated at the rim, the moment of inertia of the fly-wheel, $I = MR^2$.

Here $M = 500 \times 10^3 = 5 \times 10^5$ gm. and $R = 200/2 = 100$ cm.

∴ $I = 5 \times 10^5 \times (100)^2 = 5 \times 10^9$ gm.-cm².

(*iii*) *Kinetic Energy.*

$$\text{Kinetic energy, } E = \tfrac{1}{2} I\omega^2 = \tfrac{1}{2} \times 5 \times 10^9 \times \left(\frac{50\pi}{3}\right)^2$$

$$= 63{\cdot}57 \times 10^{11} \text{ ergs.}$$

Questions (Chapter V)

1. Define moment of inertia and explain its physical significance. What do you mean by 'radius of gyration'? [*cf.* C. U. 1946, '66 (Spl.)]

2. A body is doing a uniform angular acceleration under the action of a torque. Find a mathematical expression connecting the torque and the angular acceleration.

3. Explain the term 'angular momentum' (or moment of momentum).

4. A ball is rolling along a horizontal plane. Find an expression for its kinetic energy.

5. A thin rectangular lamina of metal is 80 cms. long and rotates about an axis in its own plane but normal to its length and equidistant from its ends. If it weighs 3·6 kgms., what is the moment of inertia, disregarding the thickness? [*Ans.* $1{\cdot}92 \times 10^6$ gm.-cm.²]

6. Calculate the moment of inertia about a transverse axis through its centre of a disc whose radius is 20 cms., its density 9 gms./cm.³, and its thickness 12 cms. [*Ans.* $27{\cdot}13 \times 10^6$ gm.-cm.²]

7. An annular ring of mass m has internal diameter d_1 and external diameter d_2. Find its moment of inertia about a diameter.

8. The mass of the earth is about 6×10^{21} tons and its radius about 4,000 miles Calculate its moment of inertia about its axis of rotation. [*Ans.* $23{\cdot}98 \times 10^{38}$ lb.-ft².]

9. An engine fly-wheel of 6 ft radius whose moment of inertia is 1,44,000 lbs.-ft.² has a steady tangential force of 1500 lb. applied to its rim. What is the resulting angular acceleration? (Take $g = 32$) [*Ans.* 2 ra./sec.²]

10. A ring whose mass may be regarded as being all equidistant from its centre rolls down an inclined plane in 12 sec. How long would a solid disc take to do the same? [*Ans.* 10·39 sec.] [Vide. Art. 119 (*d*)]

11. A solid disc rolls down an inclined plane 200 cm. long and 50 cm. high. Calculate its final linear velocity and the time of descent.
[*Ans.* 156 cm./sec. : 1·57 sec.] [Vide. Art. 119 (*d*)]

12. Explain how the moment of inertia of a bar of brass about an axis passing through its C.G. perpendicularly to its length may be determined in the laboratory by the torsional oscillation method, deducing the formula you would use for the purpose.

13. Calculate the moment of inertia of a thin spherical shell about a diameter. (P. U. 1955)

14. Derive an expression for the moment of inertia of a circular disc about (*i*) an axis perpendicular to its plane and passing through the centre and (*ii*) a diameter. (C. U. 1956)

15. Show that the kinetic energy of a thin rod of length l and mass m per unit length rotating about an axis through the middle point and perpendicular to its length with an angular velocity ω is $\frac{1}{24} m \omega^2 l^3$. (C. U. 1948)

[Hints : Here $I = Ml^2/12$

$= ml^3/12$. $\therefore$ $K.E. = \frac{1}{2} I\omega^2 = \frac{1}{24} m\, \omega^2 l^3$.]

16. What is meant by the moment of inertia of a body ? Obtain expression for the moment of inertia of an annular disc about an axis perpendicular to the plane of the disc through its centre. (Utk. U. 1966)

17. A circular disc of mass m and radius r is set rolling on a table. If ω be its angular velocity, show that its total energy is given by $E = \frac{3}{4} mr^2\omega^2$. (P.U. 1950)

[Hints : Here $I = \frac{1}{2} mr^2$

$\therefore$ $K.E. = \frac{1}{2} I\omega^2 + \frac{1}{2} mv^2$. Now put $v = \omega r$.]

18. A circular disc and a thin hollow cylinder open at both ends, each of the same radius, are allowed to roll down a smooth inclined plane starting from rest at the same height on the plane at the same instant. Calculate to show which one of the two reaches the bottom of the plane first. (Bihar U. 1959)

[*Ans.* The Disc.]

19. Calculate the moment of inertia of a uniform rectangular lamina about a line passing through a corner and perpendicular to its plane. (C.U.H. 1967)

20. Calculate the moment of inertia of a hollow sphere of uniform density ρ and outer and inner radii R and r respectively about a diameter. [Utk. U. 1970(*S*)]

CHAPTER VI

SIMPLE HARMONIC MOTION : COMPOSITION OF SIMPLE HARMONIC MOTIONS

98. Periodic Motion :—The periodic motion is a type of motion that repeats itself after a regular interval of time. The motion of the earth round the sun is periodic because the same motion is repeated after an interval of one year. Similarly, the motions of a clock pendulum and of a particle describing a circular path with uniform speed are periodic.

99. Oscillatory Motion :—When a periodic motion is reversed in direction after a fixed interval of time, it is said to be *oscillatory* or *vibratory* in nature. Thus the to-and-fro motion of a pendulum, the backward and forward motion of the piston of a reciprocating engine, etc. are instances of vibratory motion. One complete forward and backward motion is termed *a complete oscillation.*

An oscillatory motion is a periodic motion having a position of rest at which reversal in direction takes place. An oscillatory motion is therefore always periodic but a periodic motion may be oscillatory or not.

100. Simple Harmonic Motion :—Harmonic motion is an oscillatory motion executed in a symmetrical manner about a point. When an oscillatory harmonic motion is of the simplest type, *i.e.* when that motion occurs in a straight line, it is said to be *simple harmonic.* It is exactly defined with the help of its characteristics which will be explained subsequently in this chapter.

A body is said to execute a simple harmonic motion when it moves in a straight line in a forward and backward direction, the motion being repeated after a regular interval of time, and is acted on by a force, which is directed towards the mean position and is proportional to the displacement of the body from that position.

The ideal case of a simple harmonic motion may be realised by the motion of the foot of the perpendicular drawn on a diameter of a circle from a point describing the above circle with a uniform speed.

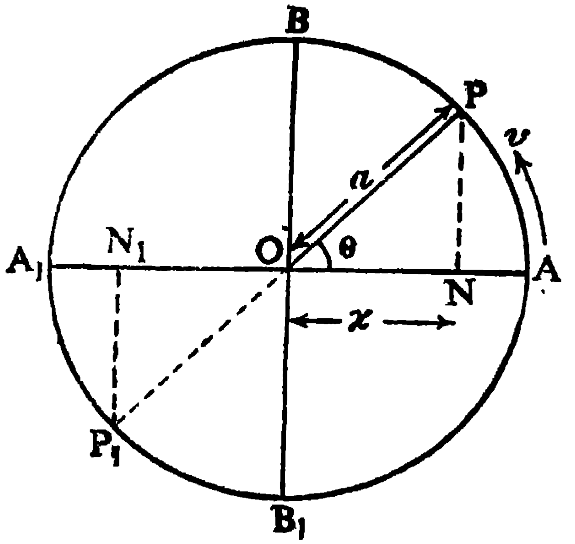


Fig. 93

Suppose a particle P starting from A describes a circle (Fig. 93) ABA_1B_1 having its centre at O moving in an anti-clockwise direction with a uniform speed v. Let AOA_1 be a diameter of the circular path through the starting point A. Now if N, the foot of the perpendicular drawn from P on AOA_1 be observed, it will be found to undergo a periodic motion along AOA_1 reversing the direction each time the moving point P crosses A or A_1 in its circular orbit. The motion of N along AOA_1 is then an ideal simple harmonic motion because it is a to-and-fro periodic motion in a straight line. The other characteristics are also satisfied in this case as will be evident subsequently. The point P is called the *generating point* and the circle it describes, with uniform speed is called the *auxiliary circle* or *the circle of reference*. The centre of the circle of reference O, is called the mean position of the vibrating point N

101. Terms associated with Simple Harmonic Motion :—

(i) **Amplitude**.—The amplitude of a particle executing simple harmonic motion is its maximum displacement measured from the mean position. Referring to the auxiliary circle (Fig. 93), the displacement, x, at any instant of time, measured from the mean position O, is given by $x=a \cos \theta$ where a is the radius of the auxiliary circle and θ is the phase (*see* below) of the vibrating particle. Evidently x is maximum when $\theta=0$, or a multiple of π and is equal to a, the radius of the circle of reference, θ being measured from the position of maximum displacement of the vibrating point in the positive direction.

(ii) **Period**.–For a particle executing $S.H.M.$, one forward motion together with a backward one is called a *complete* oscillation. *Period* is the time taken by the vibrating body to execute one complete oscillation. Referring to the auxiliary circle (Fig. 93), it is obviously equal to the time taken by the generating point P to go once round the circle of reference. If ω be the angular velocity of P, the time period is given by $T=2\pi/\omega$.

(iii) Frequency.—The number of complete oscillations made by the particle executing *S.H.M.* in one second is called its frequency. If n be the frequency, $nT=1$ or $n=\frac{1}{T}=\frac{\omega}{2\pi}$.

(iv) Phase.—The phase of a particle executing *S.H.M.* means its state of displacement and that of motion at an instant of time. Referring to Fig. 93, the speed of the generating point P is the same at every instant but the magnitude of the velocity of the vibrating point N, as also its direction, changes. Thus to know the state of N completely at any instant of time, its displacement from some standard position as well as its direction of motion must be known. Two particles are said to be in the same phase when moving in the same direction they cross their standard positions at the same time and are said to be in opposite phase when one of them moves in a direction opposite to the other at the same instant of time. But whatever is the phase of the vibrating body it is exactly repeated after the lapse of its periodic time.

Measurement of phase of a vibrating particle at any instant must be made with reference to some standard state at which the phase of the particle is known. According to Barton, the standard state is the mean position at O (Fig. 93), the vibrating point being N and the auxiliary circle $AB\,A_1B_1$. According to Rayleigh, the standard state of N is at A, the position of maximum displacement in the positive direction. When the vibrating point N is at A, the generating point P is also at A. The phase of the point N at subsequent intervals of time may be measured either ·

(1) by the angle θ which the generating point P moving along the circle of reference $AB\,A_1B_1$ has made at the centre O, since it last passed through A, its standard position.

or, (2) by the fraction of a period T which has elapsed since the generating point P last passed through the standard position A.

Thus if the phase of N at some instant of time be 60°, it may also be put at $T/6$. Make $\theta=60°$ and draw PN perpendicular from P on AA_1. N will then represent the state of the vibrating point moving from right to left. Obviously phases 60°, 420°, 780° etc. are identical as $\frac{T}{6}, \frac{7T}{6}, \frac{13T}{6}$ etc.

Again if we consider a point P_1 (Fig. 93) which is the diametrically opposite point of P, the corresponding foot of the perpendicular, N_1, drawn on the same diameter AA_1, will have the same magnitude of displacement as N, the foot of the perpendicular drawn from P. But the direction of displacement as well as its direction of motion will be opposite to those of N. So N and N_1 are said to be in opposite phases.

If the phase of N be $\theta = 60°$ as referred to above, the phase of N_1 will be $\theta = 180° + 60° = 240°$. In terms of time period phases $\frac{T}{6}$ and $\frac{4T}{6}$ are opposite. *So the phase of a vibrating particle becomes exactly opposite after an interval of half a period.*

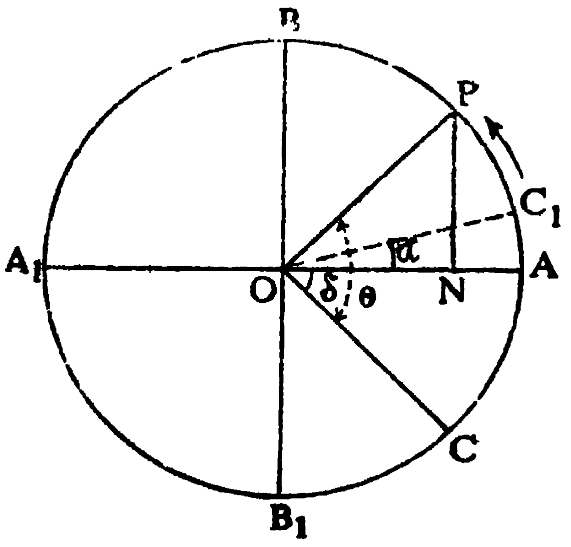


Fig. 94

(v) **Epoch.**—Epoch is the phase of a vibrating particle at the instant when the counting of time is begun. Referring to Fig. 94, N, is a vibrating point executing *S.H.M.* along AA_1 and its auxiliary circle is ABA_1B_1 and let A be its standard state. If the phase angle θ is measured from the instant when the generating point P passes, through C or C_1 and not through A, while the time is counted from the instant when P passes through the standard state A, then the phase of the vibrating point or particle N at the commencement of time is either $\angle COA = \delta$, or $\angle C_1OA = \alpha$. This angle δ or α is, by definition, the epoch of the vibrating particle, N. The equation of displacement, x, in this case will be given by,

$$\text{either, } x = a \cos \angle POA = a \cos [\angle POC - \angle COA]$$
$$= a \cos (\theta - \delta) ;$$

$$\text{or, } \quad x = a \cos \angle POA = a \cos [\angle POC_1 + \angle C_1OA]$$
$$= a \cos (\theta + \alpha).$$

102. (a) Expression for the displacement of a particle executing S.H.M. at any instant :—The displacement signifies the change in position of the vibrating particle in a definite direction from its mean position. With reference to Fig. 94, let a vibrating particle or point N, executing *S.H.M.* move from A to A_1 and back from A_1 to A in time T, called its time period. Let a be the radius of the corresponding circle of reference ABA_1B_1A. Evidently a is the amplitude of vibration. Let T be the position of the generating point at any instant of time t, while the vibrating point is at N. Let the time be counted from the instant when the generating point last passed through the standard position A. The displacement of N is given by, $x = a \cos \angle AOP$ ... ... (1

If ω be the angular velocity of P and $\angle AOP$ be described in time t, then $\angle AOP = \omega t$,

$$\therefore \quad x = a \cos \omega t \qquad \ldots \qquad \ldots \qquad \ldots \quad (2)$$

Again if T be the period of N, which is identical with the time required by P to move once round the auxiliary circle, $\omega = \frac{2\pi}{T}$.

$\therefore$ Eqn. (2) takes the form,

$$x = a \cos \frac{2\pi}{T} t. \qquad \ldots \qquad \ldots \quad (3)$$

If the phase is measured from the instant when P passes through point C or C_1 (Fig. 94), and time is counted from the instant when P passes through A, then if $\angle COA = \delta$ and $\angle C_1OA = \alpha$, eqn. (2) will be modified as,

$x = a \cos(\omega t - \delta)$, or, $a \cos(\omega t + \alpha)$, as the case may be,

where δ and α represent the epochs of the *S.H.M.*

(b) Expressions for Velocity and Acceleration of a Particle executing S.H.M.—Since the velocity of a body is the rate of change of its displacement with regard to time, the velocity can be very easily deduced from the expression for the displacement.

The displacement $x = a \cos \omega t \qquad \ldots \qquad \ldots \qquad \ldots \quad (1)$
when the phase at the commencement of time (*i.e.* epoch) is zero.

$$\therefore \quad \text{Velocity} = \frac{dx}{dt} = -a\omega \sin \omega t \qquad \ldots \qquad \ldots \quad (3)$$

But from eqn. (1), $\cos \omega t = \frac{x}{a}$.

$$\begin{aligned} \therefore \quad \text{Velocity, } v &= -a\,\omega \sin \omega t \\ &= -a\omega \sqrt{1 - \cos^2 \omega t} \\ &= -a\omega \sqrt{\frac{a^2 - x^2}{a^2}} \\ &= -\omega \sqrt{a^2 - x^2} \qquad \ldots \qquad \ldots \qquad \ldots \quad (3) \end{aligned}$$

So the velocity is zero when the displacement is maximum, *i.e.*, when $x = a$, and it is maximum $(-\omega a)$ when $x = 0$, *i.e.* when the particle is at its mean position of rest.

Again since the acceleration is defined as the rate of change of velocity with regard to time, we can directly, by differentiating the expression for velocity with respect to time, obtain an expression for the acceleration of the body doing *S.H.M.*

$\therefore$ The acceleration of the vibrating body is given by

$$f=\frac{dv}{dt}=\frac{d}{dt}(-a\omega\sin\omega t)=-a\omega^2\cos\omega t.$$

$$\text{But } a\cos\omega t=x. \qquad \therefore\ f=-\omega^2x \qquad \ldots \qquad \ldots \quad (4)$$

So the acceleration is zero when $x=0$, *i.e.* at the mean position of rest and it is maximum when $x=a$, *i.e.* at the maximum displaced position.

103. Characteristics of a S.H.M. :—The acceleration of a body executing *S.H.M.* is given by the expression, $f=-\omega^2x$. This shows that the acceleration is proportional to the displacement and moreover the negative sign indicates that the direction of the acceleration is opposite to the direction of displacement, *i.e.* the acceleration is always directed towards the mean position about which the vibration takes place. So the characteristics of a S.H.M. may be stated as follows :—

(*i*) The body executing a S.H.M. must have an acceleration (hence must be acted on by a force) directed towards a fixed point in its path, and (*ii*) this acceleration at any instant must be proportional to the displacement of the particle from that same fixed point at that instant. Thus the eqn. of acceleration may be considered as the characteristic equation of S.H.M. From the expression of acceleration, we know that

$$\omega^2=\frac{f}{x}\ ;\ \text{or, } \omega=\sqrt{\frac{f}{x}}$$ (neglecting the −ve sign because it indicates direction only.

$$\therefore\ \frac{2\pi}{T}=\sqrt{\frac{f}{x}}\ ;\ \text{or, } T, \text{ the periodic time}=2\pi\sqrt{\frac{x}{f}}$$

$$=2\pi\sqrt{\frac{\text{displacement}}{\text{acceleration}}}$$

So T, the period, is independent of the amplitude of vibration. This type of periodic motion in which time period does not depend on the amplitude is called ***isochronous motion***. Within a certain limit of amplitude, the motion of a simple pendulum is also isochronous in nature.

104. Force, Potential Energy and Kinetic Energy :—Let the particle executing *S.H.M.* be of mass m. Its acceleration, as has been found before. is given by $-\omega^2x$, where $\omega=\frac{2\pi}{T}$ and x is the displacement. So the *force* acting on the particle, when its displacement is x, is given by $F=-m\omega^2x$.

So the *work done by the particle* against the above force in moving a short distance dx, over which the force may be supposed to be constant, is $m\omega^2 x.dx$. Therefore the work done by the particle in moving through x, *i.e. the potential energy* of the particle at the position where the displacement is x is given by,

$$W_p = \int_0^x m\omega^2 x \, dx = \tfrac{1}{2} m\omega^2 x^2.$$

∴ Potential energy is zero for zero displacement, *i.e.* when the particle is at the mean position of rest. It is maximum for maximum displacement when $x = a$.

Again, as the velocity of the particle at the position of the said displacement, x, is given by,

$$v = \frac{dx}{dt} = -\omega\sqrt{a^2 - x^2},$$

the attending *kinetic energy* of the particle will be given by,

$$W_k = \tfrac{1}{2} mv^2 = \tfrac{1}{2} m\omega^2 \; (a^2 - x^2).$$

So K.E. is maximum when $x = 0$ and it is zero when $x = a$.

Hence the *total energy of the vibrating particle*,

$$W = W_p + W_k = \tfrac{1}{2} m\omega^2 x^2 + \tfrac{1}{2} m\omega^2 \; (a^2 - x^2) = \tfrac{1}{2} m\omega^2 a^2.$$

The expression is independent of x, the displacement. Thus the total energy remains constant, whatever may be the displacement, thereby showing the validity of the Principle of Conservation of Energy in a *S.H.M.*

105. Graphical Representation of the Changes in the Potential and Kinetic Energies in S.H.M. :—For a particle executing *S.H.M.* and for which epoch angle is zero, the displacement x at any instant of time is given by $x = a \cos \omega t = a \cos \theta$.

Hence its velocity, $v = \frac{dx}{dt} = -a\omega \sin \theta$.

∴ The kinetic energy of the particle $= \tfrac{1}{2} mv^2 = \tfrac{1}{2} ma^2\omega^2 \sin^2\theta$ where m = mass of the particle.

∴ $K.E. = W_k = \tfrac{1}{2} ma^2\omega^2 \sin^2 \theta = K \sin^2 \theta$...(1) where the constant, $K = \tfrac{1}{2} m\omega^2 a^2$.

Again, as shown before, its potential energy at the same instant is given by

$$W_p = \int_0^x m\omega^2 x dx = \tfrac{1}{2} m\omega^2 x^2 = \tfrac{1}{2} m\omega^2 a^2 \cos^2 \theta = K \cos^2\theta \qquad \ldots (2)$$

Now let W_k and W_p be plotted as ordinates against different values of θ, from 0° to 360°, as abscissa as shown in Fig. 95. It is found from the graph that for each value of θ, though the individual values of W_k and W_p are different, the sum of the ordinates of the two curves ($= K \sin^2\theta + K \cos^2\theta$) is a constant throughout and is equal to $K = \frac{1}{2} m r^2\omega^2 =$ the maximum ordinate of either of the curves. This constancy also indicates that a *S.H.M.* obeys the principle of conservation of energy.

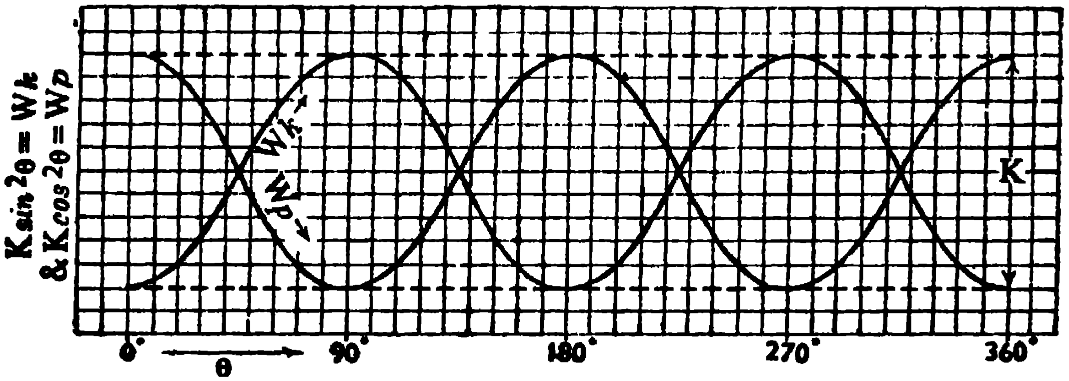


Fig. 95

106. Equation of a S.H.M. :—If m be the mass of the particle executing *S.H.M.* and x its displacement at any instant, its acceleration will evidently be $\frac{d^2x}{dt^2}$. So the force acting on the particle is $m\frac{d^2x}{dt^2}$. This is proportional to the displacement and oppositely directed to it. as force must be known from the characteristics of a simple harmonic motion.

$$m\frac{d^2x}{dt^2} = -\mu x \qquad (1)$$

where μ is the constant of proportionality. This may be rewritten as

$$\frac{d^2x}{dt^2} = -\frac{\mu}{m}x \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

The eqn. (2) is identical with the equation,

$f = -\omega^2 x$ as has been already deduced.

So here $\frac{\mu}{m}$ stands for ω^2. Substituting ω^2 in place of $\frac{\mu}{m}$ we get the equation in the form,

$$\frac{d^2x}{dt^2} = -\omega^2 x \; ;$$

$$\text{or,} \quad \frac{d^2x}{dt^2} + \omega^2 x = 0 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3)$$

This is the differential equation for a simple harmonic motion.

To solve this equation multiply both sides by $2\frac{dx}{dt}$.

$$\therefore \quad 2\frac{d^2x}{dt^2}\cdot\frac{dx}{dt}+2\omega^2x\cdot\frac{dx}{dt}=0.$$

On integration, $\left(\frac{dx}{dt}\right)^2+\omega^2x^2=k$ ... ... (4)

where k is the constant of integration.

When the displacement is maximum, *i.e.*, $x=a$, $\frac{dx}{dt}=0$.

Applying this condition to eqn. (4), we get, $\omega^2a^2=k$.

$\therefore$ Eqn. (4) takes the form,

$$\left(\frac{dx}{dt}\right)^2=\omega^2(a^2-x^2).$$

$$\therefore \quad \frac{dx}{dt}=\omega\sqrt{a^2-x^2}\ ;\ \text{or,}\ \frac{dx}{\sqrt{a^2-x^2}}=\omega dt.$$

On integration, we get,

$\sin^{-1}\frac{x}{a}=\omega t+\delta$, where δ is the constant of integration ;

or, $\frac{x}{a}=\sin(\omega t+\delta)$.

$\therefore \quad x=a\sin(\omega t+\delta)$ ... ... (5)

Here ωt is the phase of the vibrating particle and δ is the epoch.

To express the solution of the equation as given by relation (5) in a more general form, let us write relation (5) as,

$$\begin{aligned} x&=a\sin\omega t\cos\delta+a\cos\omega t\sin\delta\\ &=[a\cos\delta]\sin\omega t+[a\sin\delta]\cos\omega t\\ &=A\sin\omega t+B\cos\omega t. \end{aligned}$$

This is the general solution of eqn. (3) of a *S.H.M.*, where A and B are two constants.

The particular solutions are evidently (*i*) $x=A\sin\omega t$, and (*ii*) $x=B\cos\omega t$, both of which represent S. H. Motions.

Taking any one of those particular solutions, say solution (*i*) we see that,

$x=A\sin\omega t=A\sin(\omega t+2\pi)=A\sin\omega\left(t+\frac{2\pi}{\omega}\right)$. So after an interval of $\frac{2\pi}{\omega}$, the displacement and hence the state of motion also is exactly repeated in nature. So $\frac{2\pi}{\omega}$ is the time period, T, of the *S.H.M.*

$$T=\frac{2\pi}{\omega}=\frac{2\pi}{\sqrt{\frac{\mu}{m}}}=2\pi\sqrt{\frac{m}{\mu}}\quad\left(\because\ \frac{m}{\mu}=\omega^2\right)$$

and the corresponding frequency, $n=\frac{1}{T}=\frac{1}{2\pi}\sqrt{\frac{\mu}{m}}$.

Examples of Simple Harmonic Motion

(*1*) *If the period of a S. H. M. is 12 secs. and amplitude 10 cm., what will be the phase and displacement at a time 14 secs. after the passage of the particle through its extreme elongation ?* (C. U. 1952)

Ans. : Considering time for extreme elongation to be zero, the displacement equation becomes

$$x=a\ \cos\ \omega t=a\ \cos\frac{2\pi}{T}t.$$

Here $T=12$ sec., $t=14$ sec. and $a=10$ cm.

$\therefore\quad x=10\cos 7\pi/3=10\cos(2\pi+\pi/3)=10\cos 60^\circ=5$ cm.

The phase is evidently $7\pi/3$ and displacement 5 cm.

(*2*) *Show that a bar magnet freely suspended so as to move in a horizontal plane in the earth's field executes a. S.H.M. Hence calculate its time period and the angular displacement at any instant.*

Ans. : Let the bar magnet AB swing round, O, (Fig. 96) about the mean position NS which represents the magnetic meridian at the place of observation. The magnet is first turned through an angle θ to the position AB and then released. At this position its acceleration is $\frac{d^2\theta}{dt^2}$ and it is under a deflecting force of moment $I\frac{d^2\theta}{dt^2}$ where I is the moment of inertia of the magnet about O. Again if H be the horizontal component of the earth's magnetic field at the locality, the forces acting on the two ends of the magnet will be mH each. These forces are equal and parallel but are oppositely directed. So they form a couple which tries to restore the magnet to the meridian plane. The moment of this restoring couple is given by $mH\times$distance AA_1.

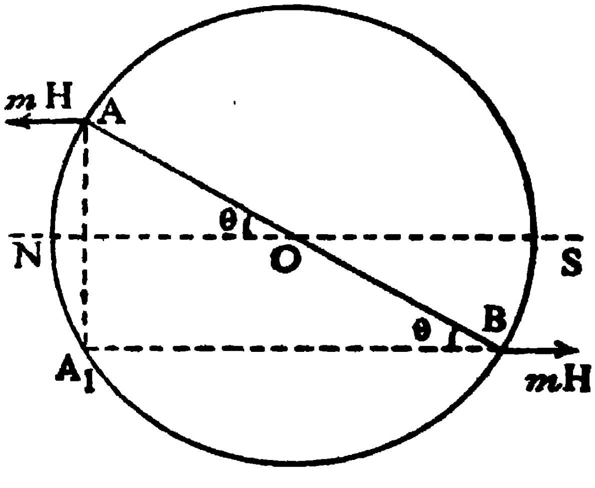


Fig. 96

But $AA_1 = 2l \sin\theta$, where $2l$ = length of the magnet.

$\therefore$ Moment of the restoring couple $= mH \times 2l \sin\theta$

$= MH \sin\theta = MH\,\theta$, when θ is small and M = magnetic moment of the magnet $= 2ml$.

Since the magnet is temporarily at rest in its displaced position AB, the algebraic sum of the moments of forces about O, must vanish.

$$\therefore \quad I\frac{d^2\theta}{dt^2} + MH\theta = 0\,;$$

$$\text{or,} \quad \frac{d^2\theta}{dt^2} = -\frac{MH}{I}\theta = -\omega^2\theta \ldots (1), \text{ where } \omega = \sqrt{\frac{MH}{I}}.$$

This is an equation of a *S. H. M.* So the motion of the magnet is simple harmonic in nature. Its time period can be easily deduced because ω in the above equation refers to the angular velocity of the corresponding generating point as is evident from the nature of the equation [See the equation of a *S.H.M.*].

$$\therefore \quad \omega = \sqrt{\frac{MH}{I}}\,; \text{ or, } \frac{2\pi}{T} = \sqrt{\frac{MH}{I}}\,; \text{ or, } T = 2\pi\sqrt{\frac{I}{MH}}.$$

Since eqn. (1) represents the differential equation of a *S. H. M.* its general solution will be of the form,

$$\theta = A \sin\omega t + B \cos\omega t \qquad \ldots \qquad \ldots \quad (2)$$

To calculate the angular displacement θ at any instant of time, we shall have to evaluate the constants A and B from the boundary conditions of the problem.

Applying the initial condition, *i.e.* when $t=0$, $\theta=0$,

we get, $0 = A \sin 0 + B \cos 0 = B$.

$$\therefore \quad \theta = A \sin\omega t \qquad \ldots \qquad \ldots \qquad \ldots \qquad \ldots \quad (3)$$

Again, when $t = T/4$, $\theta = \theta_0$ = maximum angular displacement,

$$\therefore \quad \theta_0 = A \sin \omega\frac{T}{4} = A \sin\frac{2\pi}{T}\times\frac{T}{4} = A \sin\pi/2 = A.$$

Substituting in eqn. (3), we get the expression for the angular displacement at any instant t, as $\theta = \theta_0 \sin\omega t$

$$= \theta_0 \sin\left(\sqrt{\frac{MH}{I}}\right)t.$$

(3) A quantity of gas is enclosed in a cylinder in which works a smooth heavy piston. The axis of the cylinder is vertical. The piston is thrust down to compress the gas and then let go. Show that the motion produced is simple harmonic.

Ans.: Let the piston P be at A initially and after being thrust down, let the piston be at B (Fig. 97). If P be the pressure of the gas when the piston is at A, then,

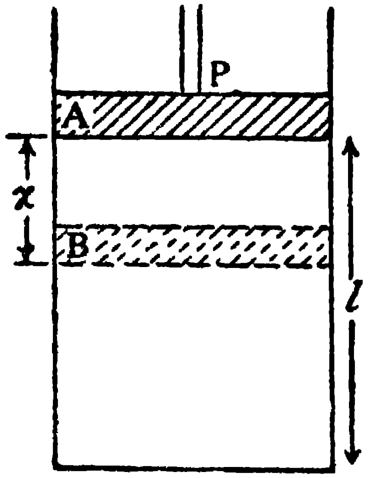


Fig. 97

$\alpha P = mg$ where α = cross-sectional area of the piston, m = its mass and g = acceleration due to gravity.

$$\therefore \quad P=\frac{mg}{\alpha}.$$

When the piston is at A, let the vertical length of the gas column be l. When the piston is thrust down to B through a distance 'x', the vertical length of the column will be $l-x$.

If the change in volume be assu to take place adiabatically, we can
$P(l\alpha)^{\gamma}=(P+p)\{(l-x)\alpha\}^{\gamma}$.

where γ is the ratio of the two specific heats of the gas, *i.e.* C nd p is the additional pressure on the gas caused by the diminuti volume.

So, cancelling α^{γ} from both sides, we get,

$$Pl^{\gamma}=(P+p)(l-x)^{\gamma}=(P+p)l^{\gamma}\left(1-\frac{x}{l}\right)^{\gamma}$$

$=(P+p)l^{\gamma}\left(1-\gamma\frac{x}{l}\right)$ approximately neglecting higher powers of $\frac{x}{l}$.

$$\therefore \quad P=(P+p)\left(1-\gamma\frac{x}{l}\right)=P+p-P\gamma\frac{x}{l}-p\gamma\frac{x}{l}$$

$=P+p-P\gamma\frac{x}{l}$ (p and $\frac{x}{l}$ each being small quantities their product can be neglected).

Hence, $p=P\gamma\frac{x}{l}=\gamma\frac{mg}{\alpha}.\frac{x}{l}$.

Thus the extra force on the piston,

$$p\alpha=\gamma.\ P\frac{x}{l}\alpha=E_{\phi}\frac{\alpha}{l}.x.$$

where $E_{\phi}=\gamma P$ = bulk modulus of adiabatic elasticity.

So the extra force is proportional to the displacement, x, which is a characteristic of *S. H. M.* Therefore the motion under consideration is simple harmonic in nature.

(4) A particle of mass m is attached to the mid-point of a string stretched vertically under constant tension T. The particle oscillates

horizontally but the ends are prevented from horizontal movements. If l be the length of the string, calculate the time period of small oscillation of the particle.

Ans. : Let SS_1 be the vertical string and O be its mid-point (Fig. 98). If the particle of mass m be attached to O and if it be given a small displacement x so as to shift the particle to P, then the force acting on the body horizontally will be the horizontal components of the tensions on the two halves of the string in the displaced position.

Hence the force acting on the particle m at P is $2T\cos\theta = 2T.\dfrac{OP}{SP} = 2T.\dfrac{x}{l/2} \quad 4T\times\dfrac{x}{l}$ acting along PO.

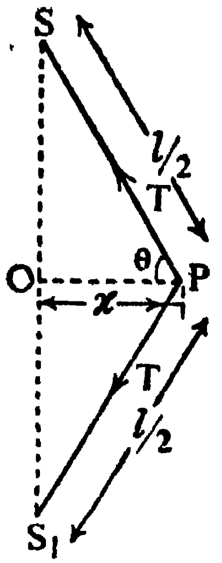


Fig. 98

This force is evidently restoring in nature.

∴ The equation of motion of the particle is

$$m\frac{d^2x}{dt^2} = -\frac{4T}{l}x\ ; \quad \text{or,} \quad \frac{d^2x}{dt^2} = -\left(\frac{4T}{ml}\right)x.$$

∴ Comparing this with the equation of a *S.H.M.* we have,

$$\omega = \frac{2\pi}{\tau} = \sqrt{\frac{4T}{ml}}\ ; \ (\tau = \text{time period})$$

$$\text{or,} \quad \tau = 2\pi\sqrt{ml/4T} = \pi\sqrt{ml/T}.$$

(5) *A vertical U-tube of uniform cross-section contains water up to a height of 30 cms. Show that if the water on one side is depressed and then released, its motion up and down the two sides of the tube is simple harmonic and calculate its time period.* (Del U. 1947)

Ans. : Since the curvature at the bend of the U-tube is not given, the mass of water in the two limbs of the U-tube may be approximately taken to be $2\times30\times\alpha$ gms. $=60\alpha$ gms. where $\alpha =$ the cross-sectional area of the tube.

If the water on one side be depressed, say through x cms., the water level on the other limb will rise by x cms. above the equilibrium level. So a difference of height of $2x$ cms. will be created between the water levels in the two limbs. This column of water will apply on the total mass of water a force which is equal to its weight. This force is equal to $2x\times\alpha\times1\times g$ dynes (density of water being unity).

∴ The acceleration produced in the water mass

$$= \frac{\text{Force}}{\text{mass}} = \frac{2x\alpha g}{60\alpha} = \frac{gx}{30}.$$

Thus acceleration is proportional to displacement, x.

$\therefore$ The motion of the water column will be simple harmonic in nature.

$$\therefore \quad \omega^2 = \frac{g}{30};$$

$$\text{or,} \quad \omega = \frac{2\pi}{T} = \sqrt{\frac{g}{30}} \text{ (where } T = \text{time period).}$$

$$\therefore \quad T = 2\pi\sqrt{\frac{30}{g}} = 2\pi\sqrt{\frac{30}{981}} = 1{\cdot}098 \text{ secs.}$$

(6) *A particle is moving with simple harmonic motion in a straight line. When the distance of the particle from the equilibrium position has the values* x_1 *and* x_2, *the corresponding velocities are* u_1 *and* u_2. *Show that the period is*

$$2\pi\{(x_2^2 - x_1^2)/(u_1^2 - u_2^2)\}^{1/2}.$$

Ans. : Let the simple harmonic motion be represented as

$$x = a \sin \omega t.$$

$$\text{Then,} \quad u = \frac{dx}{dt} = a\omega \cos \omega t.$$

$$\therefore \quad \frac{x}{a} = \sin \omega t \text{ and } \frac{u}{a\omega} = \cos \omega t.$$

$$\text{Hence,} \quad \frac{x^2}{a^2} + \frac{u^2}{a^2\omega^2} = 1.$$

$$\therefore \quad \frac{u_1^2}{\omega^2} + x_1^2 = a^2 = \frac{u_2^2}{\omega^2} + x_2^2;$$

$$\text{or,} \quad \frac{u_1^2 - u_2^2}{\omega^2} = (x_2^2 - x_1^2).$$

$$\therefore \omega = \left\{\frac{u_1^2 - u_2^2}{x_2^2 - x_1^2}\right\}^{1/2} = \frac{2\pi}{T}.$$

$$\text{Hence,} \quad T = 2\pi\{(x_2^2 - x_1^2)/ u_1^2 - u_2^2)\}^{1/2}.$$

107. Composition of Simple Harmonic Motion :—The resultant motion of a particle simultaneously acted on by two forces each of which acting separately would make the particle execute a *S.H.M.*, depends on whether (*i*) the forces acting on the particle are collinear or (*ii*) they are in mutually perpendicular (rectangular) directions.

Case. 1 When the forces acting on the particle are collinear.— The composition of *S.H.M.*'s in this case may be made either by analytical method or by the graphical method.

(A) Analytical method of Composition of two collinear S.H.M.'s—

Let the displacements of the simple harmonic motions to which the particle is subjected be represented by,

$$x_1 = a_1 \cos \omega t = a_1 \cos \theta$$

$$\text{and } x_2 = a_2 \cos (\omega t + \delta) = a_2 \cos (\theta + \delta)$$

where a_1 and a_2 are their amplitude and $(\theta + \delta)$ are their phases. Here the amplitude and phases are supposed to be different for the two *S.H.M.'s* but their period is taken to be the same.

Since the motion and hence the displacements are collinear, the resultant displacement will be obtained by the algebraic sum of the two individual displacements.

∴ The resultant displacements, x, is written as,

$$x = x_1 + x_2 = a_1 \cos \omega t + a_2 \cos (\omega t + \delta) \quad .. \quad \cdots \quad (1)$$

If the periods of component motions were different, the eqn. (1) would take the form.

$$x = a_1 \cos \theta_1 + a_2 \cos (\theta_2 + \delta) \quad .. \quad \ldots \quad \ldots \quad (2)$$

which is the equation of a *compound harmonic motion*. As the compound harmonic motion is of no general interest, we shall proceed with eqn. (1) and discuss the nature of the resultant motion. From eqn. (1), we get,

$$x = a_1 \cos \theta + a_2 \cos \theta \cos \delta - a_2 \sin \theta \sin \delta$$

$$= (a_1 + a_2 \cos \delta) \cos \theta - (a_2 \sin \delta) \sin \theta.$$

Let $a_1 + a_2 \cos \delta = R \cos \beta$,

and $a_2 \sin \delta = R \sin \beta$.

Then, $x = R \cos \theta \cos \beta - R \sin \theta \sin \beta = R \cos (\theta + \beta)$ . . (3)

where $R^2 = (a_1 + a_2 \cos \delta)^2 + (a_2 \sin \delta)^2 = a_1{}^2 + a_2{}^2 + 2a_1a_2 \cos \delta$ (4)

$$\text{and } \tan \beta = \frac{a_2 \sin \delta}{a_1 + a_2 \cos \delta}, \quad \ldots \quad (5)$$

The motion represented by eqn. (3) is evidently a simple harmonic one.

Special Cases :

(*i*) When $\delta = 0$, the resultant amplitude is $R = a_1 + a_2$ and $\beta = 0$, from eqns. (4) and (5).

Hence from (3), $x = (a_1 + a_2) \cos \theta$.

So the resultant motion is simple harmonic and the resultant amplitude is equal to the sum of the amplitudes of the two motions. If again $a_1 = a_2 = a$, then $x = 2a \cos \theta$, *i.e.* the resultant motion has an amplitude equal to double the amplitude of each of the component motions.

(*ii*) When $\delta=\pi/2$, $R=\sqrt{a_1^2+a_2^2}$ and $\beta=\tan^{-1}\frac{a_2}{a_1}$.

So $x=\sqrt{a_1^2+a_2^2}\cos(\theta+\beta)$. If further $a_1=a_2=a$ and $\beta=45°$, *i.e.* $\tan\beta=1$,

$$x=\sqrt{2}\times a\cos(\theta+\pi/4).$$

(*iii*) When $\delta=\pi$, $R=a_1-a_2$ and $\beta=0$, $x=(a_1-a_2)\cos\theta$. If further $a_1=a_2=a$, $x=0$, the resultant displacement is zero, *i.e.*, one *S.H.M.* cancels the effect to the other.

(B) Graphical method of Composition.—

The graphical method requires the construction of the time-displacement curve of each *S.H.M.* So to understand the method thoroughly, a knowledge of drawing the time-displacement curve is essential. This is given below.

Let ABA_1B_1 (Fig. 99) be the circle of reference for a simple harmonic motion. The circumference of the circle ABA_1B_1A is divided into 16 equal parts and perpendiculars are drawn from each point of division on the diameter BB_1 and they are numbered as given in the figure. The diameter A_1A which is at right angles to the diameter BB_1 is produced to X. From O_1, a point close to A, a vertical line O_1Y is drawn on the horizontal axis AX.

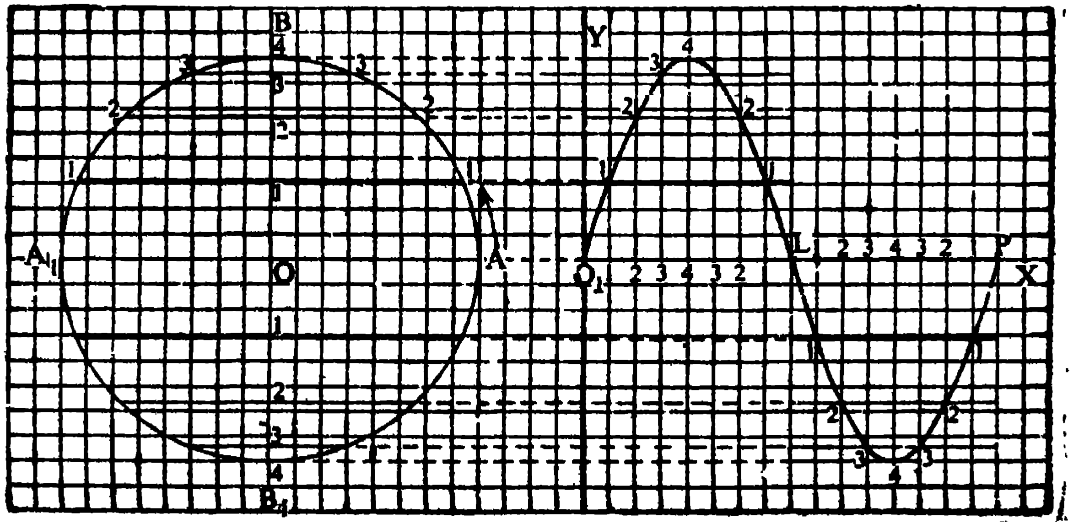


Fig. 99

Let O_1Y and O_1X represent respectively the displacement and time axes of the graph. Along the X-axis a suitable length O_1P is taken and divided into 16 equal parts also, L is marked midway between O_1

and P. O_1L is divided into 8 parts and numbered as 0 to 4 and 4 to 0. Vertical lines are drawn parallel to O_1Y from each of these points of division. Lines from O_1 to L are drawn above O_1X and similar lines are drawn below O_1X in the segment LP. If O_1P represents a time equal to one period of oscillation, each of these 16 segments on O_1P will indicate a time interval of $T/16$.

Now suppose the *S.H.M.* of the foot of the perpendicular drawn from the generating point moving along the circle of reference ABA_1B_1 be considered on the diameter BB_1. Let the time of ovservation begin when the generating point is at A and moving in the anti-clockwise direction. Evidently at the start the foot of the perpendicular will be at O and its displacement will be zero. So when $t=0$ (along x-axis), the displacement y is also zero. This point is evidently represented by O_1 in the graph. When the generating point moves to point (1) on the circumference after an interval of $T/16$, the foot of the perpendic ular moves to point (1) on BB_1 in the vertically upward direction. The displacement is evidently the distance between O and 1 on BB_1. Draw a horizontal line through this point (1) parallel to AA_1. It intersects the vertical line (1) on the graph at point (1). This vertical line again everywhere indicates an interval of $T/16$ from the start. So this point (1) on the graph indicates the time $T/16$ and its corresponding displacement. In this way points 2, 3, 4 are obtained for the ***time-displacement curve*** corresponding to positions 2, 3, 4 of the vibrating point (foot of perpendicular) while moving up vertically. Again as it moves down from 4 to 3, 2, 1, 0 etc. the points will be indicated on the graph by points 4, 3, 2, 1, L on the downward slope of the upper loop of the graph. Once these points are obtained the whole upper loop can be drawn by joining these points by a free-hand curve. Similarly, the downward motion of the vibrating point from 0 to 4 and then the upward motion from 4 to 0 is represented by the lower loop of the curve. This complete curve for a single period or any other specified time, wherein displacement is plotted against the corresponding time is called a ***time-displacement curve*** of a *S.H.M.*

Since in this case the vibrating point has been considered on diameter BB_1, the displacement expression will be $y=a \sin \theta = a \sin \omega t$. So the time-displacement curve is also a sine curve. If the vibrating point is supposed to move along the diameter AA_1, the displacement expression will be, $x=a \cos \omega t$, add the time-displacement curve would be a cosine curve. The time-displacement curve can also be drawn by taking one diameter and one half of the circumference only for reference. This will give the curve for half period—the other half being obtained from symmetry.

Once the construction of the time-displacement curve is known, the composition of two simple harmonic motion may be made in the following way :—

(a) When the motions are in the same phase.—On the same axes of reference O_1X and O_1Y the time-displacement curves for the two *S.H.M'.s* are drawn on the same scale such that

(i) the radii of the circles of reference are in the ratio of the amplitudes of the two motions ;

(ii) the semi-circumference of the reference circles of both the S.H.M.'s are divided into equal number (say, n) of divisions, each corresponding to an interval equal to a definite fraction $\left(\frac{1}{2n}\right)$ of the period—the length O_1P along the time axis being divided into twice the above number of parts (i.e. $2n$).

Now finding out the displacement along the vertical diameter (*i.e.* along y-axis) at the end of each of these intervals, the time-displacement curve for each of the component *S.M.H*'s can be drawn.

The resultant curve is obtained by taking the algebraic sum of the ordinates of the component curves at the same interval of time.

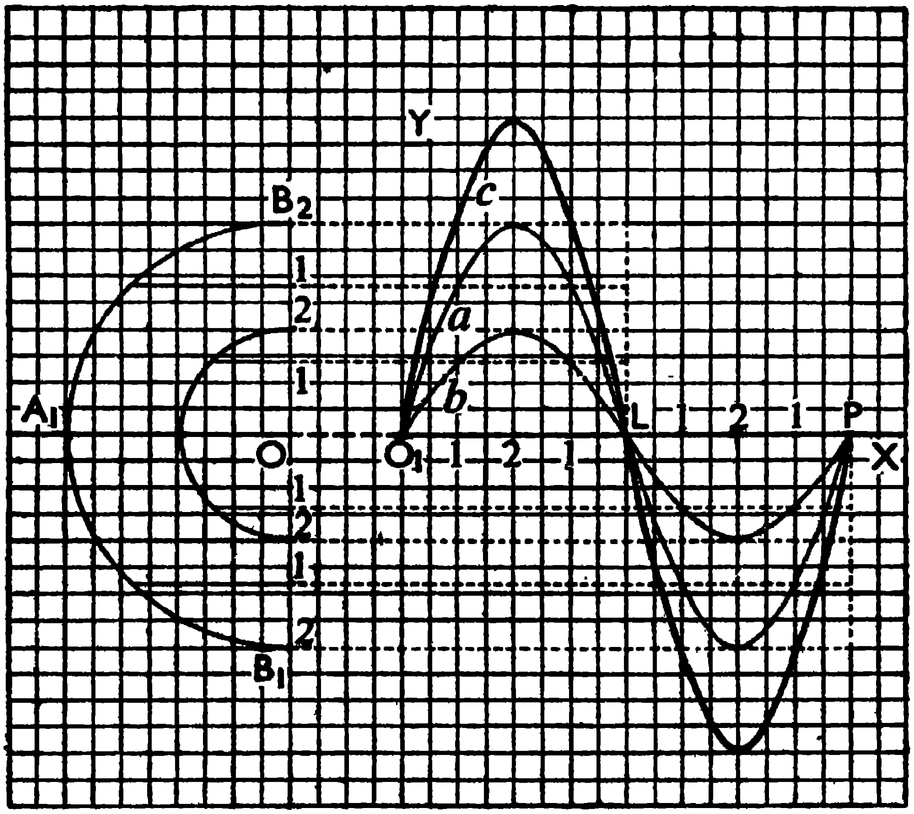


Fig. 100

In Fig. 100, the curve (*c*), shows the resultant of two collinear *S.H.M's* (*a*) and (*b*), having the same period and no initial phase difference and their amplitudes being in the ratio 2 : 1. In case the periods of the *S.H.M.*'s are different, the semi-circumferences should be divided into a number of parts having the same ratio as that of the periods. Then the time-displacement curves are drawn as usual. The thick-line curve (*c*) in Fig. 101 represents the resultant of two component vibrations (*a*) and (*b*) having the same amplitude and phase but periods in the ratio 2 : 1 (or frequency in the ratio 1 : 2). The thick-line curve (*c*) in Fig. 102 represents the resultant of two component vibrations (*a*) and (*b*), whose amplitudes are identical, phase difference is nil and the periods are nearly the same (*i.e.*, frequency also nearly the same). When the component vibrations under these conditions, are sonorous in nature, the resultant vibration represents a beat curve, (see Part II, Sound).

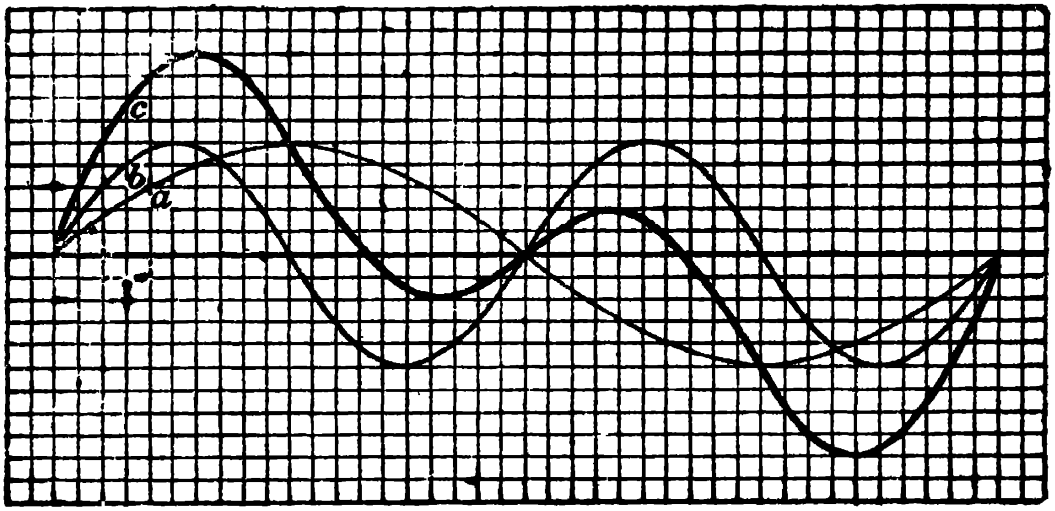


Fig. 101

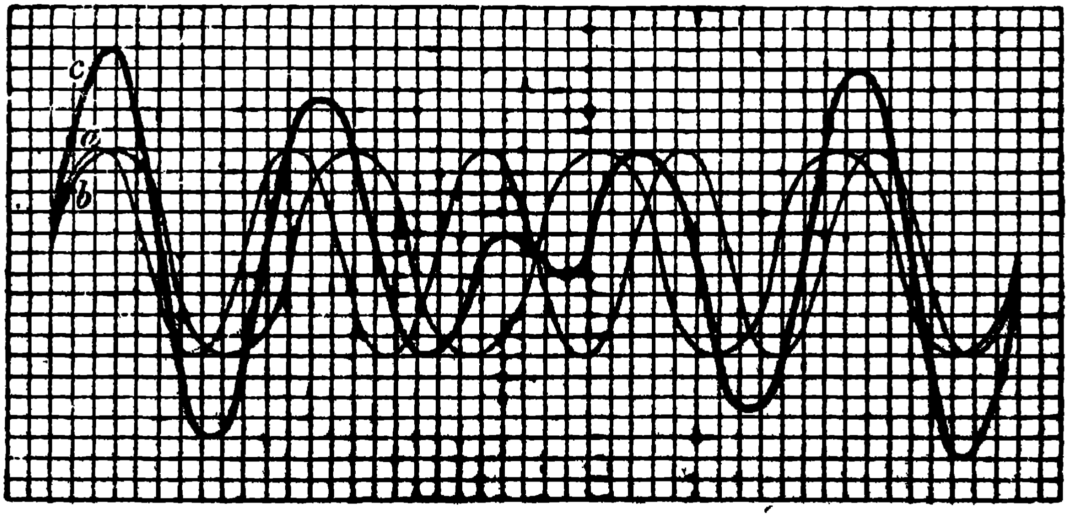


Fig. 102

(b) When there is a phase difference between the two motions.—The method is similar to the above with the difference that the origin of the displacement curve for them is not the same. The origin for the lagging motion is to be taken to the left of O_1 by an amount equal to the phase difference measured in terms of the time period, when the motions are of equal period. When the motions have different periods, the division of the circumference or semi-circumference, as also O_1X axis, has to be made into an appropriate number of parts or intervals such that the phase difference may be represented by an integral number of such intervals. The time intervals are to be numbered separately bearing in mind the different origins of the two curves. Fig. 103 shows the graphical compositions of two *S.H.M.*'s of same amplitude, phase difference $\pi/3$ and time periods in the ratio 3 : 4. The curves, (*a*) and (*b*), represent the component vibrations and the thick-line curve (*c*) the resultant.

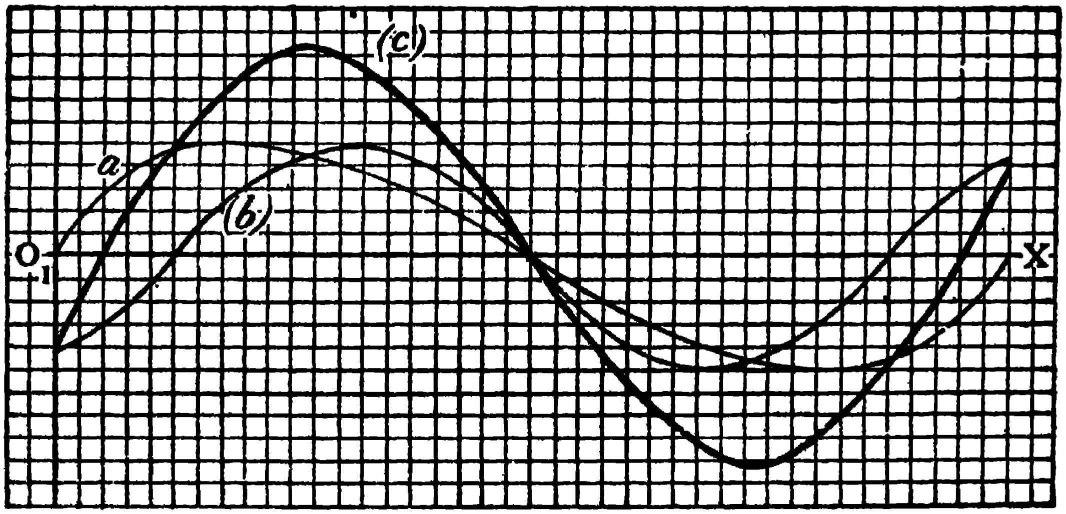


Fig. 103

It should be remembered that if the time periods are different, the phase difference does not remain constant throughout. The initial phase difference can only be introduced in investigating the resultant motion.

Case 2. When the forces acting on a particle are in mutually perpendicular directions.—

In this case also the composition may be made either analytically or graphically.

(A) Analytical method.—

(i) When the periods are identical but the amplitudes and phases differ in the two component vibrations.

Let the two rectangular *S.H.M.* s be represented by

$$x = a_1 \cos\theta \quad \ldots \quad \ldots (1)$$

$$y = a_2 \cos(\theta + \alpha) \quad \ldots \quad \ldots \quad \ldots (2)$$

From (2) $\frac{y}{a_2} = \cos\theta\cos\alpha - \sin\theta\sin\alpha = \frac{x}{a_1}\cos\alpha - \sqrt{1 - \frac{x^2}{a_1}}$

$$\left| \text{since from (1), } \cos\theta = \frac{x}{a_1} \right];$$

or, $\frac{y}{a_2} - \frac{x}{a_1}\cos\alpha = -\sqrt{1 - \frac{x^2}{a_1^2}}\sin\alpha$;

or, on squaring both sides,

$$\frac{y^2}{a_2^2} + \frac{x^2}{a_1^2}\cos^2\alpha - \frac{2xy}{a_1 a_2}\cos\alpha = \left(1 - \frac{x^2}{a_1^2}\right)\sin^2\alpha ;$$

or, $\frac{y^2}{a_2^2} - \frac{2xy}{a_1 a_2}\cos\alpha + \frac{x^2}{a_1^2}(\cos^2\alpha + \sin^2\alpha) = \sin^2\alpha$;

or, $\frac{y^2}{a_2^2} - \frac{2xy}{a_1 a_2}\cos\alpha + \frac{x^2}{a_1^2} = \sin^2\alpha \quad \ldots \quad (3)$

This is the equation of an ellipse. Hence the resultant vibration will, in general, be *elliptical*.

Special cases :

(1) When $\alpha = 0$, *i.e.* when there is no phase difference and a_1 and a_2 are different,

$\left(\frac{y}{a_2} - \frac{x}{a_1}\right)^2 = 0$; or, $y = \frac{a_2}{a_1}x$, which gives a straight line passing through the origin, inclined at an angle $\tan^{-1}\frac{a_2}{a_1}$ to the axis of x. Again if $a_2 = a_1$; the angle of inclination is $\tan^{-1} 1 = 45°$.

(2) When $\alpha = \pi/2$, but $a_1 \neq a$ the equation (3) reduces to

$$\frac{y^2}{a_2^2} + \frac{x^2}{a_1^2} = 1.$$

So under the conditions, the resultant vibration is elliptical. This ellipse has its axes of symmetry coincident with the co-ordinate axes and its semi-axes equal to a_2 and a_1.

(3) When $\alpha = \pi/2$ and $a_1 = a_2 = a$, say, the equation (3) takes the form

$$x^2 + y^2 = a^2,$$

which gives a circular motion, the radius of the circle being a.

(4) When $\alpha=\pi$, and a_2 and a_1 are different, the eqn. (3) can be written as,

$$\frac{y^2}{a_2^2}+\frac{2xy}{a_2^2}+\frac{x^2}{a_1^2}=0\,; \qquad \text{or,} \quad \left(\frac{y}{a_2}+\frac{x}{a_1}\right)^2=0.$$

$\therefore$ $y=-\frac{a_2}{a_1}x$, which gives a straight line passing through the origin and inclined at an angle $\tan^{-1}\left(-\frac{a_2}{a_1}\right)$ to the x-axis.

(ii) *When the amplitudes, phases and periods of the component rectangular vibrations are different, the periods being in the ratio 2 : 1.*

Under these conditions the two *S.H.M.'s* may be represented by the equations,

$$x=a_1\cos\theta \qquad \ldots \qquad \ldots \quad (1)$$

$$y=a_2\cos(2\theta+\alpha) \qquad \ldots \qquad \ldots \quad (2)$$

Here the phase of vibration along y-axis is taken to be in advance.

From eqn. (2), $\frac{y}{a_2}=\cos 2\theta.\cos\alpha-\sin 2\theta\sin\alpha$

$$=(2\cos^2\theta-1)\cos\alpha-2\sin\theta\cos\theta\sin\alpha$$

$$=\left(2\frac{x^2}{a_1^2}-1\right)\cos\alpha-2\sqrt{1-\frac{x^2}{a_1^2}}\cdot\frac{x}{a_1}\sin\alpha.$$

$$\left[\text{substituting } \cos\theta=\frac{x}{a_1} \text{ from (1)}\right]$$

$$\therefore \quad \frac{y}{a_2}-\left(\frac{2x^2}{a_1^2}-1\right)\cos\alpha=-2\sqrt{1-\frac{x^2}{a_1^2}}\cdot\frac{x}{a_1}\sin\alpha.$$

Squaring both sides,

$$\frac{y^2}{a_2^2}+\left(\frac{2x^2}{a_1^2}-1\right)^2\cos^2\alpha-\frac{2y}{a^2}\left(\frac{2x^2}{a_1^2}-1\right)\cos\alpha$$

$$=4\left(1-\frac{x^2}{a_1^2}\right)\frac{x^2}{a_1^2}\sin^2\alpha.$$

Adding $\left(\frac{2x^2}{a_1^2}-1\right)^2\sin^2\alpha$ to both sides.

$$\frac{y^2}{a_2^2}-\frac{2y}{a_2}\left(\frac{2x^2}{a_1^2}-1\right)\cos\alpha+\left(\frac{2x^2}{a_1^2}-1\right)^2$$

$$=\sin^2\alpha\left[\left(\frac{2x^2}{a_1^2}-1\right)^2+4\left(1-\frac{x^2}{a_1^2}\right)\cdot\frac{x^2}{a_1^2}\right]$$

$$=\sin^2\alpha \qquad \ldots \qquad \ldots \quad (3)$$

When the phase difference vanishes $\alpha=0$, the eqn. (3) can be written as,

$$\frac{y^2}{a_2^2}-2.\frac{y}{a_2}.\left(\frac{2x^2}{a_1^2}-1\right)+\left(\frac{2x^2}{a_1^2}-1\right)^2=0\ ;$$

$$\text{or,}\quad \left[\frac{y}{a_2}-\left(\frac{2x^2}{a_1^2}-1\right)\right]^2=0 \quad \ldots \quad \ldots \quad \ldots \quad (4)$$

Eqn. (4) stands for two coincident parabolas, each of which may be represented by the relation,

$$\frac{y}{a_2}=\frac{2x^2}{a_1^2}-1\ ;\qquad \text{or,}\quad y=\frac{2a_2}{a_1^2}x^2-a_2 \quad \ldots \quad \ldots \quad (5)$$

This represents a parabola symmetrical about the y-axis, its vertex being at the point $(0, -a_2)$.

(*iii*) *When the component rectangular vibrations have their amplitudes different, phases identical but the periods in the ratio 3 : 2.*

In this case, since $\alpha=0$, the component vibrations can be represented by the equations,

$$x=a_1 \cos 2\theta \quad \ldots \quad \ldots \quad (1)$$

$$\text{and } y=a_2 \cos 3\theta \quad \ldots \quad \ldots \quad (2)$$

From (1) $\cos 2\theta=\frac{x}{a_1}=2\cos^2\theta-1$; or, $\cos\theta=\left(\frac{1}{2}+\frac{x}{2a_1}\right)^{\frac{1}{2}}$

Again from (2), $\frac{y}{a_2}=\cos 3\theta=4\cos^3\theta-3\cos\theta$

$$=4\left(\frac{1}{2}+\frac{x}{2a_1}\right)^{\frac{3}{2}}-3\left(\frac{1}{2}+\frac{x}{2a_1}\right)^{\frac{1}{2}}.$$

$$\therefore\quad \frac{y^2}{a_2^2}=16\left(\frac{1}{2}+\frac{x}{2a_1}\right)^3+9\left(\frac{1}{2}+\frac{x}{2a_1}\right)-24\left(\frac{1}{2}+\frac{x}{2a_1}\right)^2$$

$$=2+\frac{2x^3}{a_1^3}+\frac{12x}{2a_1}+\frac{6x^2}{a_1^2}+\frac{9}{2}+\frac{9x}{2a_1}-6-\frac{6x^2}{a_1^2}-\frac{24x}{2a_1}$$

$$=\frac{2x^3}{a_1^3}-\frac{3x}{2a_1}+\frac{1}{2} \quad \ldots \quad \ldots \quad \ldots \quad (3)$$

The equation (3) represents a cubic curve.

(b) Graphical Method.—

Let the amplitudes of the two S. H. motions one vertical and the other horizontal be in the ratio $a_1 : a_2$ and the periods in the ratio $m : n$. To get the resultant of these two motions the following procedure is to be adopted.

(1) On the graph paper take a horizontal line and a line at right angles to it such that their lengths are in the ratio of the amplitudes *i.e.*, $a_2 : a_1$. Draw on the upper side a semi-circle with the horizontal line as the diameter and another semi-circle on the L.H.S. with the vertical line as the diameter (Figs. 104 to 108).

(2) Divide the above semi-circumferences into kn and km number of parts respectively. Here k is to be so chosen that the interval

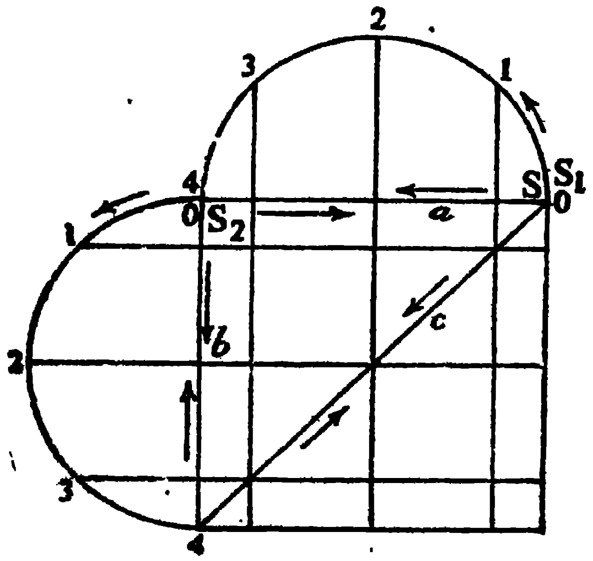
between two points of division on the semi-circumference is either equal to or an exact submultiple of the initial phase difference between the two motions.

(3) Draw straight lines through the points of division on both the semi-circumferences, so that they intersect and form smaller rectangles within the rectangle contained by the two diameters.

(4) Mark on the semi-circumferences, the starting positions of the particles bearing in mind the initial phase difference. Usually the starting position of the lagging motion is taken at the extreme end of the diameter concerned and the position of the leading motion at that instant of time is found out on the other semi-circumference by taking the phase difference into account.

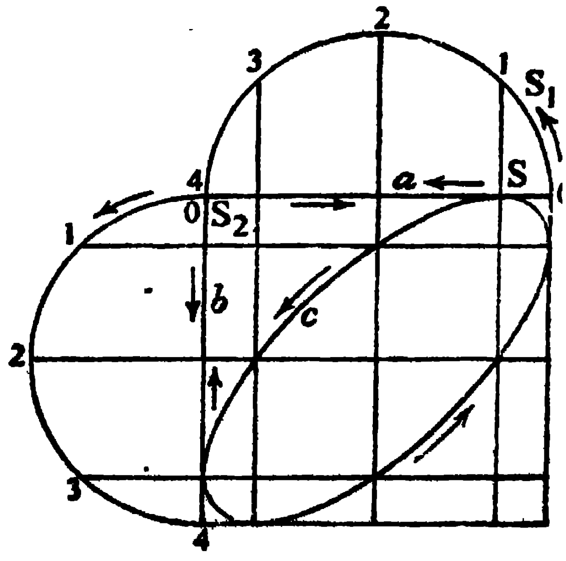
(5) The initial position of the resultant motion will be at a point in the big rectangular diagram where the horizontal and the vertical straight lines corresponding to the initial posts (positions of the generating points) of the two motions intersect.

(6) Determine the initial position according to rule (5) and then proceed along the circumferences of the semi-circles taking one division of one semi-circumference for one division of the other. The corresponding vertical and horizontal lines through these points of division intersect somewhere within the big rectangle contained by the diameters. This intersection point is a point on the resultant curve. In this way, find out the different points on the resultant curve and join them by a free-hand curve turning each time a side of the big rectangle is touched. The curve so drawn will give the resultant curve of two component *S. H. M.*'s.



Amplitudes and time periods same and phase difference nil.

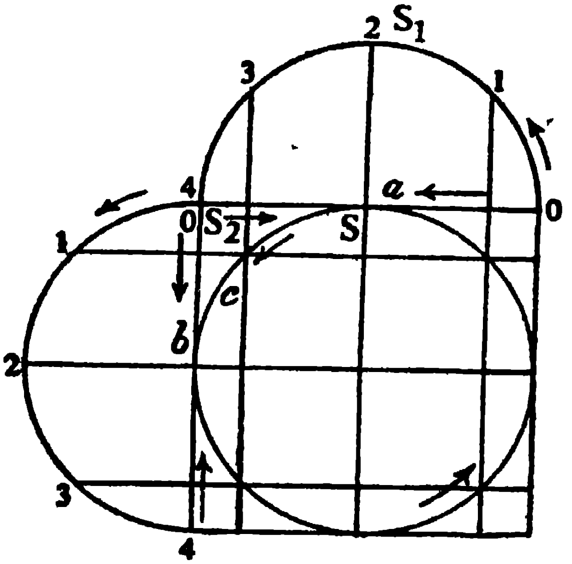
Fig. 104



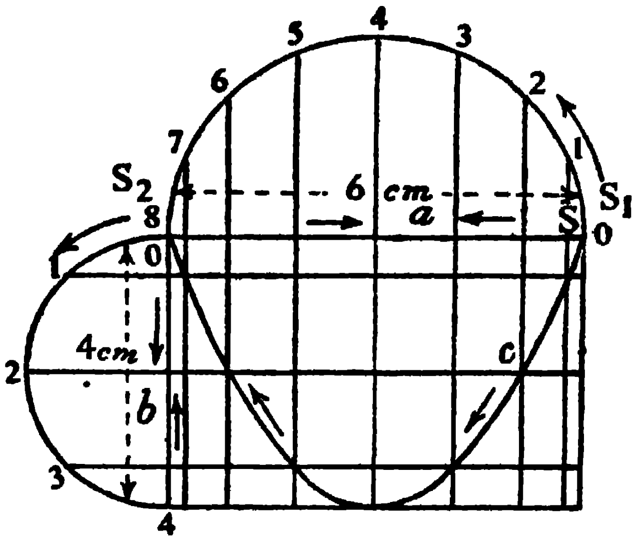
Amplitudes and time periods same and phase difference $= \pi/4$.
(S.H.M., *a* leading)

Fig. 105

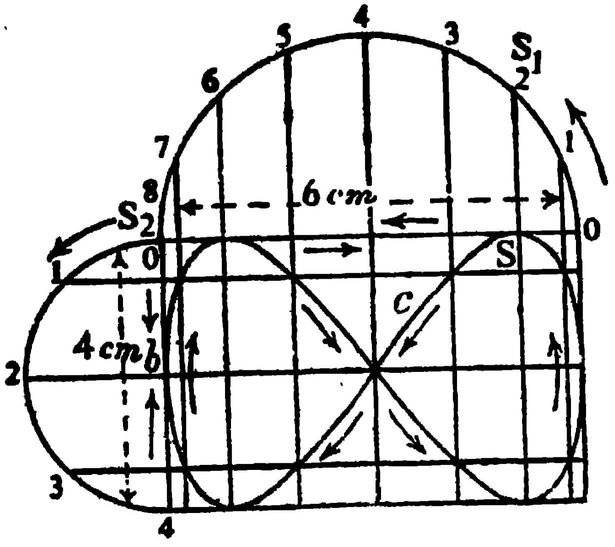
It is easy to follow that the diameters of the two circles will be equal when the two motions have equal amplitudes. The number of parts into which the semi-circumferences are to be divided will be the same when the periods of the two motions are equal. The starting points of them will be from the extreme right hand end of the horizontal diameter and the extereme upper end of the vertical diameter when there is no phase difference. If the periods of the two motions are different, the phase difference cannot remain constant. What can be done at most is that the initial phase difference can be introduced. The tracing points on the two semi-circles are taken to be moving in the anti-clockwise direction.



Amplitudes and time periods same and phase difference $= \pi/2$.
(S.H.M., *a* leading)
Fig. 106



Amplitudes are in the ratio 3 : 2(*a* : *b*), time periods 2 : 1, (frequency 1 : 2) and initial phase difference nil.
Fig. 107



Amplitudes 3 : 2 (*a* : *b*), periods 2 : 1 and initial phase difference $= \pi/4$.
(S.H.M., *a* leading)
Fig. 108

Figs. 104—108 show the different cases of two rectangular *S.H.M.*'s graphically combined. In these figures *a* and *b* are horizontal and vertical *S.H.M.*'s, S_1 and S_2 the starting positions of their tracing points and *C* and *S* their resultant and its starting point respectively.

108. Resolution of a Uniform Circular Motion :—Let a particle P move uniformly round the circle ABA_1B_1 (Fig. 109). Suppose AOA_1 and BOB_1 are two mutually perpendicular diameters of the circle. As P moves from A to A_1 and then from A_1 to A moving all along in the anti-clockwise direction, the feet M and N of P execute $S.H.M.$'s along AOA_1 and BOB_1 respectively with O as the mean position. Now the displacement $OM=r\cos\theta$ and $ON=r\sin\theta$ where r is the radius of the circular motion and θ, the angular displacement of the particle at that instant of time, this displacement being measured from the instant the particle passes through A. When $\theta=0$, OM is maximum and ON is zero and when $\theta=\pi/2$, OM is zero and ON is maximum. So the phase of N is $\pi/2$ behind that of M. Thus the uniform circular motion of P is equivalent to the two $S.H.M.$'s of equal amplitude r acting at the right angles to each other and differing in phase by $\pi/2$.

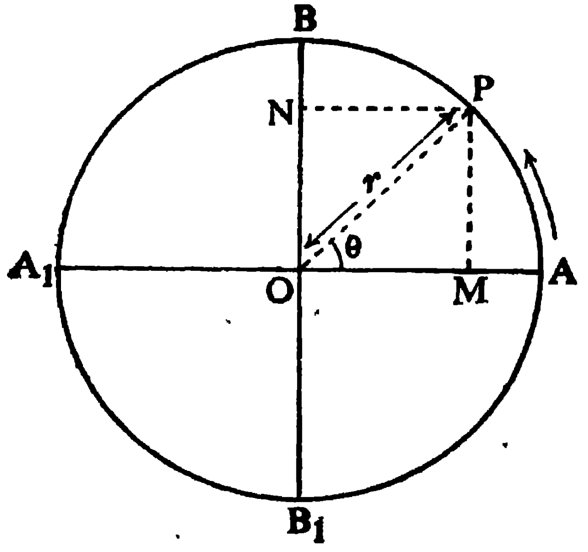


Fig. 109

109. Composition of two equal and opposite circular motions :—

(i) Analytical method.—

Let $x_1^2+y_1^2=r^2$ represent one circular motion, x_1 and y_1 indicating the displacements along the x- and y-axes respectively and r the amplitude of the motion. This circular motion is equivalent to two simple harmonic motions,

$$x_1=r\cos\theta=r\cos\omega t$$
$$\text{and } y_1=r\sin\theta=r\sin\omega t,$$

the two motions obviously acting at right angles to each other and differing in phase by $\pi/2$; ω is the angular velocity. Since the other circular motion is in the opposite direction, it will give rise to the other simple harmonic components with only the sign of ω changed. So these component $S.H.M.$'s are—

$$x_2=r\cos(-\omega t)t=r\cos\omega t$$
$$\text{and } y_2=r\sin(-\omega)t=-r\sin\omega t.$$

Combining these x-displacements as also the y-displacements, the resultant displacements are,

$$X=x_1+x_2=r\cos\omega t+r\cos\omega t=2r\cos\omega t \text{ and}$$
$$Y=y_1+y_2=r\sin\omega t-r\sin\omega t=0.$$

Thus the resultant motion is a *S.H.M.* of displacement $X=2r\cos\omega t$, having the same period as either of the circular motions and of amplitude equal to double the radius of either of these circles.

(ii) **Graphical method.**—Let P (Fig. 110) be the position of the particle executing anti-clockwise circular motion at any instant of time. This circular motion is equivalent to two *S.H.M.*'s, one along AA_1 and the other along BB_1. The displacements of P along these axes are $OM=NP$ and ON. At the same instant of time, P_1 will represent the position of the particle executing clock-wise circular motion having corresponding displacements $OM_1=NP_1$ and ON. Now as the two circular motions are equal in magnitudes, the components NP and NP_1 being equal and opposite cancel each other, and the components along the diameter BOB_1 being equal and similarly directed add up together to give resultant displacement equal to $2ON$. Since the component displacements are *S.H.* in nature, the two circular motions are equivalent to a *S.H.M.* having a displacement $2ON$ which is double the displacement of the similarly directed components of each of the circular motions. Again since displacement $y=a\sin\omega t$ along BOB_1, the resultant displacement will be $2y=2a\sin\omega t$. So the amplitude of the resultant displacement will be double that of the similarly directed component of each of the circular motions.

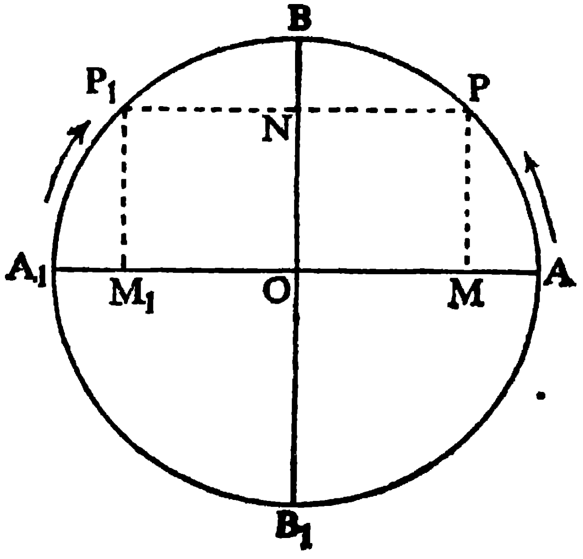


Fig. 110

110. Resolution of a S.H.M. into two equal and opposite circular motions :—

Let the *S.H.M.* be represented by the displacement equation,

$$x=r\cos\omega t.$$

This can be rewritten as,

$$(a)\quad x=\frac{r}{2}\cos\omega t,\ y=\frac{r}{2}\sin\omega t,$$

$$\text{and}\quad (b)\quad x=\frac{r}{2}\cos\omega t,\ y=-\frac{r}{2}\sin\omega t.$$

This is justified because in the direction of y-axis, two equal and opposite *S.H.M.*'s are taken whose total effect is nil.

∴ From (a) $x^2+y^2=\left(\frac{r}{2}\right)^2$, which represents a circular motion of amplitude (or radius) $\frac{r}{2}$. Again from (b), $x^2+y^2=\left(\frac{r}{2}\right)^2$ also represents a circular motion of amplitude $\left(\frac{r}{2}\right)$ but it is oppositely directed because of the negative sign of ω, which is indicated by the negative expression of y in (b). Thus it is found that a *S.H.M.*'s can be resolved into equal and opposite circular motions.

111. Lissajous' Figures :—Lissajous' figures are curves representing the resultant of two *S.H.M.*'s acting at right angles. These component *S.H.M.*'s may again be varying in amplitudes, phases and periods, or in some of them. Sometimes they may be even identical in all respects. These curves were for the first time produced by Lissajous. Hence the name. These figures may be obtained experimentally by (1) Tisley's Harmonograph, (2) Blackburn's Pendulum, (3) Kaleidophone, (4) Optical method and (5) Cathode ray oscillograph.

Production of Lissajous' Figures

(1) Tisley's Harmonograph.—

It consists of two vertical iron rods provided with adjustable weights, *A* and *B* (Fig. 111) at the lower ends. They are fitted with knife-edges about which they can oscillate inside two slots cut at right angles to the two adjacent sides of a rectangular table. Thus these rods form two pendulums whose periods can be adjusted for the given ratio with the help of adjustable weights and a stop-watch. The upper ends of the rods, which project a little above the plane of the table, support two straight strips, whose free-ends are hinged perpendicularly together at *C*. A fine vertical pencil is attached to this hinge. The tip of this pencil is free to move over a piece of paper placed on the table. Now as the pendulums formed by the rods are made to oscillate with the desired periods the resultant curve is automatically drawn by the attached pencil on the piece of paper.

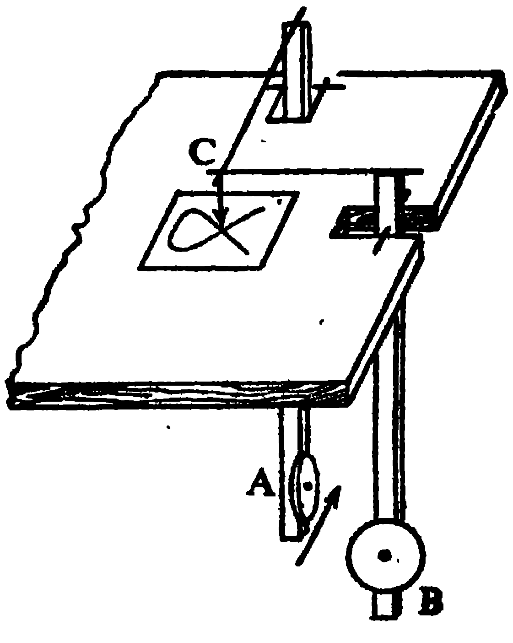


Fig. 111

(2) Blackburn's Pendulum.—

It consists of vertical rectangular wooden frame fitted on to a heavy horizontal base (Fig. 112). At the centre of the horizontal arm of the vertical rectangular frame is fitted a screw peg *P*, from which two strings are made to pass through the two corners E_1 and E_2 of the frame. The two strings are clamped together at C_1, by means of an adjustable clip. At C_2 another clip is fitted and the strings are here branched into 3 parts which hold a heavy metal ring *R*. Through the central hole of the ring passes a funnel filled up with coloured sand or any other heavy coloured powder. Just below this ring is placed a piece of paper. Now if the ring *R* be displaced slightly in the plane of the rectangular frame, it will oscillate and form a pendulum of length $C_1R = l_1$ (say). Its time period will be proportional to $\sqrt{l_1}$. Again if the ring be displaced in a direction normal to the plane of the frame, the effective length of the pendulum will be $SR = l_2$ (say) and the time period will be proportional to $\sqrt{l_2}$. Thus by adjusting l_1 and l_2 any ratio of the time periods can be set up. Since in each of the above two modes of vibration, *R* executes a *S.H.M.*, if *R* be displaced in any other intermediate direction, it will simultaneously be subjected to two *S.H.M.*'s at right angles. So if the sand in the funnel be now allowed to fall on the paper below, it will trace out the resultant curve of these two rectangular *S.H.M.*'s thus giving the required Lissajous' figure.

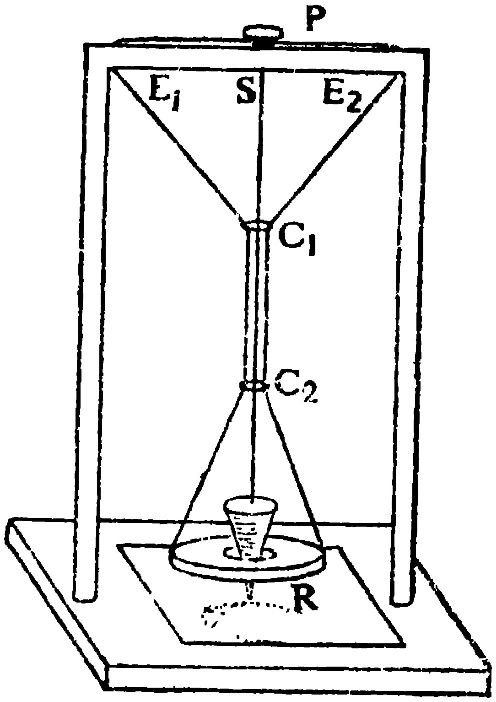


Fig. 112

(3) Kaleidophone.—

This is an arrangement devised by Wheatstone by means of which Lissajous' figures can be visualised. It consists of a rectangular metal bar fitted vertically on to a heavy metal base. At the top of the bar a polished metal bead is attached. Now since the widths in those to perpendicular directions are different, the periods of vibration in those two directions will also differ because of the different elastic forces that are called into play while the vibrations continue in those two directions. So when made to vibrate in an intermediate direction, the vibration will be the resultant of those two rectangular *S.H.M.*'s and the shining bead will indicate the resultant Lissajous' figure. For getting different types of Lissajous' curves, different rods of suitable rectangular cross-sections are to be chosen and fitted in the similar way on to the heavy base.

(4) Optical method.—

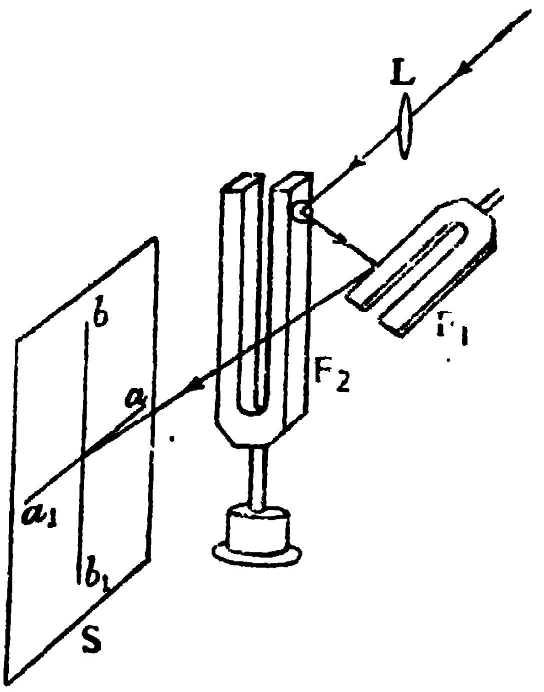


Fig. 113

Lissajous used an optical method for the production of the resultant curves of two rectangular *S.H.M.*'s. Two tuning forks (Fig. 113) - one horizontal and the other vertical—are adjusted on suitable stands. To one prong of each a small mirror is attached. A narrow beam of light is passed through a focussing lens *L* and is successively reflected from the two mirrors and is **finally** received on the screen *S*. If only the horizontal fork *F* vibrates a horizontal lines, aa_1 will be traced and if only the vertical fork F_2 vibrates, a vertical line bb_1 will be traced on the screen. These two are two separate *S.H.M.*'s having different periods depending on the frequencies of the forks. When both the forks vibrate together the spot of light will describe the locus of the resultant motion. The initial phase difference between the two motions depends on the times of excitation of the forks. The amplitudes of vibration will depend on the dimensions, the elastic constants and the magnitudes of the exciting forces—either electrical or mechanical.

(5) Cathode Ray Oscillograph (C.R.O.) Method.—

The cathode ray oscillograph is essentially a modified cathode ray tube wherein a beam of electrons possessing high energy is focussed on to a fluorescent zinc sulphide screen attached to the end of the tube which is opposite the cathode. A greenish blue spot of light is obtained on the screen. Since the electron beam is very light, it can easily respond to the high frequency variation of an electric or magnetic field near it. The nature of this variation can be visualised by the movement of the spot of light on the screen.

In the attached figure (Fig. 114) *C* is the oscillograph tube which is highly evacuated. The electrons are supplied by the filament *F* which serves as the cathode and is heated to a high temperature by passing a desired current through it. Now-a-days an indirectly heated cathode is more commonly used. G is a metal cylinder having an opening in front of F for the passage of electrons G is maintained at a negative potential with respect to F and this potential can be varied by some

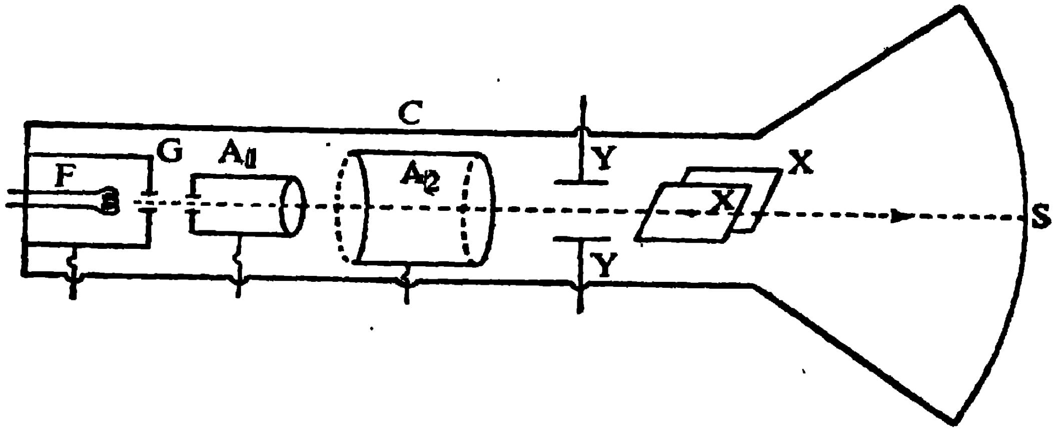


Fig. 114

suitable arrangement. A_1 and A_2 are two metal cylinders serving as the anodes and kept at high positive potentials relative to F. The components F, G, A_1 and A_2 together form the *electron-gun* of an oscillograph. S is the fluorescent screen. The interior of the tube is coated with a conducting paint, with the help of which. S is connected to A_2. Thus S and A_2 are maintained at the same potential. This makes the electron beam approach the screen without any change in the associated kinetic energy. YY and XX are two pairs of plates known as the *deflecting plates*. These are placed between S and A_2. YY-plates, when connected to a voltage source, deflect the beam in the vertical direction and similarly, XX-plates deflect it in a horizontal direction.

The intensity and hence the ***brilliance*** of the spot on the screen depends on the number of electrons reaching S per second *i.e.* on the beam current. So the brilliance can be partly controlled by adjusting the filament current and the p.d. between the anode and the cathode. which supplies the energy of the electrons producing fluorescence on the screen. Again as G is at a negative potential with respect to the filament F, the magnitude of the potential applied to G evidently controls the number of electrons passing out of the opening in G. Moreover, the fluorescent material used in the screen should have a high conversion efficiency and spectral response so that spot brilliance may effectively increase.

Focussing of the spot of light on the screen is mainly effected by applying suitable voltages to A_1 and A_2 which serve as an electron lens system. The focussing can also be done with the help of magnetic fields in places of electrostatic fields. If the spot of light is not at the centre of screen, it can be easily brought to the centre by applying a small suitable voltage to either of the two pairs of plates as the occasion demands.

When the instrument has been once adjusted, an alternating p.d. of simple harmonic type applied to YY-plates causes the spot of light to describe a vertical line on S, if XX-plates are kept at a constant p.d. This vertical line indicates the line along which the $S.H.M.$ takes place (actually electrons are subjected to this $S.H.M.$). Now if YY be kept at a constant potential and an alternating simple harmonic p.d. be applied to XX, a horizontal line will be described by the light spot, which again indicates the line of action of the $S.H.M.$ to which electrons are subjected in between the plates, XX.

In order to obtain Lissajous' figures with this apparatus, two alternating p.d.'s of desired periods (having the desired ratio) are simultaneously applied—one to the plates YY and the other to the plates XX. As a result, the electron beam will trace the curve of the resultant of these two rectangular $S.H.M$'s. When the ratio of the periods is rather high, this simple method cannot be suitably applied and as such modifications are to be introduced in the method. By this method Lissajous' figures at a frequency of the order of megacycles per second can be easily traced.

Time-displacement curve of a S.H.M.—

This can be easily obtained with the help of a cathode ray oscillograph. A simple harmonic p.d. is applied to YY-plates and the steady voltage, applied to XX-plates, is made to decrease uniformly with time. The spot of light now travels across the field and describes the time-displacement curve of the $S.H.M.$ on the screen. The uniformly varying voltage applied to XX-plates is obtained from a circuit known as the *time-base* which is one of the essential parts of the tube.

112. Utility of Lissajous' Figures—Vibration microscope :—

Lissajous' figures are practically utilised in accoustical determinations like standardisation of frequencies of tuning forks. The method is based on the fact that for two $S.H.M.$'s at right angles, the resulting curve will have a particular form for a particular ratio of the periods or frequencies, so long as the phase relation between the two motions remains constant.

If m and n are the frequencies of the two forks producing a Lissajous' figure. its form will correspond to the frequency ratio m/n. Now if in place of n, another fork of frequency $n+\delta$ is taken, δ being small, and we start both these forks at the same phase, then at start the path will be almost the same as that for frequencies m and n. But gradually the faster one will gain over the slower one and this gain will continue till at last they are again in the same phase. This happens when the faster one has made one vibration more than it would have made, had the ratio been exactly m/n. At this instant when they are in the same phase, the form of the curve will be repeated. If the time interval, t, after which the form of the curve is repeated, is noted, then $\delta=\frac{1}{t}$. Thus it is possible to know δ whence the frequency, $n+\delta$, of the second fork can be determined.

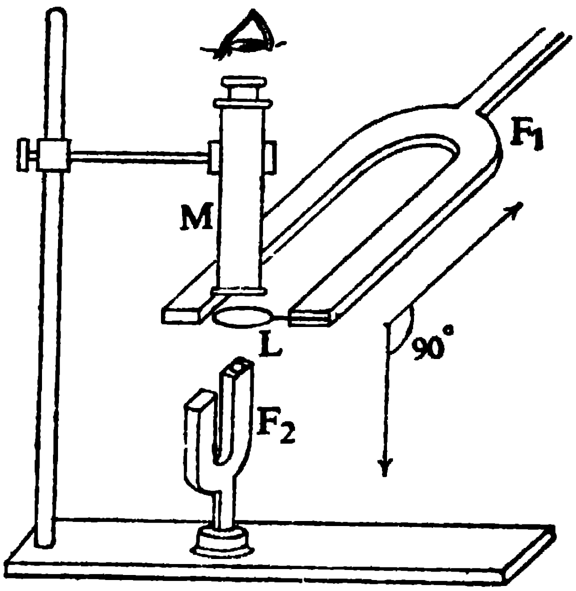


Fig. 115

The above principle has been practically applied in the vibration microscope devised by Helmholtz. Such a microscope essentially consists of a horizontal tuning fork F_1 (Fig. 115) carrying on one of its prongs a lens L supported by a frame. This lens forms the objective of a vertical microscope M, its other parts being suitably supported by a separate stand. This fork holding the objective serves as the standard of known frequency. The fork F_2 whose frequency has to be determined is placed vertically with one of its prongs below the microscope objective. This prong has a dot at its top The microscope is focussed on this dot. The motions of the prongs of the forks are evidently at right angles to each other. So when both the forks are excited, Lissajous' figures are observed on looking through the microscope. By applying the principle indicated above, the frequency determination of the vertical fork is then made.

Questions (Chapter VI)

1 Define a S. H. motion, explaining the meanings of the terms, period, amplitude and phase.

A particle is subjected simultaneously to two S. H. vibrations, of the same period but of different amplitude and phases, in perpendicular directions. Find an expression for the resultant motion and show that the path traced by the particle is an ellipse.

For what conditions the path may be a circle and a straight line ? (C. U. 1950)

2. Explain the characteristics of a simple harmonic motion and show how to find the velocity at any phase of the motion.

A particle executes simple harmonic motion of period 16 secs. Two seconds after it passes the centre of oscillation, its velocity is found to be 4 ft. per second. Find the amplitude. [*Ans.* 14·41 ft.] (Madr U. 1949)

3. Show that if the displacement of a particle executing a S. H.M. from a fixed point is given by $x=a\cos(\omega t+\delta)$, the total energy at any instant is given by $\frac{1}{2}m\omega^2a^2$ ergs, m being the mass of the particle. (C. U. 1957)

4. A body executes simple harmonic motion of amplitude 1·0 cm. and frequency 12 cycles per second. What is its velocity when the displacement is 5 mms. ? [*Ans.* 65·3 cm./sec.]

5. Deduce the equation for the simple harmonic motion of a particle.

Two simple harmonic motions, having the same period but differing in phase and amplitude, are acting in the same direction on a particle. Show that the resultant motion is simple harmonic and deduce the expression for the resulting amplitude and phase. (C. U. 1940)

6. Find the resultant of two mutually perpendicular S. H. motions which agree in period but differ in phase. Consider the important cases for phase differences varying from 0 to 2π. (Pun. U. 1935)

7. Define simple harmonic motion and show that if the displacement of a particle moving at any time is given by an equation of the form

$$x=a\cos\omega t+b\sin\omega t,$$

the motion is simple harmonic. If $a=3$, $b=4$, and $\omega=2$, find the period, amplitude, maximum velocity and maximum acceleration of the motion.

[*Ans.* (*i*) 3·142; (*ii*) 5; (*iii*) 10; (*iv*) 20] (Madr. U. 1949)

8. Derive an expression for the motion of a particle subjected to two mutually perpendicular *S. H. M.*'s having the same period but differing in phase. Hence obtain expressions when the phase difference is 0, $\pi/4$, $\pi/2$, π.
(*cf.* C. U. '67) (Bom. U. 1943)

9. Find the velocity and acceleration of a point executing simple harmonic motion.

A point describes simple harmonic motion in a line 4 cms. long. Its velocity when passing through the centre of the line is 12 cms. per sec. Find the period.
[*Ans.* 1·047 secs.] (C. U. 1949)

10. Deduce an expression for the *S. H. M.* of a particle. Explain the terms period, phase and epoch angle.

Find graphically or otherwise the resultant of two *S.H.M.*'s at right angles to each other having the same period and amplitude but with a phase difference of $\pi/4$. (Utk. U. 1949)

11. A particle executing a simple harmonic motion has a maximum displacement of 4 mms. and its acceleration at a distance of 1 mm. from its mean position is 3 mms./sec^2. What will be the velocity at a distance of 2 mms. from the mean position ? [*Ans.* 6 mm./sec.]

12. A particle moves with a uniform speed in a circle. Show that the motion may be resolved into two simple harmonic motions at right angles to each other. How do they differ in phase and amplitude ? Show how the potential and the kinetic energies of a particle executing simple harmonic motion vary.

(C. U. 1913; *cf.* Madr. U. 1947)

13. A body executing *S.H.M.* has an amplitude of 10 cm. and time period is 1·5 secs. Calculate the time taken by the body to travel a distance of $5\sqrt{3}$ cm. from its rest position.

[*Ans.* 0·25 sec.] **(Pat. U. 1959]**

14. A mass of 100 gms. is attached to the lower end of a long light spring hanging vertically from a fixed point at its upper end. This increases the length of the spring by 2 cms. Find the period of vertical oscillation when the mass is slightly pulled down and released.

[*Ans* 0·2836 sec.] **(Burd. U. 1955)**

15. A test tube of weight 6 gms. and of diameter 2 cms. is floated vertically on water by placing 10 gms. of mercury at the bottom of the tube. The tube is depressed by a small amount and then released. Find the time of oscillation.

[*Ans.* 0·4527] **(Bihar U. 1955)**

CHAPTER VII

GRAVITATION AND GRAVITY

113. Historical Notes :—One day while Newton was sitting under an apple tree in the garden of his village home at Woolsthorpe, a ripe apple, it is said, fell on his head. This simple event started him thinking why the apple should fall towards the earth. There must then exist some attractive force between the earth and any material body. Reasonings on this line ultimately led him to found the doctrine of universal gravitation.

114. Newton's Laws of Gravitation :—(1) *In the nature every material body attracts every other material body towards itself.*

(2) *The force of attraction between any two bodies varies directly as the product of their masses and inversely as the square of the distance between them.*

If m_1 and m_2 be the masses of two bodies and d the distance between them (Fig. 116) and if F be the force of attraction, which each exerts on the other, $F \propto m_1 m_2$ and also $F \propto 1/d^2$;

$$\therefore \quad F \propto \frac{m_1 m_2}{d^2}\,; \quad \text{or,} \quad F = G\frac{m_1 m_2}{d^2}\,,$$

where G is a constant known as the *Universal Gravitational Constant.*

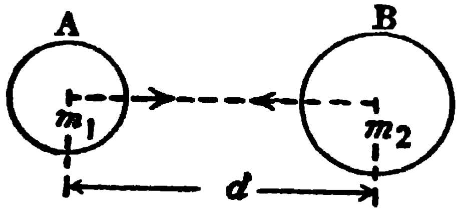


Fig. 116

Its value as determined by Boys in 1895 is $6{\cdot}6576 \times 10^{-8}$ in C.G.S. units. The most accurate value of G claimed so far is by Heyl (1930) and is $(6{\cdot}670 \pm 0{\cdot}005) \times 10^{-8}$ C.G.S. units. The dimensional formula for **G** is given by $[\mathbf{G}] = [F][d^2]/[m_1 m_2]$

$$= \frac{[MLT^{-2})(L^2)}{(M^2)} = (M^{-1}L^3T^{-2}).$$

Let m_1 and m_2 in the above equation be each equal to 1 gram and d equal to 1 centimetre, then $G = F$, which means that G is ***numerically equal to the force of attraction between two masses, each of one gram, when separated by a distance of one centimetre.***

115. Gravitation and Gravity :—The force of attraction between any two material bodies is called *gravitation*. The term is more specially applied to the attraction between two heavenly bodies.

Now gravity is the force with which the earth attracts every body on or near its surface towards its centre. If the mass of the earth is M and the mass of any object on its surface is m, the force of attraction due to gravity $= G\frac{Mm}{R^2}$ where R is the radius of the earth. So gravity is a particular case of gravitation and may be called earth's gravitation. The force of gravity on a body is called its *weight.*

116. The Acceleration due to gravity (g):—A body, if dropped from a height, falls vertically towards the earth, *i.e.* as if it would pass through the centre of the earth ; its velocity continuously increases as it falls. That is, it falls with an acceleration. Such motion is due to the attraction between the body and the earth, *i.e. due to gravity.* When a body falls freely from rest, as in the case when it is simply dropped from a height, it is experimentally found that the distance s through which it falls is proportional to the square of the time taken t. That is, $s=kt^2$, where k is a constant. This is possible only if the acceleration of the falling body is constant (*cf.* $s=ut+\frac{1}{2}ft^2$). *Thus, a body dropped from a height falls vertically towards the earth with a constant acceleration.*

If then the acceleration is constant, a heavy body as well as a light body dropped from a height should reach the ground simultaneously. That is also exactly what is found experimentally. Newton's famous *Guinea and Feather* experiment conclusively proved that all bodies starting from rest fall in vacuum with equal rapidity. That is, *the acceleration due to gravity at the same place is the same for all bodies.*

Consider, again, a body projected vertically upwards ; its velocity gradually diminishes and is finally reduced to zero after which it begins to fall downwards again. This also clearly shows the existence of an acceleration directed towards the earth due to which the upward motion of the body is gradually reduced. Thus, all experimental observations lead us to the belief that any body moving in the field of the earth's attraction is subject to a constant acceleration acting vertically downwards and its value is the same for all bodies at the same place. This acceleration is called the acceleration due to gravity and is conventionally denoted by 'g'.

We have seen in Art. 115 that the force of attraction due to gravity varies inversely as the square of the distance of a body from the centre of the earth. So the acceleration due to gravity g also varies in the same way, according to Newton's Second Law of Motion, the mass of the body being constant. Since the force of attraction on a body of mass m due to gravity is $F=G\frac{Mm}{R^2}$ (vide Art. 115) the acceleration due to gravity is $g=\frac{F}{m}=\frac{GM}{R^2}$. It is affected due to rotation of the earth on its axis (*vide* Art. 117).

The value of the acceleration due to gravity at sea-level and in latitude 45° is generally taken as the **standard of reference** for values of acceleration. The value of g at any place varies with its height above the sea-level, being less at the top of high mountain than at its bottom. *The value of g is constant at the same place,* but varies with the latitude. It is minimum at the equator and increases gradually to attain the maximum value at the poles. At the equator, the value of g is about 978 cms. per sec. per sec. and at the poles, it is about 983 cms. per sec. per sec. and the accepted mean value is taken to be 981 cms. per sec. per sec. or 32·2 ft. per sec. per. sec. It also changes from one place to another due to various local conditions.

117. Variation of 'g' (or the weight of a body) from place to place :—

The numerical value of the acceleration due to gravity may vary when its magnitude is determined at a higher altitude or at a plane below the surface of the earth. It is also found to alter from place to place even on the surface of the earth when the latitude of the place changes. The nature of these changes are discussed fully below.

(a) **Above the Surface of the Earth.**—(*i.e.* at a higher altitude). At a height h above the surface of the earth, the force of attraction on a body of mass m, according to the law of gravitation $= G\frac{Mm}{(R+h)^2}$. For such positions external to the earth, the mass of the earth may be supposed to be concentrated at its centre. So the acceleration due to gravity $g' = \left[\frac{G.M.}{(R+h)^2}\right]$, on a body above the surface of the earth is *inversely proportional* to the square of the distance of the body from the centre of the earth. If g be the acceleration due to gravity on the surface of the earth, $\frac{g'}{g} = \frac{G.M.}{(R+h)^2} \times \frac{R^2}{G.M.} = \frac{1}{\left(1+\frac{h}{R}\right)^2} = \left(1 - \frac{2h}{R}\right)$ approximately. $\therefore\ g' = g\left(1 - \frac{2h}{R}\right)$. So g' will be less, as the distance of the body above the surface of the earth increases.

(b) **Below the Surface of the Earth.**—Again, consider a body of mass m at a depth 'h' below the surface of the earth [Fig. 117]. Imagine a sphere concentric with the earth having radius $(R-h)$, *i.e.* with its surface passing through points at a distance h below the surface of the earth. It is known that the gravitational force of attraction within a hollow spherical shell is zero. Here the given body is on the surface of the inner sphere of radius $(R-h)$ but it is *inside* with respect to the portion of the earth outside the smaller sphere ; so the outer portion has no attractive force on the body. The force of attraction on the

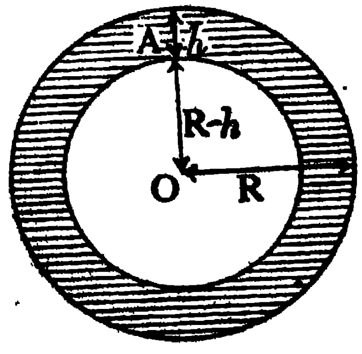


Fig. 117

body will be due to the inner solid sphere of radius $(R-h)$ towards the centre of the earth and is equal to

$$G.\frac{\frac{4}{3}\pi(R-h)^3.D.m}{(R-h)^2} = \frac{4}{3}\pi G(R-h).D.m,$$

where $\frac{4}{3}\pi(R-h)^3 \times D$ is the mass of the inner sphere, D being the mean density of earth's crust.

The force of attraction, and hence g' inside the earth, is therefore, *directly proportional* to $(R-h)$, that is, to the distance of the body from the centre of the earth. Here also if g be the acceleration due to gravity on earth's surface, $\frac{g'}{g} = \frac{4}{3}\pi G(R-h)D \Big/ \frac{G.M.}{R^2} =$

$$\frac{4}{3}\pi(R-h)DR^2/\frac{4}{3}\pi R^3 D = \frac{R-h}{R} = \left(1-\frac{h}{R}\right).$$

$$\therefore\ g' = g\left(1-\frac{h}{R}\right).$$

Thus from (a) and (b) we find that g varies inversely as the square of the distance from the centre of the earth for points on and exterior to the earth's surface, whereas for points in the interior of the earth g varies directly as the distance from the centre of the earth. This behaviour is illustrated in Fig. 118. Here g at a place linearly rises with its distance R from earth's centre up to a value equal to R_0, the radius of the earth. This limiting value of g (for $R = R_0$) is nearly 32 ft./sec.^{2}. When R is greater than R_0, g falls less slowly than the linear rate of rise for the same variation of distance. This inverse square variation is shown by the portion of the curve on the right-hand side of P.

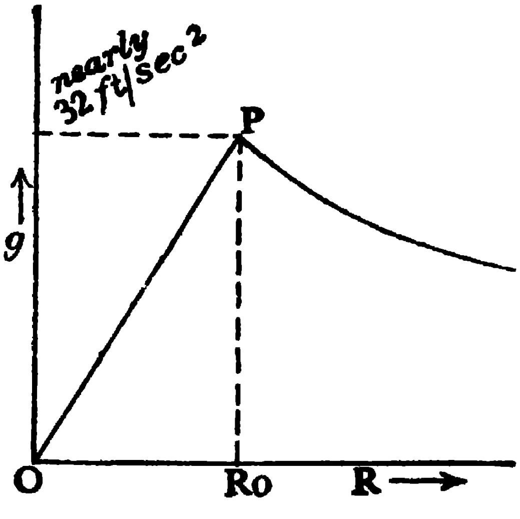


Fig. 113

So g' will be less inside the earth's surface, the greater the depth the lesser being the value of g'. Hence the *maximum value* of g is *on the surface of the earth*, and *the value of g is minimum (i.e. zero) at the centre of the earth.*

(c) On the Earth's Surface.—(*i.e.* at different latitudes). In this case the value of g varies due to two reasons—

(i) The Peculiarity in the Shape of the Earth (*effect of equatorial bulge*)—As the force of gravity on the earth's surface is inversely proportional to the square of the radius of the earth at the place, it is greatest at the poles and least at the equator, the earth being a spheroid of which the polar radius is the least and the equatorial radius the greatest, the difference between the two radii being about 13·5 miles. So the value of g or the wt. of a body, increases from the equator to the poles.

If λ be the latitude of any place, the true value of acceleration due to gravity there is given by $g=(980{\cdot}61-{\cdot}025\cos 2\lambda)$ cms./sec.^{2}

(ii) The Rotation of the Earth about its Axis.—The diurnal rotation of the earth (about its polar axis) causes every body placed on it to be subjected to a centrifugal force due to which it tends to be flung off. A part of the gravitational force, due to which the body is attracted towards the centre of the earth, and which gives the true weight of the body, is used up in counteracting the centrifugal force and the balance shows itself as the weight of the body, which is an apparent weight only.

The apparent weight and so the apparent acceleration due to gravity, which we actually measure in our experiments, increase from the equator to the poles, as shown below.

Let us assume the earth to be a uniform and homogeneous sphere of mass M. Let XX' and YY' be respectively the equatorial and the polar axes of the earth mutually intersecting at right angles at O, the centre of the earth and let P be a body of mass m at latitude λ (Fig.119). The true weight W of the body is a force F acting on P in the direction PO and is given by $W=F=\frac{G.Mm}{R^2}$, where $G=$gravitational constant, and $R=$radius of the earth. Draw PC perpendicular from P on YY'. The centrifugal force f to which P is subjected, is given by $f=\frac{mv^2}{(PC)}$, where $v=$linear velocity of P and $PC=$radius of the circle which the body P describes while the earth rotates about YY'; or, $f=\frac{m.(\omega.PC)^2}{(PC)}$ $=m\omega^2(PC)=m\omega^2R\times\cos\lambda$, where $\omega=$the common angular velocity of the earth and the body P about YY'. This force is directed from P towards C' the continuation of PC in the opposite direction. The component of this force in the direction PO', which is opposite to PO, is given by $f \cos\lambda$, *i.e.* $m\omega^2 R \cos_2\lambda$.

$\therefore$ The apparent weight of P $= W - m\omega^2 R \cos^2\lambda$ ... (1)

If g_λ be the acceleration due to gravity at P, then $mg_\lambda = mg - m\omega^2 R \cos^2\lambda$

$$= mg\left(1 - \frac{\omega^2 R \cos^2\lambda}{g}\right).$$

$$\therefore\ g_\lambda = g\left(1 - \cos^2\lambda . \frac{\omega^2 R}{g}\right)$$

$$= g\left(1 - \frac{\cos^2\lambda}{288}\right) \ldots (2)$$

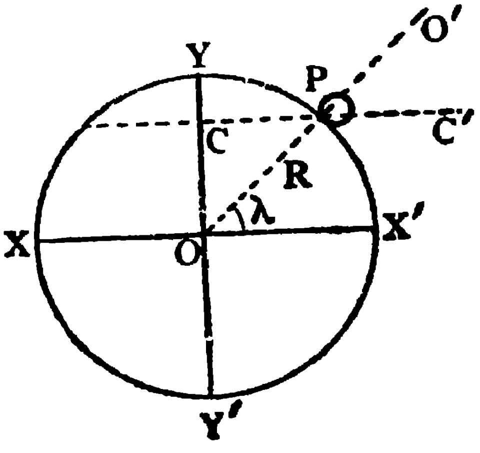


Fig. 119

Now $\lambda = 0$ at the equator, *i.e.* $\cos\lambda$ is maximum, *i.e.* the apparent weight of the body (or the acceleration due to gravity) is minimum at the equator. Again, $\lambda = 90°$ at the pole, *i.e.* $\cos\lambda = 0$, *i.e.*, the apparent weight of the body, or the acceleration due to gravity is maximum there. Thus apart from the fact that the weight of a body or the acceleration due to gravity increases from the equator to a pole due to decrease of the earth's radius, it also increases from the equator to a pole on account of the diurnal rotation of the earth.

VARIATION OF '*g*' WITH LATITUDE

Place	Latitude	Value in ft./sec.^{2}	Value in cms./sec.^{2}
Equator	0° 0′	32·09	978·10
Madras	13° 4′	32·10	978·36
Bombay	18° 53′	32·12	978·63
Calcutta	22° 32′	32·13	978·76
New York	40° 43′	32·16	980·19
Paris	48° 50′	32·18	980·94
London	51° 29′	32·19	981·17
Poles	90°	32·25	981·11

118. The weight of a Body on the Sun or the Moon :—A 1-lb. mass will weigh twentyseven times more when taken to the sun, since the gravitational force exerted by the sun is so many times greater. For similar reasons it will only weigh $\frac{1}{6}$th of a pound when taken to the moon.

119. Experimental Determination of 'g' :—The acceleration due to gravity is best determined by a pendulum experiment, though the following methods may also be used in the laboratory :

(a) By the Atwood Machine.—The machine is generally used (*i*) to determine the value of *g* at a place, and also (*ii*) to verify Newton's laws of motion.

The machine consists of a fine silk thread (i.e. *a light inextensible* thread) having two equal cylindrical brass weights of masses *P*, *P* attached at the two ends (Fig. 120) passing over a *light smooth* pulley *E*, fixed to the top of a graduated vertical stand *S*. There are two platforms *A* and *C* which can be clamped at any desired point of the stand, as also a ring *B* which is clamped somewhere in between the platforms. There is a small weight of mass *Q*, called the rider, of the shape shown in the figure, with arms projected on either side, and this is placed horizontally over the right-hand cylindrical weight *P* which, to start with, rests on the upper platform *A*, placed as near the top of the stand as possible. The ring has such a diameter that the upper weight *P*, when dropped, passes through the ring but the rider *Q* is arrested. When the upper platform is turned (in some arrangements, there is no upper platform and a spring-catch holds the lower weight *P* ; when motion is to be started, this catch is released), the system with a mass $(P+Q)$ on one side and *P* on the other begins to move. At some instant after start, the right-hand weight *P* passes through the ring *B*, when the rider *Q* is arrested and the subsequent motion of the system takes place with equal weights *P*, *P* at the either end of the string. The motion stops at the right-hand weight touches the lower platform *C*.

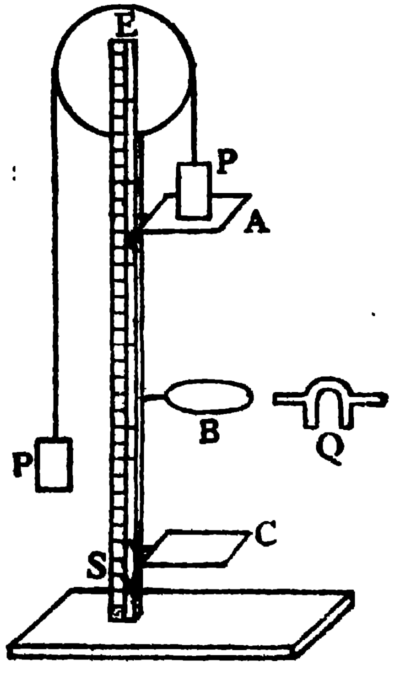


Fig. 120—The Atwood Machine

In performing an experiment distances moved through by the right-hand weight *P* during the two stages of motion, (*a*) first, from its start from the upper platform till the rider is arrested, (*b*) second, from the instant when the rider is arrested till it is stopped on reaching the lower platform C, are noted and also the times taken by the two stages of motion are recorded by stop-watch. Let the two intervals of distances be h_1 and h_2 and the times taken t_1 and t_2. Here h_1 is to be taken as the distance from the top of the upper weight P at start to the ring B and h_2 as the distance from the ring B to the top of the same weight when it is on the lower platform C.

(i) **Determination of g.**—Since the masses P, P on the two sides of the pulley are identical, the downward acceleration, f of the right-hand P and its rider attachment Q, as also the upward acceleration f of the left-hand P is slowly caused by the weight of the rider of mass Q. $\therefore$ $(P+P+Q)f=Q.g$. So the acceleration of the moving body will be given by, $f=\dfrac{Q}{2P+Q}.g$ ... ... (1)

Starting from rest, the body moved through a distance h_1 in time t_1 with the acceleration given by equation (1).

$$\text{So, } h_1=\tfrac{1}{2}.\frac{Q}{2P+Q}.g.t_1^2 \qquad \dots \qquad \dots \qquad \dots \quad (2)$$

P and Q being known, and h_1 and t_1 being noted, g is determined. The experiment may be repeated by altering h_1 at pleasure by shifting the position of the ring and noting t_1 in each case, when the value of g will be found the same.

(ii) **Verification of Newton's Laws of Motion.**—Here assume g to be known. The observed value of h_1 and t_1 will be seen to satisfy the relation (2) for all values of h_1 proving the correctness of the formula for the acceleration f, given by relation (1). This indirectly verifies the truth of Newton's Second Law of Motion because the relation (1) is deduced directly from the relation $P=mf$.

The velocity acquired by the moving body at the end of the first stage of motion is given by,

$$v^2=2.\frac{Q}{2P+Q}.g.h_1 \qquad \dots \qquad \dots \quad (3)$$

From the experiment it will be found that the value of v so determined exactly equals h_2/t_2, *i.e.* $h_2=t_2v$. The same result will be obtained on changing h_2 by altering the position of the lower platform C. This verifies Newton's First Law of Motion, for during the second stage of motion, *i.e.* from the ring B to the lower platform C, when the weights on the two sides of the string are equal, the velocity once acquired remains uniform in the absence of any resultant force on the system.

In deducing the above relation (1) which gives the acceleration f, the tension T is assumed constant throughout the string and this assumption is based on the truth of the Third Law of Motion. The experimental verification of the value of f given by the relation (1) supplies an evidence in support of the Third Law as well.

(b) **By the Falling Plate Method.**—The method is very appropriate for measurement of the frequency of a tuning fork and has therefore, been described in Part II of the book under Sound. It will be evident from the last equation of that article [Art. 103(8)] that the same experiment may be used to determine 'g' at a place, if a standard fork of known frequency is supplied.

(c) **Vertical oscillation of a body suspended by an elastic string.**—Suppose a body of mass M be suspended vertically by an elastic string. Then under the action of the weight Mg, the length L of the string will increase by l. If E be the longitudinal elasticity of the string, then,

$$E=\frac{Mg/\alpha}{l/L}, \text{ where } \alpha \text{ is the cross-sectional area of the string.}$$

Then, $$E=\frac{MgL}{\alpha l};$$

or, $$Mg=\frac{E\alpha}{L}l=Kl \text{ (where } K \text{ is a constant for the string)};$$

or, $\frac{Mg}{l}=K$ is the restoring force per unit displacement which balances the corresponding external force in the steady state.

Fig. 121

Now if the body be given a slight displacement x in the downward direction and let go, it would oscillate in a $S.H.$ manner. The restoring force acting on the body $=Kx$. The force acting on the body due to its inertia of motion is $M\frac{d^2x}{dt^2}$. These two oppositely directed forces balance each other.

Hence, $$M\frac{d^2x}{dt^2}=-Kx;$$

or, $$\frac{d^2x}{dt^2}=-\frac{K}{M}x.$$

This is the equation of a $S.H.M.$

$\therefore$ The time period of oscillation is given by

$$T=2\pi\sqrt{\frac{M}{K}}=2\pi\sqrt{\frac{M}{Mg/l}}=2\pi\sqrt{\frac{l}{g}}$$

Thus knowing the time period of oscillation, g at a place can be determined.

(d) **By the inclined plane method.**—When a body of mass m rolls down an inclined plane its kinetic energy will be partly translational and partly rotational. In Fig. 122 P and Q are two positions of the rolling body on the inclined plane, at a vertical distance h apart, P being the position from where it starts. Supposing that there is no loss of energy due to friction, the loss in potential energy at Q is equal to the corresponding gain in kinetic energy.

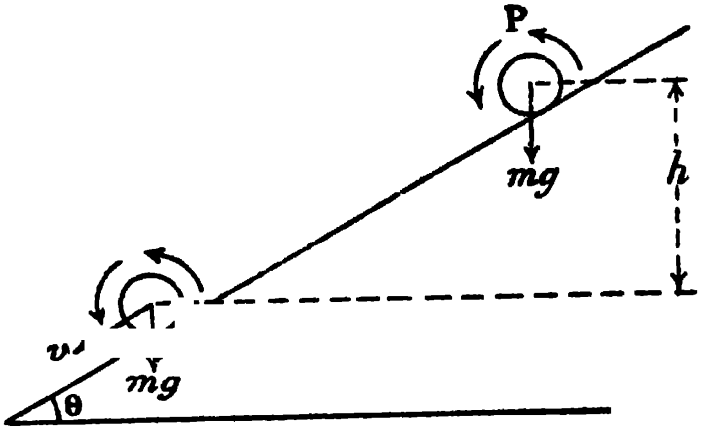


Fig. 122

$$\therefore \quad mgh = \tfrac{1}{2}mv^2 + \tfrac{1}{2}I\omega^2 \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

where I is the moment of inertia of the body about its axis, m its mass, v its linear velocity, ω its angular velocity about the axis of rotation and g and h are respectively the acceleration due to gravity and the vertical distance of fall.

If k be the radius of gyration of the body, $I = mk^2$.

$$\therefore \quad g = \frac{1}{2h}(v^2 + k^2\omega^2) \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

If the radius of the rolling body be r, then $\omega = \frac{v}{r}$. Again,

$$v^2 = 2f\frac{h}{\sin\theta}. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3)$$

($\because$ $v^2 = 2fs$ where f is the acceleration down the plane.)

$$\text{Hence,} \quad g = \frac{f}{v^2 \sin\theta}(v^2 + k^2\omega^2) = \frac{f}{\sin\theta}\left(1 + \frac{k^2}{r^2}\right) \quad \ldots \quad (4)$$

So f can be found out by actually observing time, t, of fall through the distance, s $(= PQ)$, with the help of the relation $s = \frac{1}{2}ft^2$; r is determined directly by measuring the diameter of the rolling body.

Here, $\frac{k^2}{r^2}$ is also known because $k^2 = \frac{2}{5}r^2$ (in case of a solid sphere) or, $\frac{r^2}{2}$ (for a solid cylinder or disc). θ is known from the inclination of the plane. Thus all the quantities on the right-hand side of equation (4) being known, g can be determined.

Example

How long will it take a naya-paisa starting from rest to roll a distance of 1 metre down on inclined plane which makes an angle 30° with the horizontal *($g = 981$ cm./sec.^{2})* *(Pat. U. 1958)*

Ans. : We know, $s = \frac{1}{2} ft.^2$; or, $t^2 = \frac{2s}{f}$.

But $f = g \sin\theta \Big/ \left(1 + \frac{K^2}{r^2}\right)$ from relation (4) above.

Here $I = \frac{1}{2} Mr^2$; or, $K^2 = \frac{1}{2} r^2$.

$$\therefore\ t^2 = \frac{2s\left(1 + \frac{K^2}{r^2}\right)}{g \sin\theta} = \frac{2 \times 100(1 + \frac{1}{2})}{981 \times \sin 30^\circ}$$

$$= \frac{600}{981}.$$

$$\therefore\ t = \sqrt{600/981} = 0{\cdot}7821 \text{ sec.}$$

(e) Other important methods of determining 'g' are by means of different types of pendulums—simple, conical and compound. These are dealt within the following few articles.

PENDULUM

120. Historical Note :—Galileo appears to have been the first to make use of the pendulum. One day in the Cathedral at Pisa, in 1583, while watching a swinging lamp he noticed that even when the oscillations of the lamp gradually died away, time taken by it to make one oscillation still remained the same. He timed the oscillations by beats of his pulse. This discovery, he pointed out, could be utilised to regulate clocks. In 1658 Huygens actually used the pendulum to regulate the motion of clocks.

121. Types of Pendulum :—

(a) **The Simple Pendulum.**—A simple pendulum is defined as a heavy particle suspended by a weightless, inextensible but perfectly flexible thread, from a rigid support about which it oscillates without friction. In practice, however, a small metal bob suspended from a fixed support by a very fine long thread is taken to be a simple pendulum.

(b) **The Conical Pendulum.**—Here the bob of the pendulum is made to describe a horizontal circle instead of vibrating in a vertical plane. The time period, $T = 2\pi / \sqrt{g/h}$. Hence g is found. [*vide* Art. 17 (*vi*) on Conical Governor.]

(c) **The Compound Pendulum.**—Any body capable of oscillating freely about a horizontal axis is known as a compound pendulum. The metallic rod carrying at its lower end a heavy lens-shaped mass of metal, known as the bob, acting as oscillator in a clock is an example of a compound pendulum.

(d) **The Seconds Pendulum.**—It is a simple pendulum which takes one second in making *half* a complete oscillation (*i.e.* one vibration or swing). So it has a *period* of two seconds. When it is said that a pendulum *beats* one second, it means that it takes one second to make one swing.

122. Some Terms :—

Length of a Simple Pendulum.—It is the distance L from the point of suspension up to the centre of gravity of the bob, *i.e.* the distance between A and B [Fig. 123(*a*)]. That is, it is the length of the suspension thread plus the vertical radius r of the bob. It is also called the *effective length* of the pendulum.

Amplitude.—The maximum angular displacement α [Fig. 123(*b*)] of the bob, measured, on either side, from its undisturbed position (given by the vertical position E) up to the extreme position as shown at C or D, is called its amplitude. It should not exceed 4° for the motion to be *simple harmonic.* The amplitude of a simple pendulum gradually decreases as the bob swings on account of air-resistance mainly.

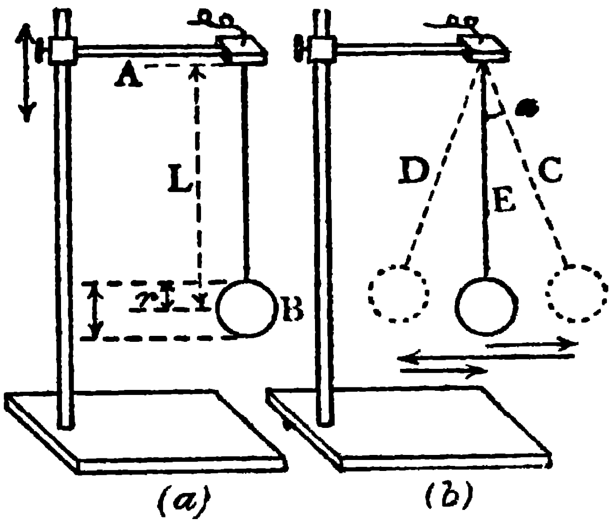


Fig. 123

Time period (or simply, **Period**)—It is the time taken by a pendulum to make ***one complete oscillation***. One complete oscillation comprises two swings—one forward, another backward. An oscillation is usually reckoned from the extreme position D [Fig. 123(*b*)] to the other extreme position C and back to D next time ; or, from the undisturbed position E (pendulum vertical) when say, it is moving to the right, until when it passes through the undisturbed position E again moving in the same direction as shown by the arrows.

One ***vibration*** means the motion from one extreme position, say, D to the other extreme position C, *i.e.* it is half of an oscillation.

Frequency.—It is the number of complete oscillations made by a pendulum per second at a place. Thus, if n = frequency, and T = time period, $nT = 1$, or, $n = 1/T$.

Phase.—The phase of a pendulum gives its stage of ***displacement*** and *motion* at any particular instant, *i.e.* it determines the position of the pendulum in the path of motion and also the direction of motion at that instant.

123. Motion of a Simple Pendulum is Simple Harmonic :—Let the bob of mass m of a pendulum of length l (Fig. 124) be displaced through an angle θ from its undisturbed position B to the position C. If g be the acceleration due to gravity at the place, the weight mg of the bob can be resolved into the two components $mg \cos \theta$ acting along CF, the direction of the string which is kept taut thereby, and $mg \sin \theta$ acting at C along CE at right angles to CF. The former is balanced by the tension of the string, while the latter tends to bring the bob back to its original position B with an acceleration $g \sin \theta$. If θ does not exceed 4°, $\sin \theta$ may be taken to be equal to θ and so the acceleration of the bob, $g \sin \theta = g\theta$ in the direction in which θ decreases. After crossing the mean position B, when the bob moves towards BD by virtue of its inertia and acquired velocity, the acceleration acts in the opposite direction, *i.e.* towards B and so the motion decreases and vanishes at the other extreme position D, when the direction of motion is reversed. ***This explains why a pendulum should oscillate at all.*** The acceleration, it is to be noted, is always directed towards the mean position B.

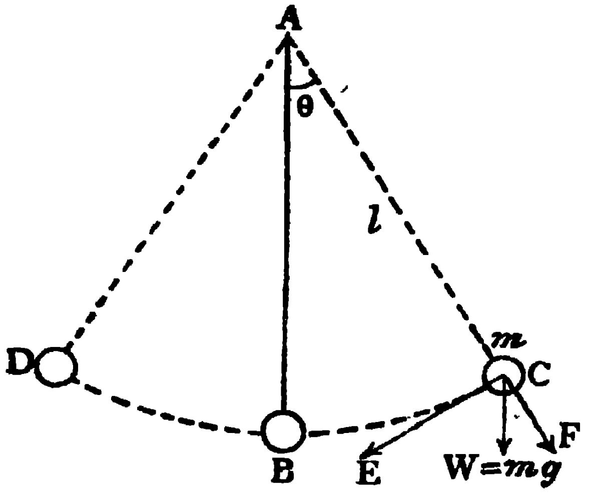


Fig. 124

$$\text{Again, } \theta = \frac{\text{arc } BC}{\text{length } AB} = \frac{\text{displacement}}{\text{length of pendulum } (l)}$$

$$\text{Acceleration} = g.\theta = \frac{g}{l} \times \text{displacement.} \qquad (1)$$

That is, the acceleration is proportional to the displacement, because g and l are constants for the pendulum at a given place.

Thus, acceleration being proportional to displacement and always directed to a fixed position B in the path of motion, the motion is simple harmonic, according to the definition of simple harmonic motion.

Though a pendulum continues to oscillate for a long time, it gradually stops due to the resistance of air and the friction at the point of suspension ; otherwise, a pendulum would have oscillated for ever, had there been no such resistances to stop it.

Mathematically, the motion of a pendulum, which is simple harmonic, is given by

$$\frac{\text{Acceleration}}{\text{Displacement}}=\omega^2, \text{ (where } \omega=\text{angular velocity)}$$

$$=\left(\frac{2\pi}{T}\right)^2, \text{ where } T=\text{time period.}$$

That is, $T^2=4\pi^2\times\frac{\text{displacement}}{\text{acceleration}}=4\pi^2\times\frac{l}{g}$, from (1) above.

$$\therefore\ T=2\pi\sqrt{\frac{l}{g}}.$$

[**Alternative method :**

From above, we know that the bob of a simple pendulum has an acceleration,

$$f=g\sin\theta=g\theta \quad \ldots \quad \ldots \quad \ldots \quad (i)$$

It acts tangentially to the arc described by the bob in a direction in which θ decreases.

If the effective length of the pendulum be 'l' the tangential velocity at any instant is $v=l\,\omega$ where ω is the angular velocity $\left(i.e.\ \frac{d\theta}{dt}\right)$.

Hence the tangential acceleration acting in the direction in which θ increases is $\frac{dv}{dt}=l\frac{d\omega}{dt}=l\frac{d^2\theta}{dt^2}$ ... ... (ii)

Hence from (i) and (ii),

$$l\frac{d^2\theta}{dt^2}=-g\theta\ ; \quad \text{or,} \quad \frac{d^2\theta}{dt^2}=-\frac{g}{l}\theta \quad \ldots \quad \ldots \quad (iii)$$

This is evidently the equation of a $S.H.M.$ as has been shown on the Chapter VI.

Hence the time period of oscillation will be given by $T=2\pi\sqrt{\frac{l}{g}}$.]

If the arc of swing be finite, a correction should be introduced for the time period. This has been dealt with in the article on compound pendulum.

124. The Laws of Pendulum :—The laws of oscillation of a simple pendulum are given by the following relation,—

$$T=2\pi\sqrt{\frac{l}{g}}.$$

where T = period of the pendulum ; l = effective length ; g = acceleration due to gravity at the place of oscillation.

Law 1. *The oscillations of a pendulum are isochronous.*—That is, a pendulum takes equal time to complete each oscillation whatever is the amplitude, provided the latter is small (within 4°). So time period is independent of the amplitude of vibration. *This is also known as* ***the law of isochronism.***

Law 2. *The period of oscillations of a pendulum varies directly as the square root of the length.* Mathematically $T \propto \sqrt{l}$, or, $l/T^2 =$ a constant for the place of observation. Thus if the length be increased four times, the period becomes double. *This is known as the law of length.*

[**Note.**—The length of the pendulum changes with temperature, so the period t of pendulum changes with temperature].

Law 3. *The period of oscillation varies inversely as the square root of the acceleration due to gravity at the place of observation. This is known as the law of acceleration.* Mathematically, $T \propto 1/\sqrt{g}$, or, $T^2 \times g =$ a constant, for the same pendulum

Thus if g be greater at the place, T will be less, *i.e.* the pendulum will vibrate more rapidly.

Law 4. *The period does not depend on the mass or material of the bob of the pendulum, provided the length remains constant. This is known as the law of mass.*

Verification of the Laws of Pendulum.—

Law 1. (*Law of Isochronism*).—To verify the first law, note with a stop-watch the total time of, say, 20 oscillations with different amplitudes, keeping the length constant. It will be found that the period T in each case remains constant.

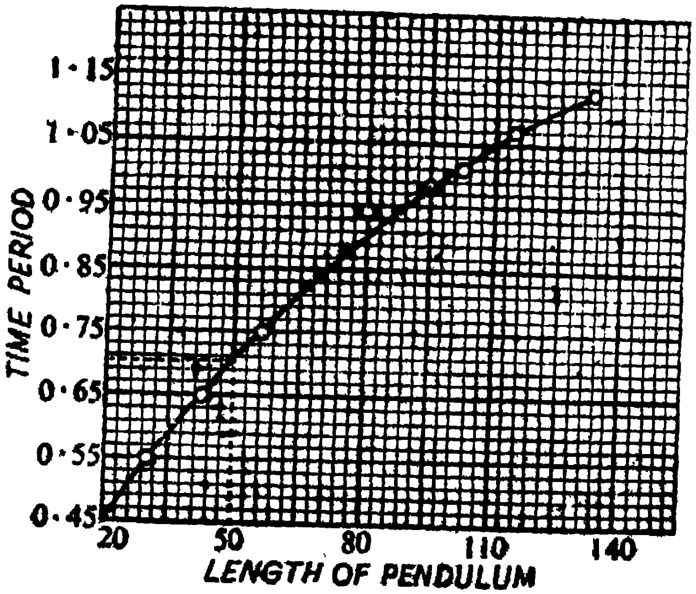


Fig. 125

It should be noted that the law is true only for small angles of amplitude (about 4°) ; so when noting the time of oscillation with different amplitudes, care should be taken not to exceed the maximum limit of 4°.

Law 2. (*The law of length*).—Find the *vertical* radius of the bob by means of a slide callipers, and hence determine the length from the point of suspension up to the centre of gravity of the bob. Observe the time taken for 20 complete oscillations and thus find T, the period.

Change the length of the pendulum and again find the period. In this way get several values of the period for the corresponding length. It will be found that $T \propto \sqrt{l}$, *i.e.* the value l/T^2 will be a constant.

Law 3. (*The law of Acceleration*).—This law can be verified by taking a pendulum to different places having different values of g. It will be seen that at a place where g is greater, the vibrations will be quicker. $T^2 \times g$ will, however, be found constant at the different places for the same length of the pendulum.

Law 4. (*The Law of Mass*).—Keeping the effective length of the pendulum the same in every case, if the bob be replaced by another of different size and of a different material, it will be found that the period T will remain unaltered.

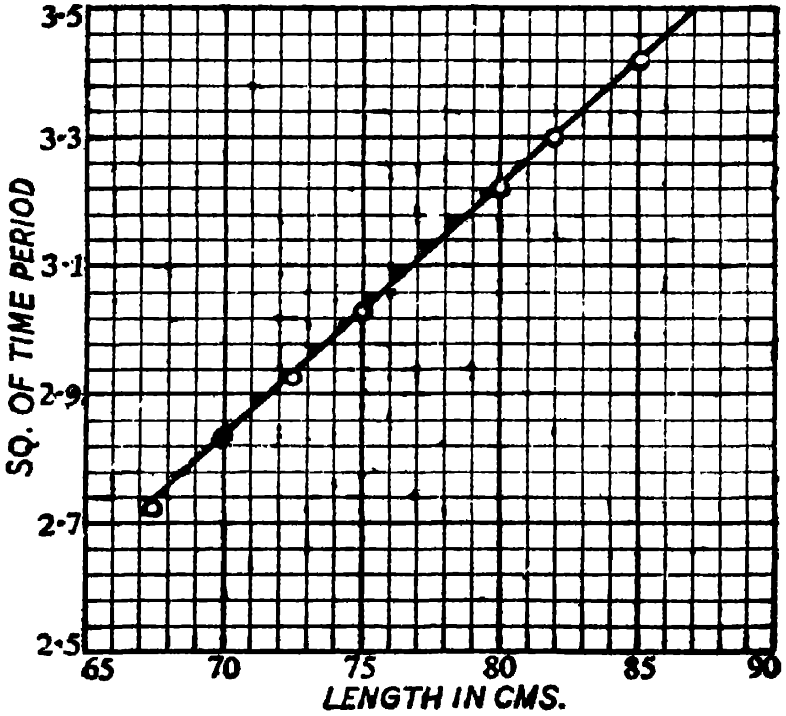


Fig. 126

By performing this experiment with bobs of different substances (such as wood, iron, brass, etc.), it can be shown that *the acceleration due to gravity at the same place is the same for all bodies.*

Graph.—Draw a graph with the length l (along the X-axis) and time period T (along the Y-axis). The relation between l and T will be an arm of a parabola (Fig. 125). The graph in Fig. 126, which is a ***straight line***, represents the relation between l and T^2. From any of these graphs, the length of the pendulum corresponding to a given time of oscillation can be determined but it is better to take the help of l and t^2 graph (straight line) for this purpose.

125. The length of a Seconds Pendulum :—The period of a seconds pendulum is 2 seconds. Hence from the formula for the period of oscillation, we have,

$$1=\pi\sqrt{\frac{l}{g}}\ ;\ \text{or, } l=\frac{g}{\pi^2} \quad \ldots \quad \ldots \quad (1)$$

So the length of the seconds pendulum changes at different places depending on the value of g.

Taking the value of g to be 981 cms. per sec. per sec., the length of the seconds pendulum becomes [from eq. (1) above],

$$l=\frac{981}{\pi^2}=\frac{981}{9·87}=99·39 \text{ cms.}$$

Taking the value of g to be 32·2 ft. per sec. per sec.,

$$l=\frac{32·2}{n^2}=\frac{32·2}{9·87}=3·26 \text{ ft.}=39·19 \text{ inches.}$$

Graph.—To determine the length of the seconds pendulum from the graph, draw the l and T^2 graph (Fig. 126) and find the length corresponding to $T^2=4$.

126. The value of 'g' by a Pendulum :—By carefully measuring the length and the corresponding period of a simple pendulum, the value of g at any place can be determined from the formula,

$$T=2\pi\sqrt{\frac{l}{g}}\ ;\ \text{whence } g=\frac{4\pi^2 l}{T^2}=4\pi^2\times\frac{l}{T^2}.$$

Thus, when the value of l/T^2 at a place is (say) 24·84, g is given by $g=4\pi^2\times l/T^2=4\times 9·87\times 24·84=980·68$ cms./sec.^{2}.

127. Loss or Gain of Time by a Clock :—The loss or gain of time depends on (*a*) *the latitude of the place* as the value of g varies with the latitude of a place. It (g) is minimum at the equator and increases gradually towards a pole. But as the time period T of a simple pendulum varies inversely as the square root of g, the period T of a pendulum will decrease as it is taken from the equator to a pole. So, pendulum clock will gradually gain time, *i.e.*, will go *fast*, when taken from the equator to a pole.

(*b*) The loss or gain of time also depends on ***the height of a place above the sea-level.*** As the value of g diminishes with the distance above and also below the surface of the earth, the time period T of a pendulum clock will increase, and so the clock will lose time, *i.e.* will go *slower* when taken to the top of a mountain or to the bottom of a mine.

(*c*) Again if the length (l) of the pendulum varies, T will directly vary as the square root of this length. When l increases, T also increases and the clock runs slow. When l decreases T also decreases and the clock runs fast. This length variation in a pendulum is gradually caused by temperature variations. The extent of these losses or gains due to variations in g and l may be found out as follows :—

We know, $T=2\pi\sqrt{l/g}$.

$\therefore$ $\log T=\log 2\pi+\frac{1}{2}\log l-\frac{1}{2}\log g$.

Differentiating, we have, $\frac{dT}{T}=\frac{dl}{2l}-\frac{dg}{2g}$ ... ... (1)

Again if N be the number of oscillations executed by a pendulum per day, then $NT=86400$ sec. $=$ a constant.

$\therefore$ Differentiating, $TdN+NdT=0$;

$$\text{or, } \frac{dN}{N}+\frac{dT}{T}=0 \qquad \ldots \quad \ldots \quad (2)$$

Thus from (1) and (2) we get,

$$\frac{dN}{N}=\frac{dg}{2g}-\frac{dl}{2l}.$$

Here dN is the number of oscillations gained by a pendulum in a day. In a seconds pendulum $N=\frac{1}{2}\times 86400=43200$. Since each oscillation in a seconds pendulum requires two seconds, the number of seconds gained by a seconds pendulum $2dN=\Delta n$ (say).

$$\therefore \quad \Delta n=2dN=N\left(\frac{dg}{g}-\frac{dl}{l}\right)$$

$$=43200\left(\frac{dg}{g}-\frac{dl}{l}\right) \qquad \ldots \quad \ldots \quad (3)$$

When g is constant, $dg=0$ and $\Delta n=-43200\frac{dl}{l}$,

$-ve$ sign indicating the fact that as l increases n diminishes.

When l is constant, $dl=0$ and $\Delta n=43200\frac{dg}{g}$.

When both these factors g and l vary Δn is given by the expression (3).

128. Measurement of the height of a Hill :—

(i) by a Pendulum Experiment.—

Suppose g and g' are the values of the acceleration due to gravity respectively at the bottom and at the top of the hill as measured by a pendulum experiment. Then, $g=\frac{G.M}{R^2}$, with usual notations where R= radius of the earth. Acceleration g' at the top of a hill of height h (Fig. 127) will be given by $g'=\frac{G.M}{(R+h)^2}$. From the above equations,

$$\frac{g}{g'}=\frac{(R+h)^2}{R^2};\ \text{or,}\ \frac{R+h}{R}=\sqrt{\frac{g}{g'}} \qquad (1)$$

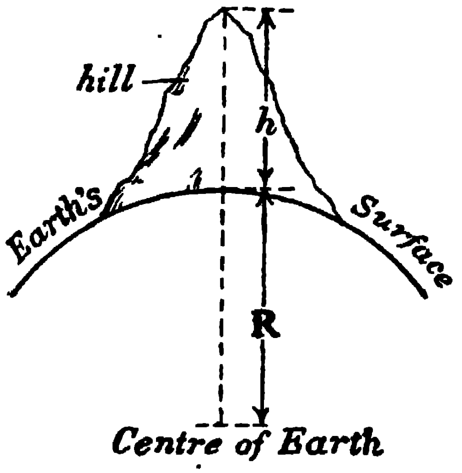


Fig. 127

Thus, if R is given h will be known when g and g' are experimentally determined.

(ii) by a Clock.—

Suppose, a clock gives correct time at the foot of a hill and loses n secs. a day at the top of it.

Then, at the bottom $1=\pi\sqrt{\frac{l}{g}}$ ... .. (2)

where l= length of its pendulum. At the top where it makes $(86400-n)$ swings in 86400 secs.,

$$\frac{86400}{86400-n}=\pi\sqrt{\frac{l}{g'}} \qquad \text{... ...} \quad (3)$$

From (2) and (3), $\sqrt{\frac{g}{g'}}=\frac{86400}{86400-n}$ ... ... (4)

From (1) and (4), $\frac{R+h}{R}=\frac{86400}{86400-n}$ ... ... (5)

Thus, h will be found if R is given and n determined.

Examples

(1) *Two pendulums of length 1 metre and 1·1 metres respectively start swinging together with the same amplitude. Find the number of swings that will be executed by the longer pendulum before they will again swing together ($g=978$ cms. per sec.^{2}).*

Ans. : Let t_1 and t_2 be the periods of oscillation of the pendulum of length 1 metre and 1·1 metres respectively ; 1 metre = 100 cms. and 1·1 metres = 110 cms.

Then we have, $t_1=2\pi\sqrt{\frac{100}{978}}$ and $t_2=2\pi\sqrt{\frac{110}{978}}$.

Suppose the pendulum of 1·1 metres length makes n_1 swings, and the other makes (n_1+n_2) swings before they again swing together;

then, $n_1 t_2 = (n_1 + n_2)\, t_1$; or, $n_1\ (t_2 - t_1) = n_2 t_1$ ... (1)

But, $(t_2 - t_1) = \frac{2\pi}{\sqrt{978}}(\sqrt{110} - \sqrt{100})$.

∴ From (1), $\frac{2\pi n_1}{\sqrt{978}}(\sqrt{110} - \sqrt{100}) = \frac{2\pi n_2\sqrt{100}}{\sqrt{978}}$;

or, $n_1 = \frac{10}{\sqrt{110} - 10}\, n_2 = \frac{10\ \sqrt{110} + 10}{10}\, n_2$

$= (\sqrt{110} + 10)\, n_2 = 20·5 n_2$ (nearly) $= \frac{41}{2}\, n_2$ (nearly).

To get a whole number, the least value for n_2 is 2 and therefore $n_1 = 41$ (nearly).

(2) *A pendulum which beats seconds at a place where g=32·2 is taken to a place g=32·197. How many seconds does it lose or gain in a day?* (*cf. Utk. U. 1970S*).

Ans.: Let t_1 be the original period, and t_2 the new period of the pendulum. In this case t_1 is equal to 2 secs. but this fact is not required.

We have, $t_1 = 2\pi\sqrt{\frac{l}{32·2}}$; $t_2 = 2\pi\sqrt{\frac{l}{32·197}}$.

Hence, $\frac{t_1}{t_2} = \sqrt{\frac{32·197}{32·2}} = \sqrt{\frac{32·2 - 0·003}{32·2}} = \sqrt{\left(1 - \frac{0·003}{32·2}\right)}$.

Because period $\propto \frac{1}{\sqrt{g}}$, we have $t_2 > t_1$, and so the pendulum will lose time.

Let n = no. of secs. lost per day. The number of secs. in a day is 24 × 60 × 60, or, 86400. ∴ $(86400 - n)\, t_2 = 86400 \times t_1$;

or, $(86400 - n) = 86400 \times \frac{t_1}{t_2} = 86400 \times \sqrt{\left(1 - \frac{0·003}{32·2}\right)} = 86400\ \left(1 - \frac{0·003}{32·2}\right)^{\frac{1}{2}}$

$= 86400\ \left(1 - \frac{1}{2} \times \frac{0·003}{32·2}\right)$ approx. $= 86400 - 4$. ∴ $n = 4$ secs.

Hence the pendulum loses 4 secs. per day.

[Alternative method :

Here $dg = g - g' = 32·2 - 32·197 = 0·003$ sec.

∴ $\Delta n = 43200 \times \frac{·003}{32·2} = 432 \times \frac{3}{322} = 4$ secs. (approx.)

Here as g decreases time period increases. Hence the pendulum loses 4 secs. per day.]

(3) *A pendulum which beats seconds at the equator gains five minutes per day at the poles. Compare the values of g at the two places.*

Ans.: Let g_1 and t_1 denote the value of g and *period* respectively at the equator, and g_2 and t_2 those at the poles.

Because the pendulum beats seconds at the equator, $t_1 = 2$ seconds.

We have, $t_1^2 = 4\pi^2 \frac{l}{g_1}$; or, $4 = 4\pi^2 \frac{l}{g_1}$; or, $g_1 = \pi^2 l$ ... ... (1)

Now, at the poles, the pendulum gains 5 minutes per day that is (5×60) seconds in $(24\times60\times60)$ secs. $\therefore$ It gains $\frac{5\times60}{24\times60\times60}$ sec. per sec. *i.e.*, it gains $\frac{1}{288}$ sec. in one vibration ; or, $\frac{2}{289}$ sec. in one complete oscillation. Because it gains $\frac{2}{288}$ sec. in one oscillation, its period, $t_2=\left(2-\frac{2}{288}\right)=\frac{574}{288}$ secs.

$$\therefore\quad\left(\frac{574}{288}\right)^2=4\pi^2\,\frac{l}{g_2},\ \text{or,}\ g_2=4\pi^2 l\times\frac{288^2}{574^2}\qquad\ldots\qquad\ldots\qquad(2)$$

$$\text{From (1) and (2), }\frac{g_1}{g_2}=\frac{\pi^2 l}{4\pi^2 l\times\frac{288^2}{574^2}}=\frac{1}{4\times\frac{288^2}{4\times287^2}}=\frac{287^2}{288^2}=\frac{143}{144}\text{ approx.}$$

[**Alternative method :**

Here $\Delta n=5\times60=300$ secs., and $dg=g_2-g_1$.

$$\therefore\quad 300=43200\times\left(\frac{g_2-g_1}{g_2}\right);\quad\text{or,}\quad 3=432\left(1-\frac{g_1}{g_2}\right);$$

$$\text{or,}\quad 432\,\frac{g_1}{g_2}=429;\qquad\text{or,}\quad\frac{g_1}{g_2}=\frac{429}{432}=\frac{143}{144}.\ \Big].$$

(4) *A pendulum of length l loses 5 secs. in a day. By how much must it be shortened to keep correct time ?*

Ans. : There are 86400 seconds in a day. As the pendulum loses 5 secs. a day, it beats $(86400-5)$, or. 86395 times in one day, *i.e.* in 86400 seconds.

$\therefore$ Time of one vibration (time of one swing), $t=\frac{86400}{86395}$ (and, not 1 sec.)

But the time of one swing, *i.e.* half oscillation is $\pi\sqrt{l/g}$.

$$\therefore\quad\text{We have }\pi\sqrt{\frac{l}{g}}=\frac{86400}{86395}.\quad\therefore\quad\pi^2\,\frac{l}{g}=\left(\frac{86400}{86395}\right)^2\qquad\ldots\qquad\ldots\qquad(1)$$

In order to keep correct time, let the length of the pendulum be shortened by x. In this case, it becomes a true seconds pendulum and its time of one vibration becomes 1 second.

$$\text{Then we have, }\pi\sqrt{\frac{l-x}{g}}=1.\qquad\therefore\quad\pi^2\,\frac{l-x}{g}=1\qquad\ldots\quad(2)$$

$$\text{From (1) and (2), }\pi^2\,\frac{x}{g}=\left(\frac{86400}{86395}\right)^2-1=\left(1+\frac{5}{86395}\right)^2-1$$

$$=\left(1+\frac{2\times5}{86395}+\text{etc}\ldots\right)-1,\ \text{from Binomial theorem}$$

$$=\frac{10}{86395},\ \text{(neglecting other terms)}$$

exact.
cms. p
Ans.
1 metre.

$$\therefore\quad x=\frac{g}{\pi^2}\times\frac{10}{86395},$$

Th

If g is taken as 981 cms./sec.^{2}, we have, $x=\dfrac{981\times 10}{9{\cdot}87\times 86395}=0{\cdot}0115$ cm.

[**Alternative method :**

Here since finally the pendulum keeps correct time. its $T=2=2\pi\sqrt{l/981}$; or $l=99{\cdot}39$ cms.

Here, $\Delta n=5$ secs.

If dl be the required shortening of length, then,

$$\Delta n=43200\,\frac{dl}{l}\,;\quad\text{or,}\quad 5=43200\,\frac{dl}{99{\cdot}39}.$$

$$\therefore\quad dl=496{\cdot}95/43200=0{\cdot}0115\text{ cm.}]$$

129. Compound Pendulum :—If a rigid body, whatever its shape or size, be supported so as to rotate freely about a horizontal axis and such that its centre of gravity does not lie on the axis of rotation, the body will oscillate to and fro, if displaced from its position of stable equilibrium and let go. Such a body is known as a *compound pendulum*.

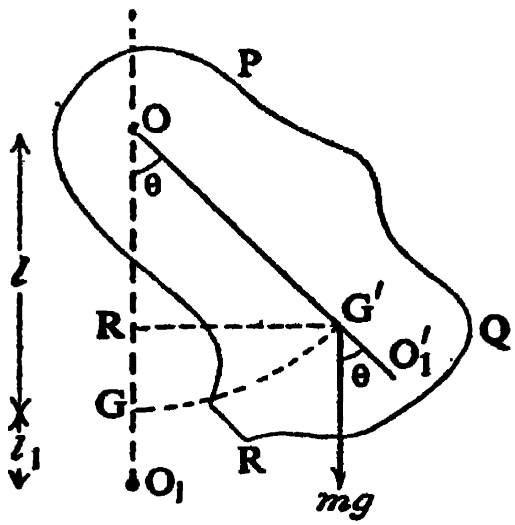


Fig. 128

Suppose the body PQR (Fig. 128) represents a compound pendulum oscillating in a vertical plane (say, the plane of the paper) round a horizontal axis (prep. to the plane of the paper) passing through the *point of suspension O*. The rest position of PQR is evidently with its centre of gravity, G, vertically below O. In course of its oscillation, let the body be displaced through an angle θ, the C.G. being displaced to the position G'. At this displaced position, the acceleration acting on the body in the direction of θ increasing is $\dfrac{d^2\theta}{dt^2}$. Hence the deflecting couple is expressed as $I\,\dfrac{d^2\theta}{dt^2}$, where I=moment of inertia of the body about the axis of rotation. Again in the displaced position, the weight mg of the body acts vertically downwards through G. This gives rise to a couple $mg\times RG'=mgl\sin\theta$ tending to restore the body to its original position, where l is the distance between O and G. This couple acts in the direction of θ decreasing.

Hence equating these two oppositely directed couples, we get,

$$I\frac{d^2\theta}{dt^2} = -mgl\,\theta \text{ (where } \theta \text{ is small)} \quad \ldots \quad (1)$$

$$\text{Or, } \frac{d^2\theta}{dt^2} = -\frac{mgl}{I}\theta \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

This is evidently the equation of a *S.H.M.* whose angular velocity ω is given by,

$$= \frac{mgl}{I}\ ; \text{ or, } \frac{2\pi}{T} = \sqrt{\frac{mgl}{I}}\ ;$$

$$\text{or, } T = 2\pi\sqrt{\frac{I}{mgl}}.$$

But here as the axis of rotation is at a distance l from $C.G.$, the value of I is given by $m\,(k^2 + l^2)$ where k is the radius of gyration.

$$\therefore \quad T = 2\pi\sqrt{\frac{k^2 + l^2}{gl}} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3)$$

Thus the time period of vibration of a compound pendulum is given by relation (3). This period is the same as that for an ideal simple pendulum whose length is $(k^2+l^2)/l$. Such a simple pendulum whose time period is the same as that of a compound pendulum is called the ***equivalent simple pendulum.***

Since k^2 is always greater than zero, the length of the equivalent pendulum is necessarily greater than l. Let us take a point O_1 such that OO_1 = the length of the equivalent pendulum $= \frac{k^2+l^2}{l}$. Then O_1 is the *centre of oscillation.* If $GO_1 = l_1$, $OO_1 = \frac{k^2+l^2}{l} = l + l_1$ and $T = 2\pi\sqrt{\frac{l+l_1}{g}}$.

If again the pendulum be suspended from O_1, the time period, $T_1 = 2\pi\sqrt{\frac{k^2+l_1^2}{l_1 g}}$.

But $\frac{k^2+l^2}{l} = l + l_1$; or, $k^2 = ll_1$.

Hence, $T_1 = 2\pi\sqrt{\frac{ll_1+l_1^2}{l_1 g}} = 2\pi\sqrt{\frac{l_1+l}{g}}$.

Hence, T (with O as the centre of suspension) is identical with T_1 (with O_1 as the centre of suspension). Hence *the centres of oscillation and suspension are interchangeable.*

Really speaking there are four points on the vertical line through $C.G.$ about which the body will oscillate with equal time periods. This is evident from the fact that the equivalent length of the pendulum $\frac{l^2+k^2}{l}$ may be supposed to be equal to the sum of two lengths l and $\frac{k^2}{l}$, any one of which may be supposed to be the distance of the $C.G.$ from the point of suspension or from the point of oscillation.

So if we draw two circles with G as centre and the above two lengths as the radii, they will intersect the line through the $C.G.$ at A and O above G and at A_1 and O_1 below G (Fig. 129).

Fig. 129

Then $GA_1=GO=l$ and $GO_1=GA=\frac{k^2}{l}$.

Hence $GA_1+GA=AA_1=\frac{l^2+k^2}{l}=GO_1+GO=OO_1$.

Thus we have four points O, A, O_1, A_1, collinear with G about which the time period is the same. It is therefore necessary to choose two points such as O, O_1 or A, A_1 in finding the length of the equivalent pendulum and not to take a symmetrical pair of points like A, O_1 or O, A_1.

Maximum and Minimum values of the time periods :

The time period of a compound pendulum is $T=2\pi\sqrt{\frac{l^2+k^2}{lg}}$

or, squaring, $T^2=\frac{4\pi^2}{g}\left(l+\frac{k^2}{l}\right)$.

Differentiating this expression with respect to l,

we get, $2T\frac{dT}{dl}=\frac{4\pi^2}{g}\;1\;\;\frac{k^2}{l^2}\Big)$.

This relation gives the variation of T with length.

If $l=0$, *i.e.* if the point of suspension coincides with the $C.G.$ the time period will be a maximum (equal to infinity) and if $k^2=l^2$, the above expression reduces to zero and the period is a minimum.

This minimum value of T is equal to $2\pi\sqrt{\frac{2l}{g}}$.

An alternative method of deducing the equation of motion of a Compound Pendulum :—

Let O be the centre of suspension of the body and OG, the vertical line passing through the point of suspension and the $C.G.$ (G) in the mean position of rest (Fig. 130). If the body be now displaced through an angle α such that G is shifted to G, and then let go, the oscillation will evidently set in. α will evidently represent the maximum angular displacement (*i.e.* amplitude).

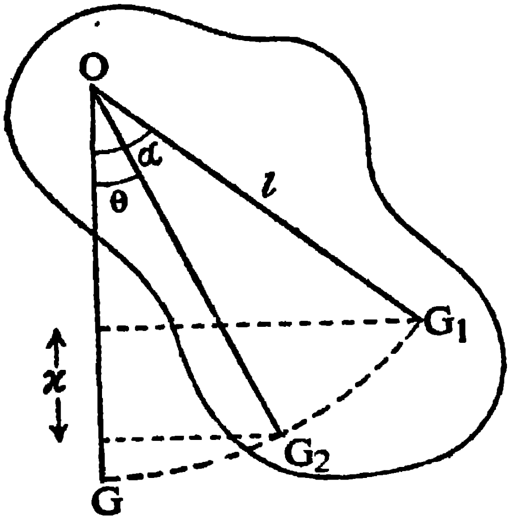


Fig. 130

Starting from the position OG_1 where it is temporarily at rest the body would gradually try to restore its equilibrium position OG. Suppose, in t secs., OG_1 reaches OG_2 making an angle θ with the vertical. Then the loss in potential energy of the body will be mgx (where x is the corresponding vertical shift) $= mg\,(l\cos\theta - l\cos\alpha)$;
$= mgl\,(\cos\theta - \cos\alpha)$.
Since the body was at rest at OG_1, the $K.E.$ at that position was zero. Hence when it comes to OG_2, it acquires a $K.E. = \frac{1}{2}I\omega^2$ at the expense of the above potential energy, where I is the moment of inertia of the body about O, i e. $m(k^2+l^2)$, m its mass and $\omega\left(=\frac{d\theta}{dt}\right)$ its angular velocity.

$\therefore\ \frac{1}{2}I\omega^2 = mgl\,(\cos\theta - \cos\alpha)$.

Differentiating with respect to t,

$$I\omega.\frac{d\omega}{dt} = -mgl\sin\theta\frac{d\theta}{dt},\ (\because \alpha \text{ is a const.})\ ;$$

or, $$I.\frac{d\theta}{dt}.\frac{d^2\theta}{dt^2} = -mgl\,\theta.\frac{d\theta}{dt},\ (\theta \text{ being small})\ ;$$

or, $$\frac{d^2\theta}{dt^2} = -\frac{mgl\theta}{m(k^2+l^2)}$$

or, $$\frac{d^2\theta}{dt^2} = -\frac{gl}{(k^2+l^2)}\theta;$$

Hence $$T = 2\pi\sqrt{\frac{k^2+l^2}{lg}} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

130. Bar Pendulum :—It is a very simple form of compound pendulum used for determining g. A metal bar RR [Fig. 131] having two sets of holes drilled symmetrically with respect to its centre G forms the compound pendulum. Here G coincides with the $C.G.$ of the uniform bar. The horizontal knife-edge K is now placed in any one of the holes. Now recording the time periods about the upper edge of each of the holes on one side of G and measuring the corresponding distances from G, curve may be drawn connecting T and this distance as shown by the right-hand half of Fig. 132. The left-hand half corresponding to the holes on the other side of G may be drawn from symmetry.

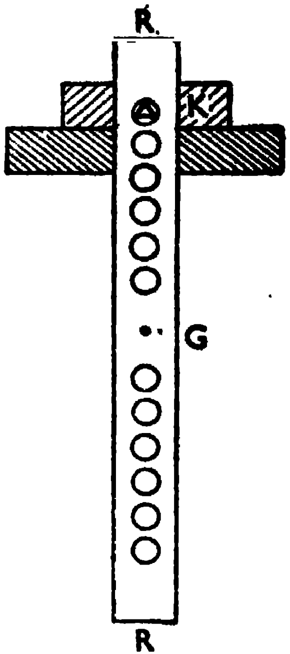


Fig. 131

As is evident from the figure, the minimum time period refers to a length LL' $(=2l)$ of the equivalent pendulum. When this length approaches zero (*i.e.* when the body is suspended from $C.G.$) the period tends to infinity. Again taking any horizontal line to cut both the curves at OA $O'A'$, we get the four possible points of suspension for the same T. Here the equivalent simple pendulum has a length OO' and AA'.

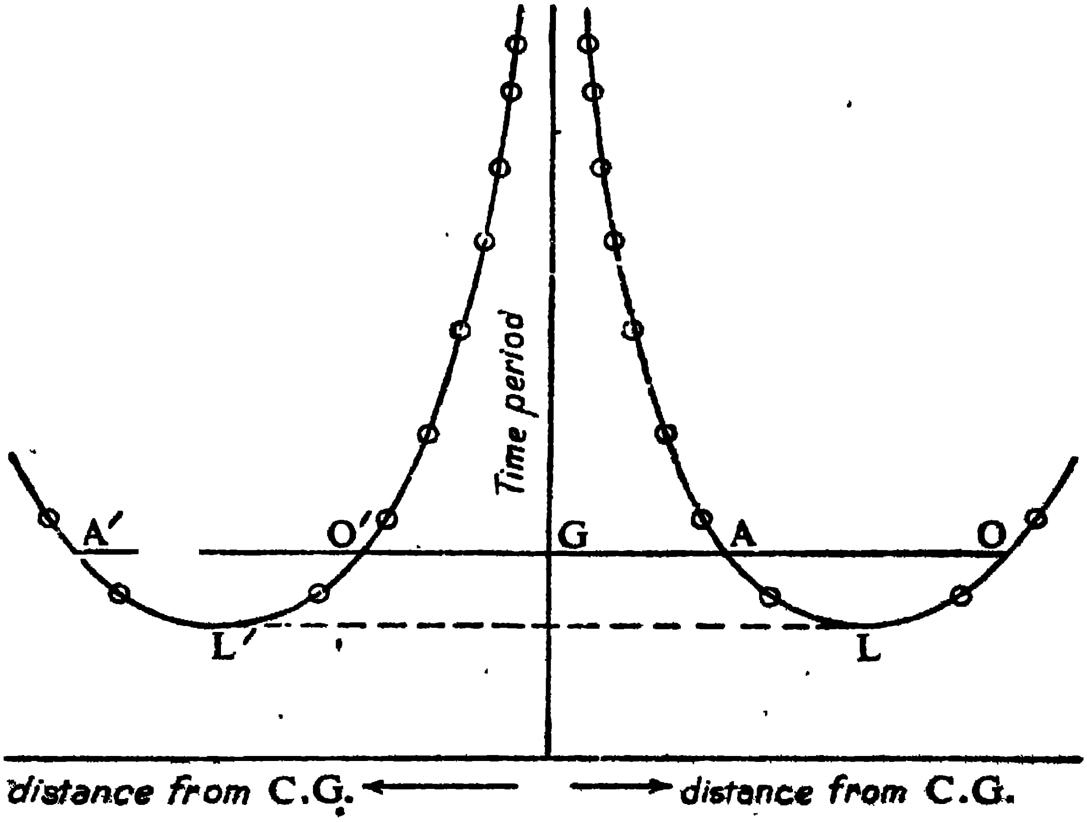


Fig. 132

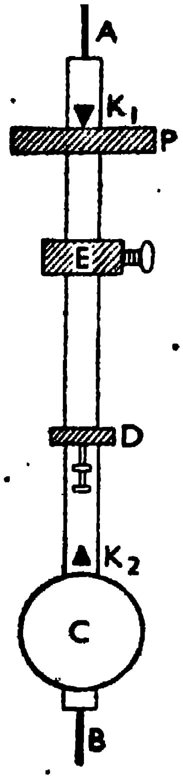


Fig. 133

181. Kater's Pendulum :—The value of the acceleration due to gravity on earth's surface is accurately determined by Kater's revisible pendulum. The pendulum (Fig. 133) consists of a rod carrying three weights, the bob C and two adjustable weights D and E. It can be supported at any of the two knife-edges K_1 and K_2 on a rigid horizontal platform P. K_1 and K_2 are turned towards the centre of gravity of the suspended system. The larger weight E is moved until the time periods about K_1 and K_2 are nearly equal and is then fixed in position. The smaller weight D is required for finer adjustments. It is moved by means of a screw until the number of swings made in 24 hours about K_1 and K_2 successively are exactly equal or at best differ by a small fraction of one vibration. K_1 and K_2 are then the reciprocal or interchangeable points within the limits of experimental error. Their distance apart is very carefully measured. This distance l between K_1 and K_2, which will be found to be asymmetric points, is the length of the equivalent simple pendulum. at the place of observation. The acceleration due to gravity will then be given by, $g = 4\pi^2 \frac{l}{T^2}$ where T is the time period of the adjusted pendulum which can be measured by a stop watch as usual. Kater, in 1917, determined g at London by this method. For finding out T he had adopted the coincidence method taking a standard seconds pendulum.

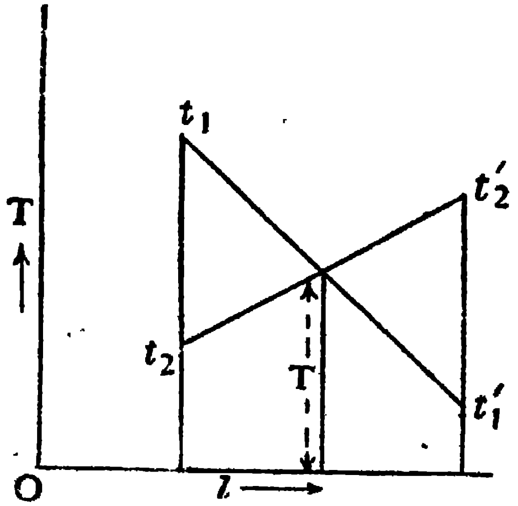


Fig. 134

* If the position of one of the knife-edges be adjustable, a much simpler method can be used to get the required result. First the adjustable knife-edge K_2 is adjusted so that the time of swing (t'_2) about K_2 is slightly greater than the time period (t'_1) about K_1.
e distance between the knife-
$'_1)$ at this position is also
Now K_2 is sufficiently
or wards K_1 such that time
) about K_1 is slightly
han the period (t_1) about
e distance l_2 between K_1 and K_2 is again measured. Now the

time periods are plotted against the distance between K_1 and K_2 (Fig. 134). The points lie on two intersecting straight lines. The co-ordinates of the point of intersection give the periodic time T and the length l of the equivalent simple pendulum.

Bessel pointed out in 1826 that the exact equality of time periods about K_1 and K_2, is not indispensably necessary. If T_1 and T_2 are nearly equal time periods about K_1 and K_2 respectively whose distances from C.G. of the pendulum are, say h_1 and h_2, then from relation (1), Art. 129, we have,

$$gh_1\frac{T_1^2}{4\pi^2}=h_1^2+k^2,$$

$$\text{and } \quad gh_2\frac{T_2^2}{4\pi^2}=h_2^2+k^2.$$

On subtraction, $\dfrac{g}{4\pi^2}(h_1T_1^2-h_2T_2^2)=h_1^2-h_2^2$;

$$\text{or, } \quad \frac{4\pi^2}{g}=\frac{h_1T_1^2-h_2T_2^2}{h_1^2-h_2^2}=\frac{1}{2}\left[\frac{T_1^2+T_2^2}{h_1+h_2}+\frac{T_1^2-T_2^2}{h_1-h_2}\right].$$

Since T_1 differs very little from T_2 the second term is very small compared to the first term and as such need not be known accurately. T_1 and T_2 are observed; h_1+h_2 is the distance between the knife-edges K_1 and K_2. While h_1-h_2, is obtained by finding the C.G. by balancing the pendulum horizontally on a sharp edge.

Examples

(1) *A metre rod is hung up as a pendulum and allowed to swing. At what point must it be suspended to have the least possible period?*

Ans. The distance of the required point from the C.G. of the rod is given by $h^2=k^2$ (when k is the radius of gyration) $=\left(\dfrac{l}{\sqrt{12}}\right)^2=\dfrac{l^2}{12}=\dfrac{100^2}{12}=833\cdot3$ cm.^{2},

∴ $h=28\cdot8$ cms. from the centre of gravity on either side of it.

(2) *What is the length of the equivalent simple pendulum which consists of a sphere of 10 cms. radius suspended by a light string of length 50 cms.?*

Ans. The length of the equivalent simple pendulum is $L=\dfrac{k^2+l^2}{l}$.

Here $k^2=\frac{2}{5}r^2=\frac{2}{5}\times10^2=40$ cms.^{2}; and $l=50+10=60$ cms.

∴ $L=\dfrac{k^2+l^2}{l}=\dfrac{40+60^2}{60}=60\cdot67$ cms.

(3) *A circular disc of radius 20 cms. oscillates as a pendulum about a point on its circumference. Find the position or positions of the centre of oscillation.*

(Punj. U. 1955)

Ans. The moment of inertia of the disc of mass m about its centre is

$$I=\tfrac{1}{2}Mr^2=Mk^2.$$

$$\therefore k^2=\frac{20\times20}{2}=200.$$

Period of oscillation about the point on its circumference is,

$$t=2\pi\sqrt{\frac{l^2+k^2}{lg}}=2\pi\sqrt{\frac{20^2+200}{20\times980}}=2\pi\sqrt{\frac{3}{98}}.$$

If the length of the equivalent pendulum be L, then

$$t=2\pi\sqrt{\frac{L}{g}}.$$

$$\therefore\ \frac{L}{980}=\frac{3}{98};\ \text{or,}\ L=30\ \text{cms.}$$

Thus the centre of oscillation is 30 cms. vertically below the point of suspension, *i.e.* 10 cms. below the centre of the disc. Since any point on the circumference may be taken as the point of suspension, the point of oscillation may also be any corresponding point lying on a circle of radius 10 cms. described with the centre of the disc as the centre.

132. Errors in the determination of 'g' by a Compound Pendulum:—In the various methods adopted for finding 'g' by a compound pendulum, the time period is accurately determined. But the expressions used for T in those experiments only refer to ideal conditions. The observed value of the time period T_1 is really connected with the ideal value T by a relation of the form,

$$T=T_1(1-\alpha-\beta-\gamma-\delta-\theta-\phi-\psi),$$

where α, β, γ, δ, θ, ϕ and ψ are small correction terms.

(1) α, arises due to the finite arc of swing, because the theoretical deduction of T supposes the arc of swing to be infinitely small. In actual practice the arc of swing has always a finite value. This increases the time period. Theoretically it can be shown that the true time period T_0 (for an infinitely small arc of swing) can be expressed in terms of the observed time period T (for a finite arc of swing) by the relation $T_0=T\left(1-\frac{\alpha_1\alpha_2}{16}\right)$ where α_1 and α_2 are half swings in radians at the start and at the end of the experiment respectively.

(2) β is due to the buoyancy of the air surrounding the pendulum.

(3) γ arises due to the motion of the air (air-drag) as the pendulum vibrates.

The corrections for air effect, *i.e.* β and γ, can be eliminated by using Repsold's pendulum with a suitable compensating device. The usual procedure now-a-days is, however, to swing the pendulum in a reduced pressure, at which the air-effect is a linear function of the pressure. Thus taking observations at two or three different pressures, the effect at zero pressure can be extrapolated from the graph.

(4) δ is damping correction due to the viscous drag of the air.

This is corrected from the observation of the logarithmic decrement from the swings.

(5) θ is due to the change in length of the pendulum for variation of temperature.

(6) ϕ arises due to the possible yielding of the support on which the knife-edges are swinging.

As ϕ is due to movement of the support caused by a horizontal force equal to the weight of the pendulum, it may be estimated by hanging the pendulum over a pulley by a string horizontally attached to the support and observing the horizontal displacement by a microscope.

(7) ψ is due to the fact that the knife-edges are not perfectly sharp but are rather rounded cylinders.

This effect may be avoided by having a plane bearing on the pendulum and a fixed knife-edge on the support.

Over and above these corrections the detailed treatment of which is beyond the scope of this book, a correction should also be introduced for the rotation of the earth about its axis.

133. General Remarks on variation of g and gravity survey :—The actual variation of the acceleration due to gravity on the surface of the earth is sometimes found not to conform to the theoretical calculations. Actually Airy in 1854 found that g at the bottom of a coal pit, 1200 ft. deep, was greater than at the top. This observation was subsequently confirmed by other experiments too. That g at first increases with the depth is now accounted for by the fact that the central portions of the earth are much denser than the crust. "Variations of density from its normal value in a locality by reason of the nature and position of a mineral deposit, affect to a measurable extent, the space-variations of the gravitational force." From the survey of gravity by suitable methods (e.g. Eötvös balance or a gradiometer) geophysical prospecting (*i.e.* location of geological deposits) has been made possible and thus a new avenue for industrial enterprise has been opened up.

Duffield in 1914 made a survey of g at sea. His results appear to show a defect of gravity over very deep oceans and on the edge of a continental mass, especially if there is a costal mountain range. The later gravity expeditions at sea, wherein the submarine with its comparatively steady motion allows the use of the compound pendulum with nearly the same accuracy as on land, also confirm the general observations of Duffield excepting in a few cases where the local irregularities may in all probability account for the deviations.

The local changes in acceleration due to gravity are generally too small to be measured by ordinary pendulums because the accuracy of them is limited to within 10^{-4} cm./sec.^{2} [or, 0·1 milligal where

1 milligal $= 10^{-3}$ cm./sec.^{2}]. So more sensitive instruments are to be used for this purpose. The instruments usually used are :—

(i) **Invariable Pendulum.**—This is a compound pendulum of invar-steel suspended in a partially evacuated chamber, where a constant specified pressure is maintained to ensure a known air-correction. The temperature correction though negligible is also taken into account in the final evaluation of 'g'.

(ii) **Gravity Meters.**—

(a) **Threlfall and Pollock Gravity meter.**—It is one of the most sensitive type of gravity meters. It consists of a quartz thread horizontally stretched and fixed at one end. The other end is attached to an adjustable screw provided with a pointer moving over a circular graduated scale. Near about the midpoint of the thread a small weight mounted on a pin is attached and the pin is set horizontally in an unstable equilibrium position. So a slight change in g will upset the horizontality of the pin. By adjusting the screw again the horizontal position can be restored. If the screw readings be previously calibrated with reference to known values of g, g at any place can be directly obtained from the pointer position.

(b) **Boliden Gravity Meter.**—It consists of two springs supporting a mass carrying two movable plates of two condensers at the two ends. The upper condenser is included in an oscillatory circuit of known frequency. The lower condenser is meant for mere calibration of the instrument. As g at a place changes, due to change in the gravitational force acting on the mass, the separation of the plates of the upper condenser changes. Thus the capacity as well as the frequency of the oscillatory circuit also change. If the instrument be previously calibrated, a knowledge of this change in frequency would give the variation of g at the place.

(c) **Gulf Gravity Meter.**—This is also a sensitive gravity meter depending on the same principle as a spring balance. Here a flat metallic ribbon spring is used and observations are made by lamp and mirror arrangement. The sensitivity attained is of the order of 0·05 milligal.

(iii) **Eötvös Balance.**—It is the most sensitive type of gravity balance and is extensively used for gravity survey. It claims an accuracy of the order of 10^{-6} to 10^{-9} milligal. It consists of a rectangular torsion beam suspended from a torsion head by means of a fine suspension of platinum-iridium, fixed at the C.G. of the beam. A small weight is suspended by a fine platinum wire from one end of the beam and it is counter-balanced by a sliding weight mounted at the other end. When g varies, the system experiences a couple, whose value and hence the variation in g can be had from the constants of the instrument and from the deflection of the beam which is recorded by a lamp and telescope arrangement.

(iv) **Horizontal Pendulum.**—The horizontal pendulum is an arrangement by which very minute temporal variations in the acceleration due to gravity (g) can be accurately determined. It was devised in 1832 by Hengler. Later on it was modified by Galitzin and others to act as a sensitive *seismograph*, which is used for recording the tremors of an earthquake.

A horizontal pendulum consists of a rod RB fitted with a spherical or cylindrical bob at C. The rod is held in a position slightly inclined to the horizontal plane by two light strings Ra and Sb attached to two rigid supports. This type of suspension is known as the Zollner suspension The $c.g.$ of the suspended system lies at C inside the bob itself and its centre of oscillation lies at O, the point of intersection of RB and ab. The line aOb makes an angle ϕ with the line Va along which the force of gravity acts [Fig 135].

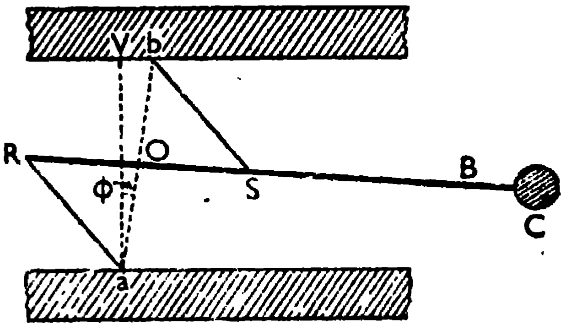


Fig. 135

If the pendulum is slightly displaced to R_1C_1 [Fig. 136] through an angle θ in its plane of oscillation, the torque acting on it due to this angular motion is $I\frac{d^2\theta}{dt^2}$ where I is the moment of inertia of the moving system. At the displaced position the force mg acting on the bob due to gravity in a direction parallel to Va can be resolved into two components, $mg \cos \phi$ acting parallel to Oa and $mg \sin \phi$ acting parallel to RC. The component $mg \sin \phi$ evidently acts in the plane of oscillation and as such it may be resolved into components, $mg \sin \phi \cos \theta$ along RC, and $mg \sin \phi \sin \theta = mg \sin \phi . \theta$ along C_1C when θ is very small. The second component exerts a restoring couple $mg \sin \phi \times \theta \times l$ on the

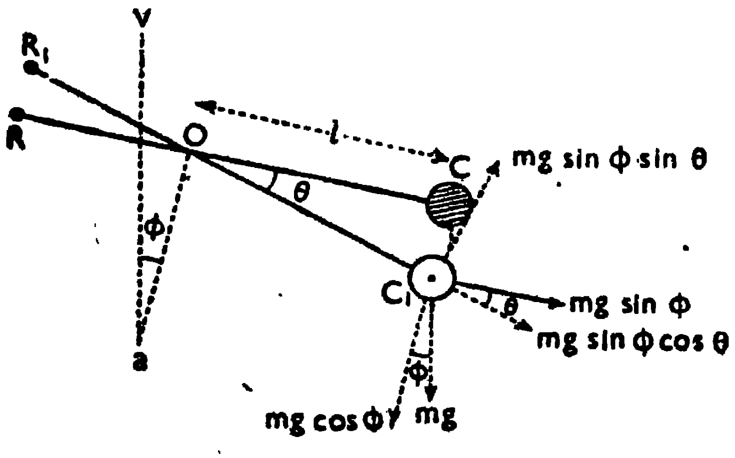


Fig. 136

moving system where l = the length OC. This couple is equal and opposite to the torques calculated above.

$$\text{Hence, } I\frac{d^2\theta}{dt^2} = -(mgl \sin\phi).\,\theta\ ;$$

$$\text{or, } \frac{d^2\theta}{dt^2} + \frac{mgl\sin\phi}{I}\theta = 0.$$

This is the equation of a S.H.M. Hence the time period of oscillation is given by,

$$T = 2\pi\sqrt{\frac{I}{mgl\sin\phi}} = 2\pi\sqrt{\frac{mk^2}{mgl\sin\phi}} = 2\pi\sqrt{\frac{k^2}{gl\sin\phi}}.$$

If the same pendulum is made to oscillate in a plane making $\phi = 90^\circ$, $\sin\phi = 1$ and the new time period is $T' = 2\pi\sqrt{\frac{k^2}{gl}}$.

$$\text{Hence}\quad \left(\frac{T'}{T}\right)^2 = \sin\phi.$$

Thus ϕ and hence the inclination of the line of gravity can be accurately determined. Any change observed in ϕ indicates the variation in the direction of g.

Note :—To use the *horizontal pendulum* as a seismograph to record the horizontal component of an earthquake shock. Galitzin's modified form is very convenient. As shown in Fig. 135, the rod or the *'boom'* makes a small angle with the horizontal and is supported by two strings. The boom extends beyond the bob and carries a flat copper coil moving in a magnetic field. When an earthquake occurs, the boom moves causing the coil also to move in a magnetic field. The current induced in the coil is proportional to the angular velocity of the boom. The coil is connected to a sensitive and critically damped galvanometer the movement of whose moving part is recorded on a moving film by the usual lamp and mirror arrangement. The developed trace on the film is called the seismogram.

The motion of the boom is rendered critically damped by electromagnetic damping. Here the rod carries a copper disc which moves in a magnetic field. The induced eddy current damps the motion of the boom. The overall magnification of this arrangement may be of the order of 1000 or more.

134. Determination of the Universal Gravitational Constant (G) :—

(i) Bouguer's Mountain Experiment.—Bouguer in 1749 carried out an experiment on Mount Chimborazo in Peru to determine G. The principle of his measurement lies in the observation of the deflection of a vertical plumb line towards a large mountain due to the gravitational attraction of the latter. If m be the mass of the bob of the plumb line, and M and R the mass and radius of the earth respectively the vertical downward force on the bob $= G\frac{Mm}{R^2}$. Again if V, ρ and d refer to the volume of the mountain, its mean density and the distance between the bob and the $C.G.$ of the mountain respectively, then the sideways deflection force is $G\frac{V\rho m}{d^2}$.

Hence the deflection θ of the plumb line is given by,

$$\tan\theta = \frac{GV\rho m}{d^2} \Big/ G\frac{Mm}{R^2} = \frac{V\rho R^2}{M\,d^2}.$$

As g at the locality which is known from a separate experiment is given by $G\frac{M}{R^2}$, we have,

$$\tan\theta = \frac{V\rho G}{gd^2}\ ;\ \text{or,}\quad G = \frac{gd^2\tan\theta}{V\rho}.$$

Thus G is determined. G, determined by this method is not reliable because of the uncertainties is determining V, ρ and d.

Maskelyne in 1774, repeated the experiment at the request of the Royal Society, in a more precise and thorough manner. But the result obtained is neither dependable because of a few uncertainties in the measurements.

(ii) Airy's Mine Experiment.—In this experiment the periods of oscillation of a pendulum at the top and the bottom of a mine of known depth were determined. Hence the value of g were calculated. By mere comparison of the two values of g, the magnitude of G and M can be approximately found out. Airy's value of G was $5{\cdot}7 \times 10^{-8}$ $C.G.S.$ units which is far too short of the standard value.

(iii) Cavendish's Method.—Cavendish in 1797, performed his famous torsion-balance experiment for the determination of G.

the gravitational constant. From a knowledge of G, the mass of the earth was also calculated, knowing the values of the radius of the earth, R and the acceleration due to gravity, g, at the place. So it is sometimes stated that "Cavendish is said to have weighed the earth." The actual arrangement of his experiment is as follows :

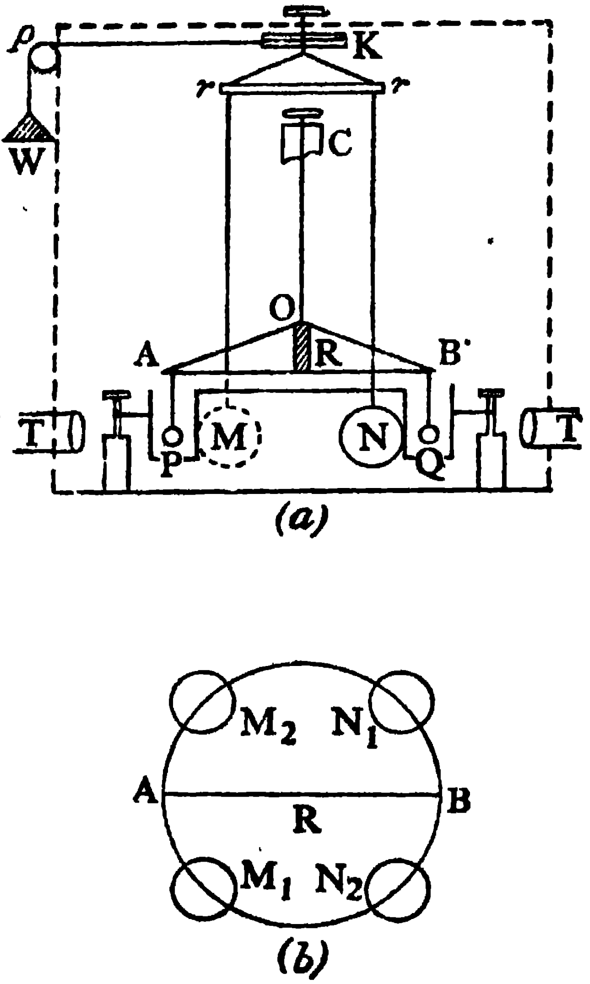


Fig. 137

At the end of a 6 ft. beam, AB [Fig. 137 (a)], were hung two small lead balls P and Q each of two inches diameter. The beam was strengthened by two braces OA and OB connected to O in a short stout upright RO, attached to the middle point of AB. The beam was suspended at O from a torsion head C by means of silver-copper torsion fibre CO, 3 feet long. Two large lead spheres, M and N, each of one foot diameter, were suspended at equal distances near the balls P and Q, from the two ends of a rod, rr. rr could be rotated through any angle about a vertical axis with the help of a wheel K, a pulley p and the weight W shown in the figure. Thus MN could be made to take up positions either M_1N_1 or M_2N_2 on either side of PQ [Fig. 137 (b)]. The centres of the four balls P, Q, M, N, all lay on the circumference of a horizontal circle of radius 3 feet. The whole arrangement was entirely enclosed in a glass case, and rr could be rotated from outside and so also the torsion head C by an arrangement not shown in the figure. The observations were made with two telescopes fitted in the walls of the room.

In this position M_1N_1 of the large balls, attraction took place between P and M_1 as also between Q and N_1. These forces constitute a couple which turned the beam AB through an angle. The equilibrium position, which was difficult to be kept steady, was determined by the method of oscillation from the observation of the swings. The beam carried at each end a vernier that could move freely on a fixed scale. The vernier readings on the fixed scales were taken from outside the closed room by telescopes. The deflecting balls M and N where rotated to the positions M_2 and N_2 such that PM_1 and PM_2 were equal and the new equilibrium positions were again estimated as before. In this way, the mean angle of twist α of the beam due to interaction between the two balls at each end was known. Hence G could be determined according to the following calculation.

The deflecting force brought into existence due to gravitational attraction at each end of the rod $AB = G\frac{m_1 m_2}{d^2}$ where m_1, m_2 respectively are the masses of big ball and a small and d is the distance between the centres of these balls when the mean deflection of the beam, is θ.

If l is length of the beam, the moment of the deflecting couple

$$= G\frac{m_1 m_2}{d^2} \times l.$$

In the position of equilibrium this couple is balanced by the torsional couple of the suspension wire, *i.e.* $c\theta$ where c = moment per unit twist $\left(= n\frac{\pi r^4}{2l_1} \right)$ where n = modulus of rigidity of the wire, l_1 being its length and r its radius.

$$\therefore \quad G\frac{m_1 m_2}{d^2} \times l = c\theta \; ;$$

$$\text{or,} \quad G = \frac{c\theta \, d^2}{m_1 m_2 l}.$$

Precautions.—1. To avoid draughts due to temperature fluctuation and air currents, Cavendish housed the apparatus in a glass case and observations and adjustments were made from outside the case.

2. To avoid possible stray electrostatic attractions from outside, the glass case was gilt-covered. This precaution also served partly to equalise the temperature within the case.

Defects in Cavendish's experiment.—(1) The torsion fibre was thick and hence θ was small. (2) The vernier method of measuring deflection is not accurate. (3) Temperature fluctuations and hence draughts could not be altogether avoided in the large apparatus used. (4) Each large sphere also attracted the more distant small sphere and thus a counter gravitational couple tending to decrease θ was present but not accounted for. (5) Thus rods supporting the large masses M and N tended to increase θ. (6) The attraction on the beam AB also tended to increase θ.

(iv) Boys' method of measuring G.—Sir C. V. Boys modified Cavendish's original experimental arrangements whereby most of the above defects were considerably rectified or minimised.

Defect (1) was much reduced by using a fine quartz torsion fibre which is very elastic but nevertheless strong.

Defect (2) was much rectified by using the mirror and scale arrangement, the glass beam being used as the mirror.

Defect (3) was effectively minimised by using a small apparatus.

Defect (4) was almost eliminated by placing P and M at a level different from the level of Q and N.

Defect (5) was removed dispensing with supporting rods and revolving the top of the case to bring M and N in the proper position.

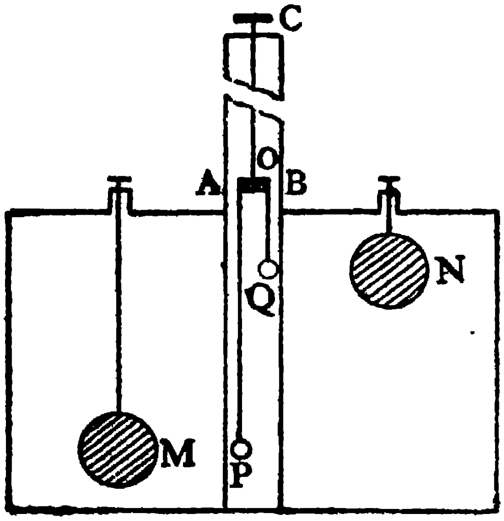


Fig. 138

Defect (6) was made negligible by making the beam AB very short.

Boys' torsion balance (Fig. 138) consists of a central vertical fibre of quartz suspended from a torsion head C. The fibre carries a horizontal glass beam AB. From two ends of this beam two quartz fibres of unequal lengths hang vertically, each supporting a small gold sphere (·25″ in diameter). This suspension system is hung inside a cylindrical glass tube and is thus protected from draughts. Further, the air pressure inside can be adjusted to any desired value. Outside the tube two equal spheres of lead M and N, (4·5″ in diameter) are suspended at equal distances from the axis. The centres of M and N are respectively situated in the same horizontal planes as the centres of the gold balls P and Q. Each lead sphere exerts an attractive force on each gold ball. But as the levels are different the attraction of a lead sphere for a gold ball in the same level will be much greater than that for the other. The attraction produces a torque about the axis and the central fibre is twisted through an angle θ. The suspension system comes to rest when the resultant moment of the displacing forces due to gravitation is equal and opposite to the moment of the restoring couple due to the twist of the fibre. The positions of the lead spheres are so chosen as to make the moment of the deflecting forces as large as possible. Deflection θ is measured by scale and telescope arrangement, the mirror being the horizontal glass beam AB itself. The spheres M and N are then swung to the reverse side of the gold spheres and the deflection again observed. If C be the couple per unit twist of the suspension,

$C\theta = \frac{GmM}{d^2} l$, where m = mass of P or Q, M = the mass of M or N, d = distance between the centres of the attracting spheres P and M or Q and N, l = the effective arm of the mechanical couple. C is obtained from a knowledge of the moment of inertia I of the suspended system and the time of torsional oscillation T from the relation, $T = 2\pi\sqrt{\frac{I}{C}}$, where I = moment of inertia of the oscillating system and C = couple per unit twist.

The mean value of G obtained by this method was $6{\cdot}6576 \times 10^{-8}$ $C.G.S.$ units.

Calculation of the values of *d* and *l*.—Let the masses MM and mm occupy the positions C, D and A, B as represented in Fig. 139, in the equilibrium condition of maximum deflection ϕ. The axis of rotation of AB passes through its mid-point O. Let $CD = 2b$, $AB = 2a$ and ϕ = the angle between CD and AB. The force of attraction between M and m

$$= F = G\frac{Mm}{BD^2}.$$

The couple exerted by this force $= G\frac{Mm}{BD^2} \times OL$.

Due to attraction on both the smaller spheres, the total couple

$$= G\frac{Mm}{BD^2} . 2OL.$$

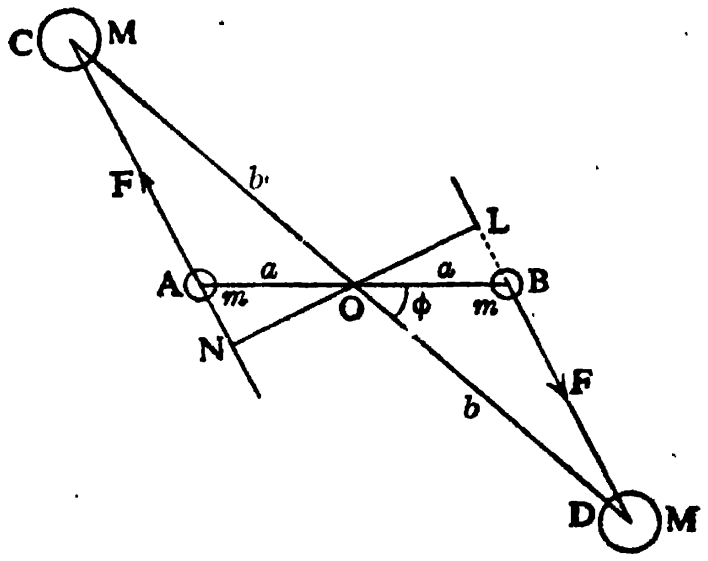


Fig. 139

Hence, in the above relation of $C\theta$, $l = 2OL$ and $d = BD$.

Here, $d = BD = (a^2 + b^2 - 2ab\cos\phi)^{\frac{1}{2}}$ (from the properties of a triangle).

Again, $OL \times BD = 2 \times$ area of $\triangle OBD = b \times a \sin\phi$;

$$\text{or, } l = 2OL = \frac{2ab\sin\phi}{(a^2 + b^2 - 2ab\cos\phi)^{\frac{1}{2}}}.$$

Hence, $$C\theta = \frac{2GmM\,ab\sin\phi}{(a^2+b^2-2ab\cos\phi)^{\frac{3}{2}}}$$

(v) **Poynting's Balance method.**—Poynting's method is one of directly weighing the attraction between two lead spheres by means of a balance. The principle of the method can be understood from the diagrammatic arrangement shown in Fig. 140. The attracting lead sphere M weighing about 350 lbs. is carried on a platform which can be revolved about an axis O such that it may in turn be brought under either of the attracted spheres P and Q of lead, each weighing about 50 lbs. M is counterpoised by the weight N so that during the revolution of the platform the base of the instrument is not disturbed. The deflection of the balance beam is measured by a suitable optical arrangement (*e.g.* a mirror and scale arrangement, the former being attached to the pointer).

Suppose ϕ is the deflection produced in moving M from below P to below Q and l is the arm of the balance, the couple per unit deflection will then be given by $\frac{GmM\times 2l}{d^2\phi}$ where m = mass of P or Q, M = mass of M, and d = the vertical distance between the centres of P and M. The balance is then calibrated by a rider of known weight, m_1g. If the rider produces a deflection θ when placed at a distance l_1 along the arm of the balance, the couple for unit deflection is $\frac{m_1gl_1}{\theta}$.

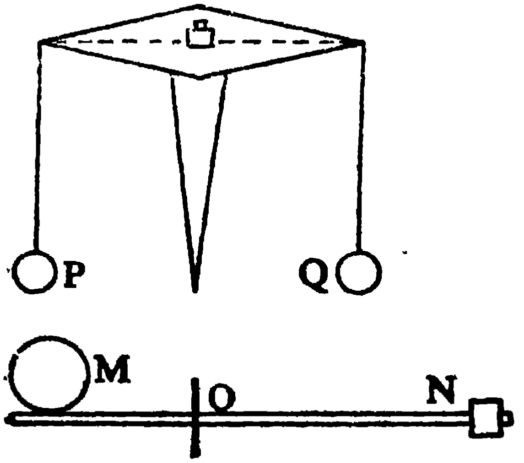


Fig. 140

∴ Equating these two couples, $\frac{2GmMl}{d^2\phi} = \frac{m_1gl_1}{\theta}$,

or, $G = \frac{m_1gl_1d^2\phi}{2mMl\theta}$. Hence G is determined. The mean value of G obtained by this method was $6{\cdot}6984\times10^{-8}$ C.G.S. units.

***(vi) Heyl's method of determining 'g'**—During the years from 1927 to 1930, Heyl in America carried out a series of torsion beam experiments for the accurate determination of the universal gravitational constant G.

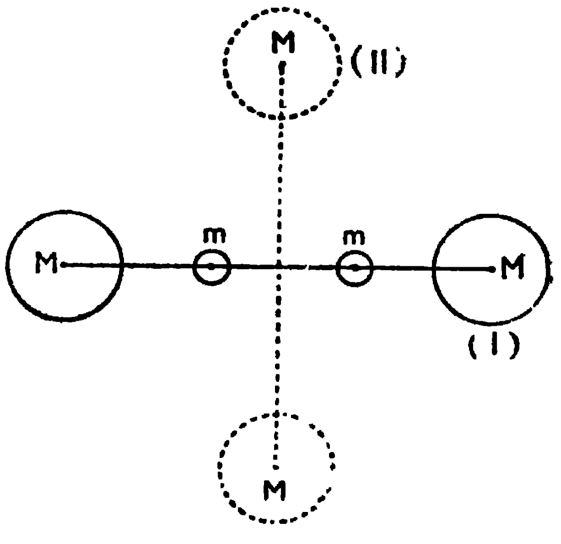


Fig. 141

A rigid beam (20 cms. in length) is horizontally suspended at its centre by a fine tungsten filament (1 metre long). Two small metallic spheres mm, (each of mass 50 gms.) are fixed to the ends of the beam. Two large attracting masses MM about 70 kgms. each) in the form of two metallic cylinders have their centres of gravity in the same horizontal plane as those of the spheres. These bigger masses can be swung either on to such a position that the centres of gravity of the four masses lie in a straight line [Fig. 141(I)], or on to a position where the line joining the $C.G.$'s of the two bigger masses intersect that joining the $C.G.$'s of the two smaller masses perpendicularly [Fig. 141(II)].

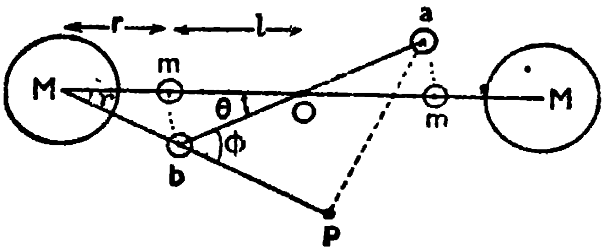


Fig. 142

In one set of experiments the time period of oscillation, T_0, of the torsion beam is first determined in absence of the attracting masses. This time period is given by $T_0 = 2\pi\sqrt{I/C}$, (1) where I is the moment of inertia of the oscillating system and C is the torsional couple per unit radian twist in the suspension wire. Then the attracting masses MM are set in the same line with mm as in Fig. 141(I), and the time period of oscillation again determined. Here the controlling couple is partly due to the torsion in the suspension and partly due to the couple arising out of the gravitational attraction. Hence the time period T_1 is given by,

$$T_1 = 2\pi\sqrt{\frac{I}{C+C_1}} \qquad \ldots \quad (2)$$

where C_1 is the couple per unit radian twist due to the attraction of large masses, the expression for which evidently includes the universal gravitational constant G.

Hence eliminating I from the two expressions (1) and (2) C_1 can be found provided C is previously determined for the same wire by suspending a system of known moment of inertia and finding the corresponding time period of oscillation. Thus G can be evaluated if the expression for C_1 be known.

To evaluate C_1, let us refer to Fig. 142 wherein the force on each of the small spheres in the deflected position ab is $\frac{GMm}{r^2}$. Hence the couple for a small twist θ is,

$$\frac{GMm}{r^2}(aP) = \frac{GMm}{r^2} 2l \sin\phi, \text{ where } ab = 2l.$$

But as is evident from the geometry of the figure,

$$l.\theta = r.\gamma \text{ ; or, } \gamma = \theta\frac{l}{r}.$$

$$\therefore \quad \phi = \theta + \gamma = \theta\left(1 + \frac{l}{r}\right).$$

Since ϕ is small, $\sin\phi = \phi = \theta\left(1 + \frac{l}{r}\right)$.

$\therefore$ When the angular displacement is very small the couple for a twist of θ radians due to the gravitational attraction

$$= \frac{GMm}{r^2} \times 2l\,\theta\left(1 + \frac{l}{r}\right).$$

$\therefore$ Couple per unit twist, $C_1 = \frac{GMm}{r^2} 2l\left(1 + \frac{l}{r}\right)$.

Thus eliminating I from (1) and (2), as referred to above, and substituting for C_1, we have,

$$T_0^2/T_1^2 = \frac{+C_1}{C} = 1 + \frac{2GMml}{Cr^2}\left(1 + \frac{l}{r}\right) \quad \ldots \quad \ldots \quad (3)$$

Hence from (3) G can be determined.

In some later set of experiments, once the time period T_1 was determined with the centres of gravity of the four masses in the line and then the time period T_2 was determined with the line joining the centres of gravity of larger masses at right angles to the line joining the other two centres of gravity.

Here $T_1 = 2\pi\sqrt{\frac{I}{C+C_1}}$ and $T_2 = 2\pi\sqrt{\frac{I}{C+C_2}}$.

From these two relations C can be eliminated.

Hence finding expressions for C_1 and C_2 in terms of G and other known quantities in the above manner, G can be determined. The mean value of G obtained by Heyl from a series of experiments is found to be

$$G = (6\cdot670 \pm \cdot005) \times 10^{-8} \text{ C.G.S. units.}$$

135. Velocity of escape of a projectile : Artificial satellites and planets.—A new era dawned in the history of science and technology when on October 4, 1957, Russia launched the first ***artificial satellite*** Sputnik No. 1. Since then a few more Russian and American satellites have been placed in different orbits round the earth. The first successful launching of an artificial planet by Russia on January 2, 1959 was definitely a step further towards the possibility of interplanetary flights. These satellites, planets and space-ships are all rocket-driven and are successfully placed in their precalculated orbits with the help of rockets moving with the tremendous speeds. To understand the theory of these rocket-propelled artificial planets and satellites it would be better to consider first of all, the limiting velocity that must be imparted to a body in order that it may escape from the earth in spite of its gravitational attraction.

Velocity of Escape.—In Art. 143 (*i*) it has been shown that the gravitational potential on the surface of the earth is given by $-\frac{GM}{r}$, where G = universal gravitational constant, M = the mass of earth and r = its radius. Here the negative sign indicates the intrinsic negative nature of the gravitational field So in order to take a body of mass m outside the attractive influence of the earth's gravitational field, the amount of work [or energy] that must be expended should exceed the potential energy of m on the surface of the earth, which is given by $G\frac{Mm}{r}$.

If the body be imparted a velocity v in the vertically upward direction its kinetic energy will be $\frac{1}{2}mv^2$. If this energy exceeds the above potential energy, the body will escape from the earth. So that limiting velocity of escape, v_e, is obtained by equating this kinetic energy with the potential energy above. This escape velocity is also known as the ***second cosmic velocity.***

$$\therefore\quad \tfrac{1}{2}mv_e^2 = G\ Mm/r\ ;$$

$$\text{or,}\quad v_e^2 = \frac{2G\ M}{r} = \frac{2G\ M}{r^2}\times r = 2gr\left[\because\quad g=\frac{GM}{r^2}\right].$$

Hence the velocity of escape is given by $v_e = \sqrt{2gr}$. Since the radius of the earth, $r = 6{\cdot}4\times10^8$ cm. and $g = 980$ cm./sec.^{2}, $v_e = \sqrt{2\times980\times6{\cdot}4\times10^8} = 1{\cdot}12\times10^6$ cm/sec. = 7 miles/sec. nearly = 25000 m.p.h. (approx.). So if a missile is projected with a velocity exceeding 25000 m.p.h. it will never return to the earth.

Further mathematical calculations regarding the velocity of projectiles moving in a field of force where inverse square law is true, reveal the following important facts which are of primary importance for the launching of artificial planets and satellites.

(*a*) If the velocity, v of the projected body be such that $v^2 < \frac{v_e^2}{2}$ or, $v < 4·9$ miles per sec., the trajectory will be an ellipse passing partly through the earth with one of its foci at the centre of the earth. So bodies projected with such velocities will describe an elliptical path as shown in Fig. 143 (*I*) and ultimately come back to the earth. (*b*) If $v^2 = \frac{v_e}{2}$, *i.e.* if $v = 4·9$ miles per sec. = 18000 m.p.h. nearly the trajectory will be a circle, with its centre at the earth's centre as shown in Fig. 143 (II). (*c*) If $> \frac{v_e^2}{2}$ but $< v_e^2$, *i.e.* if v lies between 18000 m.p.h. and 25000 m.p.h , the path will be an ellipse with the centre of the earth as a focus. The ***perigeal*** distance (the minimum distance in the orbit) of the body moving in this orbit is always greater than the radius of the earth. The orbit in this case will be as shown in Fig. 143 (*III*). The above limits of velocity of projection are always adhered to while setting an artificial satellite in its orbit. (*d*) When $v = v_e = 25000$ m.p.h., the path of the projectile will be a parabola as shown in Fig. 143 (*IV*) and the projectile will just escape from the earth. (*e*) If $v > v_e$, the trajectory will be hyperbolic in nature and the body will never return to the earth. Fig. 143 (*V*) shows such an orbit. For placing an artificial planet in its orbit, the velocity of projection must be always sufficiently above the escape velocity (*i.e.* 25000 m.p.h.).

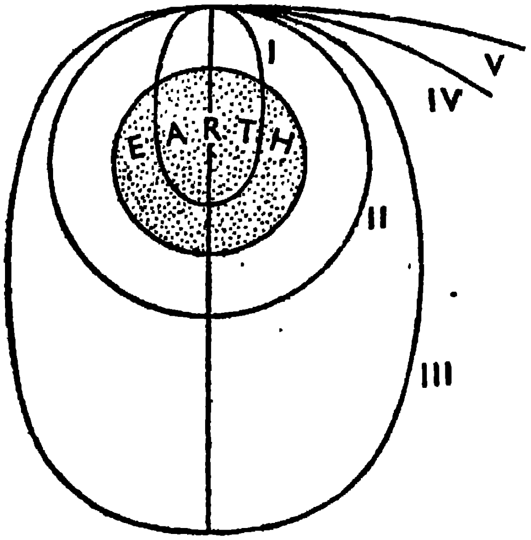


Fig 143

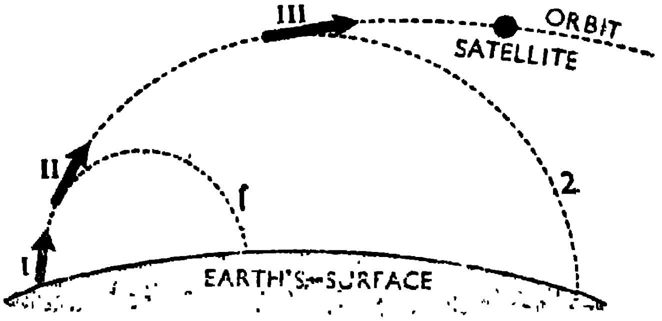


Fig. 144

From the above discussions it is clear that essential condition for the launching of an artificial satellite is that it should be imparted a speed in between 18000 m.p.h. and 25000 m.p.h. and that for an artificial planet the speed must fairly exceed the upper limit given above, *i.e.* it should be much more than 25000 m.p.h. Thus the main problem for the space flight is the attainment of these abnormally high speeds, which have been ultimately reached by means of multiple-stage rockets. Such rockets have been used for launching the artificial moons and planets.

In the first Russian satellite (Sputnik I), a three-stage carrier-rocket was used. The satellite was placed in the nose of the carrier-rocket and sealed behind a protective cone. Requisite fuel and control instruments were placed in the rocket The lowermost component rocket launched the whole missile to a previously calculated height. Though the rocket was launched vertically, its axis gradually curved away from the vertical course with the help of a special mechanical device. The first rocket imparted to the missile about [illegible]th of requisite speed. The component part of the first rocket fell to the ground describing an elliptic path [Fig. 144 (1)] at the end of its upward journey. Then the engine of the second rocket began to operate and made the satellite reach the next higher stage. This rocket imparted about $\frac{1}{3}$rd of the required speed to the missile. The component parts of this rockets also fell to the ground at the end of its upward journey [Fig. 144 (2)]. The engine of the third stage rocket then begun to work and ultimately gave the requisite orbital speed to the satellite. When the speed was attained the engine stopped and the protective cone as well as the engine itself separated from the satellite. The satellite then went on revolving round the earth with a gradually changing orbit.

In the case of artificial planets, a multiple-stage carrier-rocket is needed so that the final speed imparted to the planet may be sufficiently above 25000 miles per hour. In 1961 Russia was reported to have successfully launched an interplanetary missile towards Venus. Russia as well as the United States of America have launched quite a large number of lunar and interplanetary rockets in order to gather more and more precise informations regarding the space flight. In September, 1968 the Soviet lunar space-craft Zond-5 successfully circled the moon, took photographs of the moon's surface from various positions and safely returned to the earth. Since then the American space-ships have already carried human explorers to the moon and the Russian rockets have placed exploring machines on the moon's surface which have supplied us with varieties of scientific informations regarding the moon. Space-ships equipped with scientific laboratories have also been successfully launched towards Mars and Venus which have also sent a lot of informations about these planets. The success of such attempts is bringing the days of interplanetary flights with living passengers nearer and nearer.

Note.—*The rarity of hydrogen in the earth's atmosphere* may be accounted for from the standpoint of the escape velocity calculated above. This velocity is nearly 7 miles per sec. From the Kinetic Theory of Gases, it is found that the R.M.S. velocity of hydrogen a N. T. P. is about 1 mile per sec. But there was a time in the past when the surface temperature of the earth was much higher. At that temperature the R.M.S. velocity of hydrogen might have reached a value of about

2 to 3 miles per second. So it is most likely that quite a good fraction of the total number of hydrogen molecules in the atmosphere at that time possessed a velocity equal to or greater than the escape velocity (*cf. Maxwell's law of velocity distribution*) and ultimately escaped from the earth's atmosphere. So by a gradual process of slow escape, hydrogen and helium became rarer in the atmosphere while the earth was still hot. The absence of any gaseous atmosphere about the moon or about any of the satellite of the planets belonging to the solar system is a fact corroborated by scientific observations. This is due to the small values of 'g' on those satellites and hence small values of the escape velocity. The high speeds of the gas molecules automatically exceeded these comparatively small escape velocities and got out of the satellite

136. (a) Mass of the Earth :—Let M be the mass of the earth and its radius be R. By the law of gravitation, the force of attraction on a mass m on the earth's surface $=\frac{G.M.m}{R^2}=$ the weight of the mass $=mg$, whence $M=g\frac{R^2}{G}$ ... (1). Taking $R=4000$ miles, $G=6{\cdot}65\times10^{-8}$ c.g s. units and $g=981$ cms./sec.^{2}, $M=6{\cdot}1\times10^{27}$ gms. approximately. Thus knowing the radius of the earth and the value of G, and determining the value of g experimentally, the mass of the earth can be obtained.

(b) Density of the Earth.—Let ρ be the mean density of the earth. Then $\rho=\frac{\textit{mass of earth}}{\textit{volume of earth}}=\frac{M}{\frac{4}{3}\pi R^3}=\frac{3M}{4\pi R^3}=\frac{3gR^2}{G\times4\pi R^3}=\frac{3g}{3\pi G.R.}$ from equation (1) above. Taking $g=981$ cms./sec.^{2}, $G=6{\cdot}65\times10^{-8}$ c g.s. unit, and $R=4000$ miles, $\rho=5{\cdot}46$ gms./c.c.

Thus, from the knowledge of G, g and R, the mean ρ of the earth can be calculated. Henry Cavendish determined it (1797-98) and his result for the mean density of the earth, namely 5·5 gms./c.c., has been confirmed by later experiments carried out in different ways. Since the mean density of most substances on the surface of the earth is much smaller than this, it means that heavy substances, particularly metal ores are contained within the earth.

Examples

(1) *If the mean value of acceleration due to gravity on earth's surface be 981 cms /sec.^{2}, $G=6{\cdot}5\times10^{-8}$ c.g.s. units and the radius of the earth be 6×10^8 cms. calculate the mass of the earth and its mean density.*

(*cf.* C. U. 1941 ; *cf.* Pat. U. 1952)

Ans. : We know $g=G\frac{M}{R^2}$; or, $M=\frac{gR^2}{G}$.

$$\therefore \text{ Mass of the earth}=\frac{981\times(6\times10^8)^2}{6{\cdot}5\times10^{-8}}=\frac{981\times36\times10^{24}}{6{\cdot}5}.$$

$$=5{\cdot}43\times10^{27}\text{ gms.}$$

Again $M=\frac{4}{3}\pi R^3\rho$ ($\rho=$density)

$$\therefore \rho=\frac{3M}{4\pi R^3}=\frac{3\times5{\cdot}43\times10^{27}}{4\times\pi\times216\times10^{24}}=\frac{3\times5{\cdot}43\times10^3}{4\times\pi\times216}=6{\cdot}0\text{ gms./c.c.}$$

(2) *A shell is projected horisontally from a rocket when it is at a height of 600 kms. above the surface of the earth. Calculate the velocity of projection necessary for the shell to circle round the earth as a satellite.* (C.U.H. 1960)

($G=6{\cdot}66\times10^{-8}$ C.G.S., $M=6\times10^{27}$ gms. and $R=6400$ kms.)

Ans. : The height of the rocket $=R+h=(6400+600)=7000$ kms. $=7\times10^{8}$ cms. As the shell revolves in a circular orbit, the force on it due to gravity must be equal to the centrifugal force *i.e.* $G\frac{Mm}{(r+h)^2}=\frac{\text{[illegible]}}{R+h}$.

$$\therefore \quad v=\sqrt{GM/(R+h)}=\sqrt{\frac{6{\cdot}66\times10^{-8}\times6\times10^{27}}{7\times10^{8}}}=7{\cdot}556 \text{ kms./sec.}$$

(3) *Calculate the mass of the sun, given that the distance between the sun and the earth is $1{\cdot}49\times10^{13}$ cms., and $G=6{\cdot}66\times10^{-8}$ c g.s. units. Take the year to consist of 365 days.* (P. U. 1942)

Ans. : The force of gravitational attraction between the sun and the earth is $F=G\frac{Mm}{r^2}$ (where M = mass of the sun and m = mass of earth and r = intervening distance).

This must be equal to the centripetal force acting on the earth which is given by $m\omega^2r=m\left(\frac{2\pi}{T}\right)^2r$.

$\therefore$ Equating the two forces, $M=4\pi^2r^3/T^2G$.

Here $T=365\times86400$ secs., $G=6{\cdot}66\times10^{-8}$ c.g.s. units and $r=1{\cdot}49\times10^{13}$ cms.

$$\text{Here} \quad M=\frac{4\pi^2\times(1{\cdot}49\times10^{13})^3}{(365\times86400)^2\times6{\cdot}66\times10^{-8}}=1{\cdot}972\times10^{33} \text{ gms.}$$

137. Kepler's Laws and Newton's Law of Gravitation :—Kepler enunciated his laws of planetary motion during the first two decades of the 17th century (1600-1619). The three laws due to Kepler may be stated as follows :—

(1) The path of a planet is an elliptic orbit, the sun being at one of the foci of the ellipse.

(2) The radius vector from the sun to the planet describes equal areas in equal times or the areal velocity of the planet is constant.

(3) The square of the time period of revolution of the planet is proportional to the cube of the major axis of its orbit.

Kepler, however, could not account for these laws and it was left for Newton to give a mathematical as well as a theoretical interpretation of these laws from the theory of gravitation.

For simplification if we suppose that a planet is moving in a circular orbit of radius R, with the sun at its centre, the second and third laws can be readily deduced. As a circle is merely a degenerate case of an ellipse our supposition is not at variance with the first law.

If M and m be the masses of the sun and the planet, the gravitational attraction is $G\frac{mM}{R^2}$. This attractive force supplies the centripetal force necessary to maintain the dynamical equilibrium. The centripetal force is evidently $m\omega^2 R = m\left(\frac{2\pi}{T}\right)^2 R$ (where ω and T refer to the angular velocity and the time period).

$$\therefore \quad G\frac{mM}{R^2} = m\,\frac{4\pi^2 R}{T^2}\,;$$

$$\text{or,} \quad T^2 = \frac{4\pi^2 R^3}{GM}.$$

$$\therefore \quad T^2 \propto R^3.$$

Thus the third law is deduced.

Again since T is constant as evident from the above expression, ω is also constant.

So, in unit time, angle described $= \omega = \frac{2\pi}{T} = \sqrt{GM/R^3}$. As the area πR^2 of a circle corresponds to an angle 2π at the centre, the angle ω corresponds to an area

$$= \frac{\omega}{2\pi} \times \pi R^2 = \frac{R^2}{2}\sqrt{GM/R^3} = \text{constant}.$$

$\therefore$ Area swept out in unit time (*i.e.* the areal velocity) is also constant. Thus the second law is also deduced.

(As the rigorous deductions are beyond the scope of this book, they have been intentionally avoided.)

138 Qualities and Limitations of gravitation—Newton's law of gravitation is commonly regarded as universally true. Kepler's laws of planetary motion—deduced empirically from the astronomical observations of Tycho Brahe and others are satisfactorily explained from Newton's gravitational law without the help of any subsidiary hypotheses. This shows its applicability to interplanetary distances. Its applicability to terrestrial distances of the order of inches, yards or miles is proved beyond doubt by the general agreement of values of G obtained from various methods of its determination on the surface of the earth.

In recent years, however, observations on the motion of the perihelion of Mercury and the bending of rays of light passing close to the Sun—facts which Newtonian dynamics failed to explain—led Prof. Einstein to point out two inherent difficulties in Newton's enunciation of the law of gravitation First, the mass of a body is not a constant but changes with its velocity. Secondly, the distance between two bodies is only apparent and varies according to the system of space-time co-ordinates chosen. These new postulates—as embodied in

Einstein's theory of Relativity—cannot be dismissed of, for without them one cannot account for the shift in the spectral lines in the solar spectrum relative to their positions in the spectrum of a terrestrial source—which was successfully demonstrated in 1925 by W. S. Adams of Mt. Wilson Observatory. Thus for exactness and philosophical satisfaction, Einstein's theory of relativity is perhaps essential, but for all except the smallest (inter-atomic or smaller) distances and the distances of interstellar magnitudes as well as a few other outstanding phenomena, the applicability of Newton's gravitational law cannot be questioned.

The physical nature of the intervening medium between two bodies plays an important role in electricity and magnetism. But Austin and Thwing found no change in the value of G by interposition slabs of various materials between the attracting masses in a modified Boys' experiment carried out by them and, moreover, there is a general agreement between results of pendulum experiments in which different materials, used for the base, were interposed between the bob and the earth. These facts go to prove that gravitation is free from *permeability*.

Again the results of torsion balance experiments performed for determining G with different interacting materials as also Bessel's hollow bob experiment indicate that gravitation is independent of the nature of the attracting material or, in other words, has no ***selective property***. Crystals have, in general, directive properties towards heat, light and electricity. It is, therefore, necessary to know if the gravitational field between two bodies depends on the relative orientations of their crystallographic axes. Experiments carried out by Mackenzie, Poynting, Gray and others failed to prove any such ***directivity***, at least for distances large compared to molecular dimensions.

Again for moderate range of temperatures, the gravitational force between two interacting masses has been found by Poynting and Philips, Landbolt and others to remain unaltered.

Thus the action in a gravitational field is a function of the magnitudes of the interacting masses and their separating distance only—a conception masterly represented by Newton's Law of Gravitation.

Gravitational Field and Potential

139. Gravitational Field :—The gravitational field is the space surrounding a mass or a system of masses over which it can exert its attractive force on other bodies. Theoretically the field extends upto infinity around the mass but for all practical purposes it is limited upto a small distance beyond which the force is neither appreciable nor measurable.

Field Strength or Intensity of Field.—The intensity or the strength of the gravitational field at a point in that field is defined as the force that would be experienced by a unit mass placed at that point in the field. So if the attracting body be of mass M, and if r be the distance of the point from M, then the intensity of the field at the point is given by,

$$F = G\frac{M}{r^2}.$$

140. Gravitational Potential :—

The *gravitational potential* of a point in gravitational field is measured by the amount of work done in bringing a unit mass from infinity upto that point.

In bringing a body from infinity towards another massive body no work is required to be done by any external agency. It is the gravitational force of attraction that does the work. It is thus a negative work. It is analogous to the electric potential at a point due to a negative charge or the magnetic potential at a point due to a south pole.

The *difference of potential* between two points in a gravitational field is defined in a like manner as the amount of work done in transferring a unit mass from one point to the other.

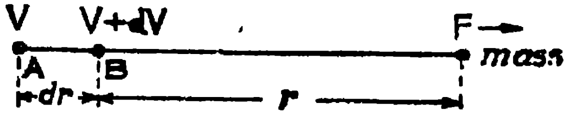


Fig. 145

If in a gravitational field acting in the direction of the arrow mark, two points A and B be considered at an infinitesimally small distance dr apart and if their respective potentials be V and $V+dV$, then the work done in transferring a unit mass from A to $B = F.dr$ where F is the average intensity of the field between B and A. The potential difference between A and B is $V_A - V_B = -dV$, which also is the work done on unit mass,

$$\therefore \quad +Fdr = -dV\,;$$

$$\text{or,} \quad F = -\frac{dV}{dr}.$$

Hence the intensity of the gravitational field at a point is equal to the negative potential gradient or the negative space-rate of variation of gravitational potential. Here because the force is attractive in nature, the gradient of potential, which is essentially negative may be equated to the negative force F and for convenience of calculation may be written as, $F = \frac{dV}{dr}$.

141. Gravitational Lines and Tubes of Force :—A line of force in a gravitational field is a curve, the tangent at any point of which gives the direction of the intensity of the field there. Theoretically we can imagine an infinite number of lines of force in the space surrounding a gravitating body and as such mathematically they cannot be properly dealt with. So the idea of tubes of force has been introduced and these tubes have been arbitrarily defined in such a manner that they may be used as a mathematical tool to solve many problems of field of force.

A tube of force consists of a bundle of lines of force packed to such a density that the number of such tubes per unit area surrounding a point in the field gives the intensity of the field at that point. This packing of the lines should be such that the tubes may touch each other laterally and fill the whole space in the field. The tubes, by their directions, indicate the direction of the intensity at different positions of the field and by the closeness with which they are packed, the strength of the field. These tubes tend to shrink in a lengthwise direction and at the same time possess a tendency of lateral expansion. Again no two tubes of force can intersect. This is so because the tangent at any point of a tube represents the resultant direction of the intensity. Now if two tubes intersect at a point, two tangents can be drawn at that point touching the two tubes and as such the resultant intensity there will have two separate directions at the same position. This is impossible and so two tubes cannot intersect. These tubes are however not the same as the Faraday tubes in electrostatics. They may be considered almost analogous to Maxwellian tubes excepting for the fact that the influence of medium is practically nil in the case of gravitational field whereas the medium plays a very important role in the other kinds of field (electrostatic or magnetic).

142. Equipotential surface :—An equipotential surface is such a surface at every point of which the potential is the same. Now if we choose an equipotential surface CD and if A and B be any two points on it, then from the property of the surface under consideration $V_A - V_B = 0$ [Fig. 146]. Consider a tube of force intersecting CD at an angle θ at the point A so that the intensity of the field acts along the tangent AF to this tube. So the component of intensity along the surface is $F \cos \theta$, which is the force per unit mass along AB. $\therefore$ The potential difference between A and B, i.e. the work done in transferring unit mass from A to B, is given by $V_A - V_B = F \cos \theta \times AB$.

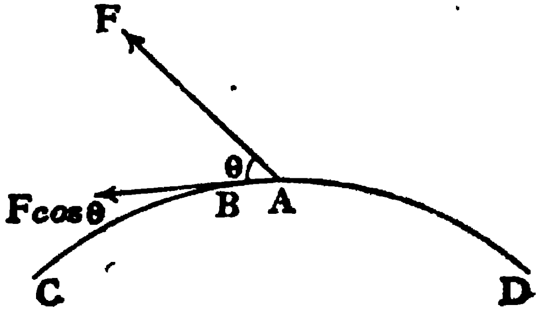


Fig. 146

But from above $V_A \sim V_B = 0$.

$\therefore$ $F \cos\theta \times AB = 0$.

But $F \neq 0$ and $AB \neq 0$; so, $\cos\theta$ must reduce to zero,

$\therefore$ $\theta = 90°$.

So the tubes must intersect an equipotential surface normally.

Again no two equipotential surfaces can intersect because if that were possible, at the point of intersection we could draw two separate normals to the two surfaces, giving the two resultant directions of intensity at the same point, which is impossible.

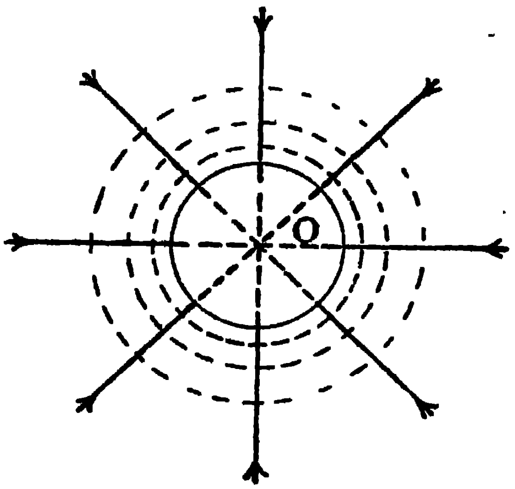


Fig 147

If we consider a body in the form of a uniform sphere, the mass of the sphere, may be supposed to be concentrated at its centre O [Fig. 147]. So if concentric spherical surfaces are drawn surrounding the original sphere the potential at every point of such a surface will have the same value. The sections of these spherical surfaces are shown by the dotted circles in the figure. These represent the equipotential surfaces due to the uniform sphere in the plane of the figure. Hence the lines of force which will be normal to these surfaces should be represented by the radial straight lines as shown in the figure. Now as the force is attractive in nature, the lines of force in the gravitational field will be directed towards the centre of the gravitating sphere.

[N.B. For a detailed discussion of tubes of force, Gauss' theorem and other relevant propositions relating to a field of force, the students should read the corresponding chapter under Electrostatics (vol. II).]

143. Calculation of gravitational Potential and Intensity :—

(i) Gravitational Potential due to a point mass. – Let a point mass M be situated at P (Fig. 148). Now suppose we are to find out the potential of the gravitational field at the point B at a distance r from P.

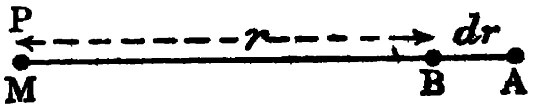


Fig. 148

The intensity F of the gravitational field at B, i.e. the force on unit mass at $B = G\frac{M}{r^2}$ acting along BP. Now if we consider another point A at an infinitely small distance dr from B, the mean intensity between B and A may be taken to be identical with that at B. Hence the work done in moving unit mass from A to B is $G\frac{M}{r^2}.(+dr)$, because the displacement dr is in the direction of the intensity. This is evidently the potential difference (dV) between B and A. $\therefore$ $+dV=+\frac{GM}{r^2}dr$.

Hence from the fundamental definition of potential at a point,

$$V=+GM\int^{r}\frac{dr}{r^2}=GM\left[-\frac{1}{r}\right]^{r}=-\frac{GM}{r}$$

The negative sign here indicates the intrinsic negative nature of the gravitational potential.

(ii) Potential and attraction (intensity) due to a thin uniform circular disc at a point on its axis.—

Let O (Fig. 149) be the centre of the uniform circular disc of radius a and m be its mass so that the mass per unit area of the disc$=m/\pi a^2$. On the axis of the disc, a point P is considered at a distance x from O.

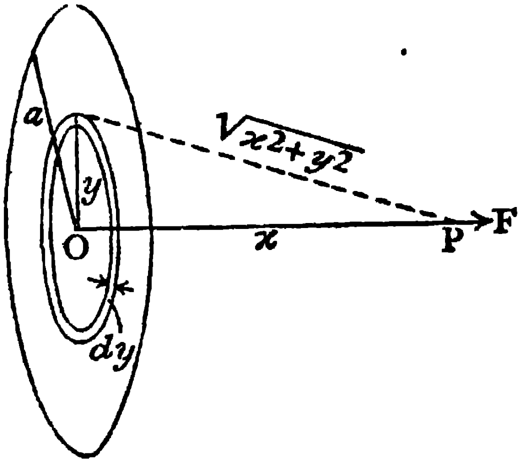


Fig. 149

In order to calculate the potential and intensity of the gravitational field at P, let us consider an arbitrary elementary ring of radius y and of thickness dy in the disc. The mass of this ring is,

$$\frac{m}{\pi a^2}\times 2\pi y dy=\frac{2mydy}{a^2}.$$

Each point of this elementary ring is at a distance, $\sqrt{x^2+y^2}$ from P. So the gravitational potential at P due to this ring is $-G\times\frac{\text{Mass}}{\text{distance}}$, *i.e.* equal to $-\frac{2mG}{a^2}ydy\times\frac{1}{\sqrt{x^2+y^2}}$.

So the potential at P due to the whole disc is,

$$V_p=-\frac{2mG}{a^2}\int_0^a\frac{ydy}{\sqrt{x^2+y^2}}=-\frac{2mG}{a^2}\left[\sqrt{a^2+x^2}-x\right].$$

Since V_p is a function of only one variable, *i.e.* the space co-ordinate x, the intensity F at P, which is the gradient of V_p, is directed along the axis of the disc towards $\overrightarrow{PO}$ because of the attracting nature of the field. The magnitude of F is given by,

$$F=\frac{dV}{dx}=\frac{2mG}{a^2}\left[1-\frac{x}{\sqrt{a^2+x^2}}\right]$$

$$=\frac{2mG}{a^2}(1-\cos\theta)$$ where θ is the angle subtended by a radius of the disc at the point P on the axis.

If ρ be the surface density of the disc, $m=\pi\rho a^2$. Hence $F=2\pi G\rho(1-\cos\theta)$.

When $\theta=90°$, $\cos\theta=0$. Hence, $F=2\pi G\rho$.

Thus the intensity at a point on the axis of a uniform plate of infinite extent is $2\pi G\rho$.

Corollary.—

The Intensity at a point on the axis of a homogeneous solid rod of circular cross-section.—

Let ρ, a and l be the density, radius and length of the rod respectively and let d be the distance of the point P on the axis where intensity is to be found, from the near end AE (Fig. 150).

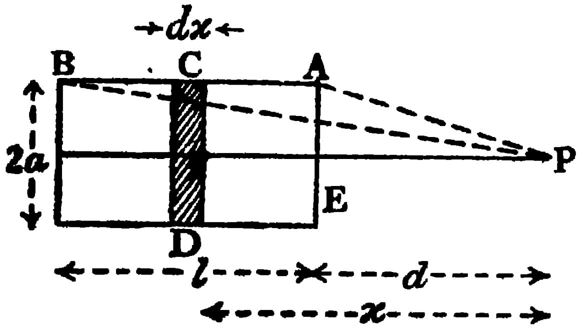


Fig. 160

At a distance x from P consider an elementary strip or disc CD of the rod of thickness dx. The mass of this circular disc $=\pi a^2.dx.\rho$.

Hence from the last article the intensity at P due to the elementary circular disc is,

$$2\pi\rho G dx\left[1-\frac{x}{\sqrt{x^2+a^2}}\right]$$

∴ The intensity due to the whole cylindrical rod

$$=2\pi\rho G\int_d^{d+l}\left[1-\frac{x}{\sqrt{x^2+a^2}}\right]dx.$$

$$=2\pi\rho G\left[l-\sqrt{(l+d)^2+a^2}+\sqrt{a^2+d^2}\right]$$

$$=2\pi\rho G[l-PB+PA].$$

(iii) Potential and intensity due to a thin spherical shell (i.e. thin hollow sphere).

Let O (Fig. 151) be the centre of a thin shell of mass M, of radius a and of uniform surface density ρ. P is a point at a distance r from the centre of the shell. Consider a circular ring JJ of width $ad\theta$, having its centre at L on OP, where θ is the semi-angle subtended by JJ at O, $d\theta$ is the angle subtended by the width of the ring at O. Evidently the radius of the ring $= JL = a \sin\theta$.

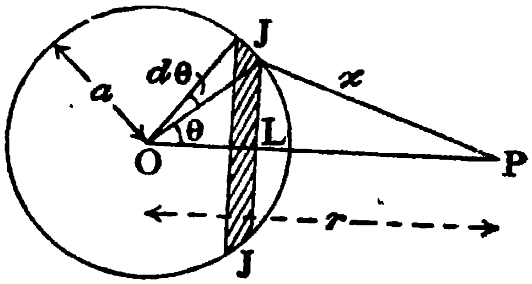


Fig. 151

Hence the surface area of the ring $= 2\pi a \sin\theta \times a d\theta$,

$\therefore$ The mass of the ring $= 2\pi a^2 \sin\theta \, d\theta \, \rho$.

So neglecting the influence of other bodies the potential at P due to the ring $JJ = \frac{-G.\, 2\pi a^2 \sin\theta \, d\theta\rho}{x}$, where $x =$ distance JP.

From Fig. 151, $x^2 = a^2 + r^2 - 2ar\cos\theta$.

Differentiating, $2x\,dx = 2ar\sin\theta\,d\theta$;

$$\text{or, } \sin\theta\,d\theta = \frac{x\,dx}{ar}.$$

Hence the potential at P due to the ring

$$= -G\frac{2\pi a^2\rho}{x} \cdot \frac{x\,dx}{ar} = -G\frac{2\pi a\rho}{r}dx.$$

This expression on integration over the whole shell will give the potential at P due to the shell.

(*a*) *When P is outside the thin shell.*—

(*i*) Potential at $P = V_p = -\int_{x=r-a}^{x=r+a} \frac{2\pi a\rho G}{r}dx = -\frac{2\pi a\rho G}{r}\Big[r+a-(r-a)\Big]$

$$= \frac{4\pi a^2\rho G}{r} = -G\frac{M}{r}. \qquad \ldots (1)$$

Due to a point mass M at O, the potential at P is also $-G\frac{M}{r}$.

Hence *for an external point, a spherical shell is equivalent to a point mass concentrated at the centre.*

(*ii*) Intensity at $P = F = \frac{dV}{dr} = \frac{GM}{r^2}$.

So for an external point the intensity is such that the whole mass may be supposed to be concentrated at the centre.

(*b*) *When P is within the shell.*

(*i*) In this case the distance r of P is less than the radius a of the shell. Hence the upper and lower limits of integration will be $(a+r)$ and $(a-r)$ respectively.

$$\therefore\quad V_p = -\int_{x=a-r}^{x=a+r} G\frac{2\pi a\rho}{r}dx = -G\frac{2\pi a\rho}{r}\Big[(a+r)-(a-r)\Big]$$

$$= -G\frac{4\pi a\rho}{r}\times r = -G\frac{4\pi a^2\rho}{a} = -G\frac{M}{a} \qquad \ldots \quad (2)$$

This expression is independent of r which proves that the potential inside a hollow sphere is constant everywhere.

Again when P is just on the surface of the shell, since $r=a$,

$$V_s = -G\frac{2\pi a\rho}{r}\int_{x=0}^{x=2a} dx = -G\frac{2\pi a\rho}{r}\times 2a = -\frac{GM}{r} = -\frac{GM}{a} \qquad \ldots \quad (3)$$

Thus the potential on the surface of the shell, is the same as that inside the shell. Again when P is on the surface of the shell, the whole mass may still be supposed to be concentrated at the centre of the shell.

(*ii*) *Intensity at P.—*

(*a*) *When P is on the shell—*

Intensity at P is $F_p = +\frac{dV}{dr} = +\frac{d}{dr}\left(-G\frac{M}{r}\right) = \frac{GM}{r^2}.$

Here also the whole mass may be taken to be concentrated at O.

(*b*) *When P is inside the shell—*

Intensity at $P = F_p = \frac{d}{dr}\left(-\frac{GM}{a}\right) = 0,$ since G, M and a are constant. So the intensity at any point inside the shell is zero.

(iv) Gravitational Potential and intensity (or attraction per unit mass) due to a thick-walled hollow spherical shell (also of a solid sphere).—

(a) At a point outside the sphere.—Let Fig. 152 represent a thick hollow sphere of internal radius, a and of external radius b.

Let P be an external point at a distance r from the centre O of the sphere.

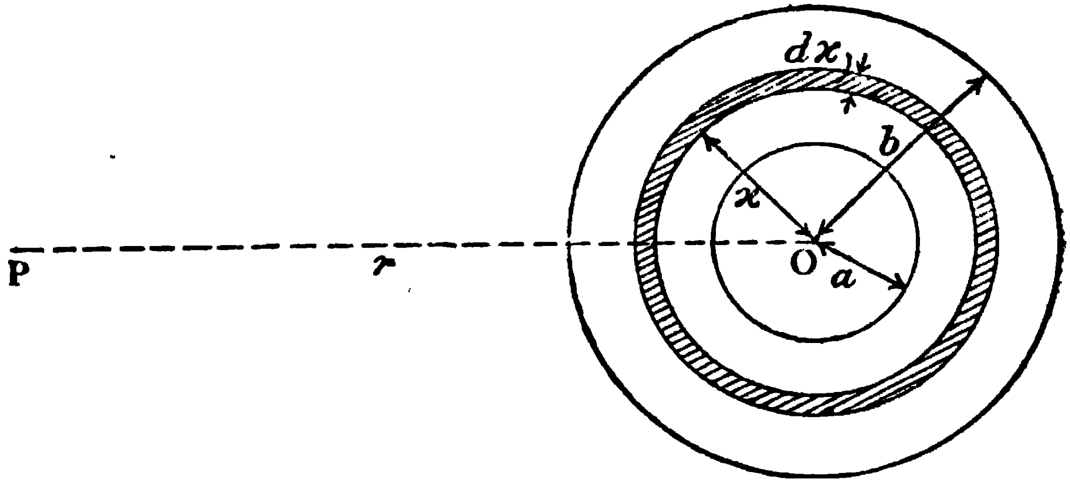


Fig. 152

Consider an elementary thin concentric spherical shell with O as the centre. Let x and dx be the radius and thickness of this shell respectively. The volume of this shell $= 4\pi x^2 dx$.

Hence its mass $= 4\pi x^2 dx \times \rho$, where $\rho =$ density of the material of the shell.

Potential at P due to this thin elementary shell $= -G.\dfrac{\text{Mass}}{\text{distance}}$

$$= -\frac{G.4\pi\rho x^2 dx}{r}, \text{ (since the whole mass is supposed to be concentrated at } O)$$

$$= -\frac{4\pi G\rho x^2 dr}{r}.$$

Hence due to the thick shell the potential at P is given by.

$$V_p = -\frac{4\pi G\rho}{r}\int_a^b x^2 dx = -\frac{4\pi G\rho}{r}.\frac{(b^3-a^3)}{3} = -\frac{GM}{r} \qquad \ldots (1)$$

(where $M =$ mass of the whole shell)

Intensity at P :— $F_p = \dfrac{d}{dr}\left(-\dfrac{GM}{r}\right) = \dfrac{GM}{r^2} = G\dfrac{4\pi\rho(b^3-a^3)}{3r^2}$.

For a Solid Sphere :

For a homogeneous solid sphere of radius b and of density ρ, we get by the same method, the potential at an external point, $V_p =$

$$-\frac{4\pi G\rho}{}\int^{b} x^2 dx = -\frac{4\pi b^3 G\rho}{3r} = -GM$$

Here also the sphere behaves as if its whole mass is concentrated at the centre.

The intensity at P is evidently, $F_p = \frac{dV}{dr} = \frac{GM}{r^2} = \frac{4\pi b^3 \rho G}{3r^2}$

(b) At a point inside the cavity of the hollow thick sphere.—Let P_1 be a point at a distance r from the centre O of the thick shell of radii a and b. Let ρ be the density of the material. Since the potential inside a thin shell is the same as on its surface the potential at P_1 due to an elementary thin shell of thickness dx at a distance x from O is given, as above, by $-\frac{4\pi\rho G x^2 dx}{x}$,

$$\therefore \quad \text{Pot. at } P_1 \text{ (inside)} = V_i = -\int_a^b 4\pi\rho G x\,dx = -2\pi\rho G(b^2 - a^2)$$

$$= -G.\frac{\frac{4}{3}\pi\rho(b^3 - a^3)}{\frac{2}{3}(b^3 - a^3)}(b^2 - a^2) = -\tfrac{3}{2}GM\frac{b^2 - a^2}{b^3 - a^3} = -\tfrac{3}{2}GM\frac{b+a}{b^2+ab+a^2}. \quad (2)$$

Intensity at $P_1 = F_i = \frac{dV_i}{dr} = \frac{d}{dr}[-2\pi\rho G(b^2 - a^2)] = 0$,

since all the quantities within the bracket are constant.

(c) When P is within the material of the thick shell.—

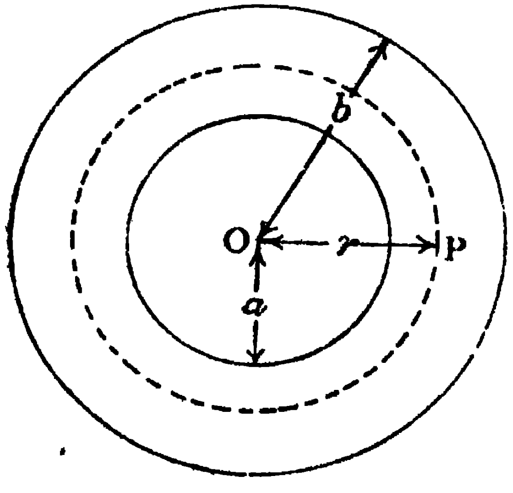


Fig. 153

Let P be at a distance r from the centre of the spherical shell and lie within the material of the shell of internal and external radii a and b respectively.

With r as radius and O as the centre (Fig. 153) a sphere is drawn, on the surface of which lies P.

Now as P is just inside a spherical shell of internal and external radii r and b; the potential at P due to this shell is from relation (2),

$V_1 = -2\pi\rho G(b^2 - r^2)$.

Again since P lies on the surface of a spherical shell of radii a and r, the pot. at P due to this shell is, according to relation (1),

$V_2 = -\frac{4\pi\rho G(r^3 - a^3)}{r \quad 3} = -\frac{GM_1}{r}$, (where M_1 = mass of the inner shell).

Remembering that the mass of the shell is $M = \frac{4}{3}\pi(b^3 - a^3)\rho$, we can write the total potential at P as,

$$V = V_1 + V_2 = -2\pi\rho\ G\left[b^2 - r^2 + \frac{2(r^3 - a^3)}{3r}\right]$$

$$= -\frac{2\pi\rho G}{3r}\left[3b^2 r - r^3 - 2a^3\right]$$

$$= -\frac{4\pi(b^3 - a^3)\rho G}{6r}\left[\frac{3b^2 r - r^3 - 2a^3}{(b^3 - a^3)}\right]$$

$$= -G\frac{M}{r}\left[\frac{3b^2 r - r^3 - 2a^3}{2(b^3 - a^3)}\right],$$

Intensity at P (attraction per unit mass)

$$= F = \frac{d}{dr}\left[-\frac{GM}{2(b^3 - a^3)}\left(3b^2 - r^2 - \frac{2a^3}{r}\right)\right]$$

$$= \frac{GM}{2(b^3 - a^3)}\left[2r - \frac{2a^3}{r^2}\right] = \frac{GM}{r^2}\frac{(r^3 - a^3)}{(b^3 - a^3)}.$$

Since the intensity due to the outer shell at $P = 0$, [Ref. case (b) above], the intensity at P is simply $F = \frac{dV_2}{dr} = GM_1\frac{d}{dr}\left(-\frac{1}{r}\right) = \frac{GM_1}{r^2} = \frac{G.\frac{4}{3}\pi\rho\,(r^3 - a^3)}{r^2}$ $= \frac{GM}{r^2}\cdot\frac{(r^3 - a^3)}{(b^3 - a^3)}$, [$\because$ $M = \frac{4}{3}\pi(b^3 - a^3)\rho$].

For a Solid Sphere (at an internal point)

The potential at an internal point of a homogeneous solid sphere of radius, b, can be similarly calculated by dividing the sphere into two parts. If r be the distance of the point P, from the centre of the sphere, one part will be a shell external to P of inner and outer radii r and b and the other part will be a solid sphere of radius r, on the surface of which P lies.

$\therefore$ The potential at P is V_p = potential due to the outer shell + potential due to the solid sphere of radius r

$$= -\left[2\pi\rho G(b^2 - r^2) + \frac{4\pi\rho G}{r}\left(\frac{r^3}{3}\right)\right].$$

$$= -2\pi\rho G\left[b^2 - r^2 + \tfrac{2}{3}r^2\right] = -\frac{2\pi\rho G}{3}(3b^2 - r^2).$$

Intensity at P is $F_p = \frac{dV}{dr} = -\frac{2\pi\rho G}{3}(-2r)$

$$= \tfrac{4}{3}\pi r\rho G.$$

$$= \tfrac{4}{3}\pi b^3 \rho G.\ r/b^3 = G\frac{M}{b^3}\ r.$$

Examples

(1) *Show that the acceleration due to gravity at the bottom of a mine of depth 5 miles is $\frac{799}{800}$ of its value at the surface assuming the earth to be of uniform density throughout and the radius of the earth 4000 miles.* (*C. U. 1911*)

Ans.: Acceleration (g) due to gravity on the surface is given by $g = G.\frac{M}{(4000)^2} = \frac{G.\frac{4}{3}\pi\,(4000)^3.\rho}{(4000)^2} = \frac{4}{3}\,G.\ \pi.\ 4000.\rho.$

Acceleration (g_1) at a depth of 5 miles is given by,

$$g_1 = G.\frac{M_1}{(4000-5)^2} = G.\ \tfrac{4}{3}\pi\ \frac{(4000-5)^3.\ \rho}{(4000-5)^2} = \tfrac{4}{3}.\ G.\ \pi\ (4000-5)\ \rho = \tfrac{4}{3}.\ G.\ \pi.\ 3995\ \rho.$$

$$\therefore\quad \frac{g_1}{g} = \frac{\frac{4}{3}\,G.\ \pi\ 3995\ \rho}{\frac{4}{3}.\ G.\ \pi.\ 4000\ \rho} = \frac{3995}{4000} = \frac{799}{800}.$$

(2) *The potentials of two homogeneous spherical shells of the same surface density at their respective centres are in the ratio 3 : 4. If the two shells coalesce into a single one the surface density remaining unchanged, what is the potential at an internal point of this shell?* (*C. U. 1913*)

Ans.: Let the radii of the two shells and the combined shell be respectively r_1, r_2 and r. Suppose that the common surface density for all of them be ρ. Let the potentials of the shells at their centres be given by V_1, V_2 and V.

Then, $\quad 4\pi r_1^2.\ \rho + 4\pi r_2^2.\ \rho = 4\pi\ r^2\ \rho$ or, $r_1^2 + r_2^2 = r^2$.

Now $V_1 = -G\ \frac{M_1}{r_1} = -G\ \frac{4\pi r_1^2\ \rho}{r_1} = -G 4\pi r_1 \rho$

and $V_2 = -G\frac{M_2}{r_2} = -G\ \frac{4\pi r_2^2\ \rho}{r_2} = -G 4\pi r_2 \rho.$

$\therefore\quad \frac{V_1}{V_2} = \frac{r_1}{r_2} = \frac{3}{4}$; or, $\frac{r_1^2}{r_2^2} = \frac{9}{16}$. $\quad\therefore\quad r_1^2 : r_2^2 : (r_1^2 + r_2^2) = 9 : 16 : 25$;

or, $\quad r_1^2 : r_2^2 : r^2 = 9 : 16 : 25$; *i.e.* $r_1 : r_2 : r = 3 : 4 : 5$.

But $V_1 : V_2 : V = (-G.4\pi\, r_1\ \rho) : (-G.4\pi r_2\ \rho) : (-G.4\pi r.\ \rho) = r_1 : r_2 : r = 3 : 4 : 5.$ (Complete evaluation of V is not possible from the given data.)

(3) *The potentials of two homogeneous spherical shells at internal points are in the ratio of 3 : 4. Find the ratio of their radii.* (*C. U. 1914*)

Ans.: $\quad V_1 = -G.\ \frac{4\pi\ r_1^2\ \rho}{r_1} = -G.4\ \pi\ r_1.\ \rho.$

and $\quad V_2 = -G.\ \frac{4\pi r_2^2\ \rho}{r_2} = -G.4\pi.r_2\ \rho.$

$\therefore\quad V_1 : V_2 = \frac{r_1}{r_2}$. Again $V_1 : V_2 = 3\ .\ 4.\quad \therefore\ r_1 : r_2 = 3 : 4.$

(4) *If the earth were a solid sphere of iron of radius 6·37 million metres and of density 7·86 gms./c.c., what would be the value of g at its surface, taking G $= 6{\cdot}658 \times 10^{-8}$ c.g.s. units?* (*C. U. 1952*)

Ans. : $g = G\frac{M}{R^2} = G.\frac{\frac{4}{3}\pi R^3.\rho}{R^2} = G.\frac{4}{3}\pi.R.\rho$

$$= 6{\cdot}658 \times 10^{-8} \times \tfrac{4}{3} \times 3{\cdot}14 \times (6{\cdot}37 \times 10^8) \times 7{\cdot}86$$
$$= 1396 \text{ cms./sec.}^2$$

(5) *Assuming the earth to be a sphere of uniform density 5·5 gms./c.c. and of radius* $6{\cdot}4 \times 10^8$ *cms., calculate the value of g on its surface, given* $G = 6{\cdot}7 \times 10^{-8}$ *c.g.s. units.*

Ans. : $g = G.\frac{M}{R^2} = G.\frac{4}{3}.\pi\frac{R^3.\rho}{R^2} = G.\frac{4}{3}\pi R.\rho$

$$= (6{\cdot}7 \times 10^{-8}) \times \tfrac{4}{3} \times 3{\cdot}14 \times (6{\cdot}4 \times 10^8) \times 5{\cdot}5$$
$$= 987{\cdot}7 \text{ cms./sec.}^2.$$

(6) *If the moon's distance from the earth is 240 000 miles, the earth's radius is 4000 miles, g at the surface of the earth is 32·2 ft./sec.² find the time taken by the moon to complete one turn in its orbit round the earth.* (*log* $\pi = 0{\cdot}4972$)

Ans. : Let M = mass of the earth,

R = Radius of „ „ $= 4000 \times 1760 \times 3$ ft.
R_1 = „ of „ moon's orbit $= 240{,}000 \times 1760 \times 3$ ft.
g = acceleration due to gravity = 32·2 ft./sec.² on the earth's surface.
g_1 = acceleration at the moon, and
v = linear velocity of the moon in its orbit.

Now $g = G\frac{M}{R^2}$; or, $GM = gR^2$.

Again $g_1 = \frac{v^2}{R_1} = G\frac{M}{R_1^2}$; or, $v^2 = \frac{G.M.}{R_1^2} \times R_1 = \frac{gR^2.R_1}{R_1^2} = gR_1.\left(\frac{R}{R_1}\right)^2$.

But $T^2 = \left(\frac{2\pi R_1}{v}\right)^2$, where T = time period of rotation of the moon

$$= \frac{4\pi^2 R_1^2}{v^2} = \frac{4\pi^2.R_1^2}{gR_1} \times \left(\frac{R_1}{R}\right)^2 = 4\pi^2.\frac{R_1}{g}.\left(\frac{R_1}{R}\right)^2 \text{ sec}^2$$

$= \dfrac{4\pi^2\frac{R_1}{g}\left(\frac{R_1}{R}\right)^2}{60^2 \times 60^2 \times 24^2}$ sq. days = 748·9 sq. days, on putting values of π, R_1, R and g. ∴ $T = 27{\cdot}36$ days.

(7) *The radius of the moon's orbit is 240,000 miles and the period of revolution 27 days, the diameter of the earth is, 8,000 miles, and the value of gravity on its surface is 32 ft./sec.². Verify the inverse square law from it.*

Ans. : If G = Gravitational Constant and M = mass of the earth, R = radius of the earth, R_1 = radius of the moon's orbit round the earth, v = velocity of the moon in ft./sec., and T, its time period,

normal acceleration of the moon $= \frac{v^2}{R_1} = \frac{GM}{R_1^2}$; or, $GM = v^2R_1$.

Acceleration due to gravity, on the surface of the earth, g, is given by

$$g = \frac{GM}{R^2} = \frac{v^2R_1}{R^2} = \frac{\left(\frac{2\pi R_1}{T}\right)^2 R_1}{R^2} = \frac{4\pi^2 R_1}{T^2} \times \left(\frac{R_1}{R}\right)^2$$

Put $R = 4000 \times 1760 \times 3$ ft., $R_1 = 240{,}000 \times 1760 \times 3$ ft., $T = 27 \times 24 \times 60 \times 60$ secs.

On calculation, g is found to be 32 ft /sec.^{2} approx. ; this proves the inverse sq. law, as the calculation is done on the assumption of the same.

(8) *How many times faster than the present speed would the earth have to rotate on its axis in order that the apparent weight of bodies at the equator should be zero ? (Radius of the earth = 4000 miles.)*

Ans. : *If* m, ω, v, R are respectively the mass, the angular velocity of rotation, linear velocity and radius of the earth and if the earth has to move n times faster than as at present such that the apparent wt. mg of a body of mass m at the equator should be zero.

$$mg=\frac{m(nv)^2}{R}=\frac{m(n\omega R)^2}{R}=mn^2\ \omega^2\ R.$$

$$\text{or,}\quad n=\sqrt{\frac{g}{\omega^2 R}}=\sqrt{\frac{g}{R}}\times\frac{1}{\omega}=\frac{\sqrt{32\cdot2}}{\sqrt{4000\times1760\times3}}\times\frac{1}{2\pi(24\times60\times60)}$$

$$=17 \text{ approximately.}$$

(9) *A straight tunnel is bored from the surface to the centre of a solid sphere of radius* r *and density* ρ. *Find the amount of works done in carrying a small mass* m *from the centre to the surface.*

Ans. : If the mass m is situated at a distance x from the centre at an instant of time, the force on it is given by, $G\,\frac{4}{3}\frac{\pi x^3.\,\rho\, m.}{x^2}$. The work done ($dW$) in carrying it through an infinitesimal distance dx is given by, $dW=G.\,\frac{4}{3}\frac{\pi x^3\ \rho.m.}{x^2}dx.$

∴ The total work done in carrying it from the centre to the surface, W,

$$\text{is given by,}\quad W=\int_0^r G\,\frac{4}{3}\,\frac{\pi x^3.\rho\, m}{x^2}.\ dx=G.\tfrac{4}{3}.\ \pi.m\,\rho\left(\frac{x^2}{2}\right)_0^r$$

$$=\tfrac{2}{3}.G.\pi.m\ \rho.r^2.$$

(10) *A straight frictionless tunnel is bored through the earth from one point of its surface to another. Show that an object will travel along the tunnel with simple harmonic motion. Hence find the period of the motion if the value of* g *at the surface is 981 cm./sec.*^{2} *and* $r=6{\cdot}38\times10^6$ *metres.* (*cf. B.H.U. 1939 ; cf. Pat.U., 1955 ; C U. 1946*)

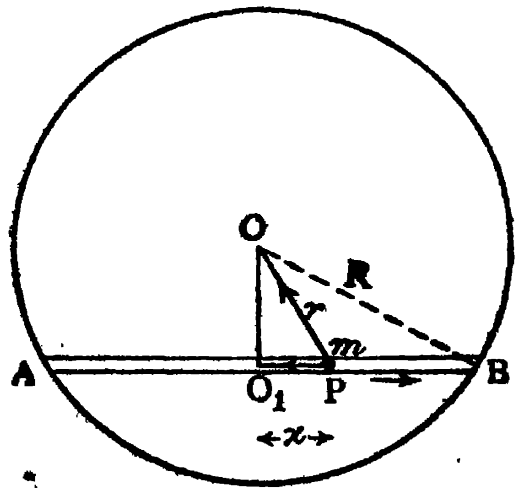


Fig. 154

Ans. : Let the earth be a sphere of radius R and let O be its centre (Fig. 154). Suppose AB represents a straight frictionless tunnel connecting two points, A and B, on the surface of the earth. OO_1 is the perp. drawn from O on to the tunnel. Supposing a body of mass m to be moving in the direction AB along the tunnel, let us consider the state of motion of m at a point P at a distance x ($=O_1P$) from the centre of the tunnel. Let OP be equal to r.

The force acting on m due to its inertia of motion $=m\frac{d^2x}{dt^2}$ (along O_1B) (considering O_1 as the mean position of rest). Again due to gravitational attraction, the force acting on m will be solely due to the inner sphere of radius, r, the shell exterior to it having no attractive influence at P.

This force is equal to $\frac{4}{3}\frac{\pi r^3 \rho . mG}{r^2} = G\,\frac{4}{3}\pi\rho m r$

$$= G.\frac{\frac{4}{3}\pi R^3 \rho m r}{R^3} = \frac{G.Mm}{R^3}\,r$$

$$= \frac{mg}{R}\,r \text{ acting along } PO, \left(\text{since } g = G\,\frac{M}{R^2}\right).$$

The component of this force acting along the tunnel is $\frac{mg}{R}\,r \cos OPO_1$ $= \frac{mg}{R}.\,r.\,\frac{x}{r} = \frac{mg}{R}\,x$ (acting along PO_1).

This component is evidently proportional to the displacement x and acts as the restoring force.

Hence the equation of motion of the body is

$$m\,\frac{d^2x}{dt^2} = -\frac{mg}{R}\,x\,; \quad \text{or,} \quad \frac{d^2x}{dt^2} = -\frac{g}{R}\,x.$$

This is an equation of S.H.M. So the motion of the body in the tunnel is simple harmonic.

Here $\omega^2 = \frac{g}{R}$; or, $\omega = \sqrt{\frac{g}{R}} = \frac{2\pi}{T}$; or, $T = 2\pi\sqrt{\frac{R}{g}}$.

Hence period of motion, $T = 2\pi\sqrt{\frac{R}{g}} = 2\times\pi\sqrt{\frac{6{\cdot}38\times10^8}{981}} = 84{\cdot}4$ min. (approx.)

Note : – If the tunnel be bored through the centre of the earth, the problem is still simpler. Here at a distance r from the centre, the equation of motion of the body will be

$$m\,\frac{d^2r}{dt^2} = -\frac{mg}{R}\,r\,;$$

$$\text{or,} \quad \frac{d^2r}{dt^2} = -\frac{g}{R}\,r$$

and the period is again given by the same expression, $T = 2\sqrt{\frac{R}{g}}$.

(11) *Calculate the period of revolution of a small satellite close to the surface of the earth.*

Ans. : Supposing the orbit of the satellite to be circular and almost of the radius equal to the radius of the earth R, the centrifugal force necessary for its circular motion is $m\omega^2 R$. This is supplied by the gravitational force $G\,\frac{Mm}{R^2}$, where M = mass of the earth and m = mass of the satellite,

$$\therefore \quad m\omega^2 R = G\,\frac{Mm}{R^2}\,; \quad \text{or,} \quad \omega^2 = G\,\frac{M}{R^2}\Big/R.$$

But $G\,\frac{M}{R^2} = g$, the acceleration due to gravity.

$$\therefore \quad \omega^2 = \left(\frac{2\pi}{T}\right)^2 = \frac{g}{R}\,; \quad \text{or,} \quad T = 2\pi\sqrt{\frac{R}{g}}.$$

With the standard values of R and g as given in example (10), $T=84{\cdot}4$ min. (approx.).

(12) The radius of the earth is $6{\cdot}37\times10^8$ cms., its mean density $5{\cdot}5$ gms./c.c. and the gravitational constant $6{\cdot}66\times10^{-8}$ C.G.S. units. Calculate the earth's surface potential. (*Agra U. 1940*)

Ans. : We know, $V=\frac{GM}{r}\cdot G\frac{4}{3}\pi\frac{r^3\rho}{r}=\frac{4}{3}G\pi r^2$

$=\frac{4}{3}\times6{\cdot}6\times10^{-8}\times3{\cdot}14\times(6{\cdot}37)^2\times10^{16}\times5{\cdot}5$

$=62{\cdot}27\times10^{10}$ ergs per gm.

(13) A uniform solid sphere of mass M and radius R is cut in two by a diametrical plane. Show that the resultant attraction between the two havles is $3GM^2/16R^2$.

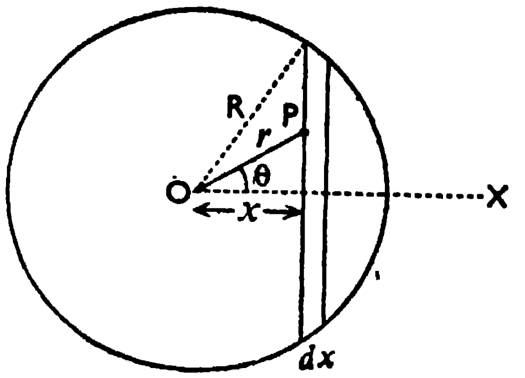


Fig. 154(*a*)

Ans. : Let us consider a strip of thickness dx at a distance x from the centre O of the sphere. The area of the strip $=\pi\ (R^2-x^2)$. If ρ be the density, the mass of the strip is $\pi\ (R^2-x^2)\ dx.\rho$.

At a point P, r cms. from O, the force per unit mass is $G\frac{M}{r^2}$ $=G.\frac{4}{3}\pi r^3\rho/r^2=\frac{4}{3}G\pi r\rho=F$ (say).

Its component along OX is $F\cos\theta=\frac{4}{3}G\pi\rho r.\frac{x}{r}=\frac{4}{3}G\rho\pi x$.

The sine components will cancel out by taking corresponding elements above and below OX. Similar is the case with all point masses on the strip.

$\therefore$ The attraction on the strip along $OX=\frac{4}{3}G\pi\rho x\times\pi(R^2-x^2)dx\rho$.

$\therefore$ Total attraction between the hemispheres $=\frac{4}{3}G\pi^2\rho^2\int_0^R[R^2x-x^3]dx$

$$=\frac{4}{3}G\pi^2\rho^2\left[\frac{R^4}{2}-\frac{R^4}{4}\right]=\frac{1}{3}G\pi^2\rho^2R^4=\frac{3}{16}G\frac{M^2}{R^2}.$$

Questions (Chapter VII)

1. Describe the principle of a compound pendulum and explain how the value of g can be found out with its help. [Utk.U. 1970 *S* ; Luck. U. 1951]

2. Distinguish between a simple and a compound pendulum. For a given pendulum, show that the centres of oscillation and suspension are interchangeable.

How is the value of 'g' determined with the help of a compound pendulum ? [Ag. U. 1948 ; *cf.* Mad. U. 1951]

3. Enunciate and verify the laws of a simple pendulum. Find out an expression for its time period.

What do you mean by a seconds pendulum ? Calculate its length in cms. at a place where $g=981$ cms./sec.^{2}. (log $\pi=0{\cdot}4972$) [*Ans.* 99·88 cms.]

4. Fully discuss and explain the causes of variation of 'g' (or the weight of a body) on the surface of the earth.

Does g vary with the altitude of a place ? If so, explain how.

5. How can you relate the value of g at a place with the distance of that place from the centre of the earth ? Graphically represent this relation and explain the graph.

6. Describe various methods of determining 'g' in short.

7. A clock pendulum swings with a period of 1 sec. where $g=32{\cdot}26$ ft./sec.^{2} How many seconds will it lose in 24 hrs., if it is taken to a high mountain where $g=32{\cdot}10$ ft./sec.^{2} ?

[*Ans.* : 214 secs.] (Gau. U. 1958)

8. (*a*) Show that in a compound pendulum there are four points collinear with the centre of gravity having the periodic time about them equal.

[B.H.U. 1948 ; Mad. U. 1948]

(*b*) Give the general theory of compound pendulum and apply it to the determination of the value of 'g' by Kater's pendulum.

Discuss its merits and demerits. (C. U. 1968)

9. How by means of a bar pendulum you would determine the intensity of gravity ? Give details of your determination.

10. Give the theory of Kater's pendulum and find an expression for the acceleration due to gravity in terms of two nearly equal periods of oscillation about the two parallel knife-edges. Indicate the sources of error in an experimental determination of g. (Bom. U. 1940, '41 ; Pun. U. 1948)

11. A compound pendulum is formed by suspending a heavy ring of radius 490·5 cms. from of a string. Find its period. (Raj. U. 1950)

[*Ans.* : 6·283 secs.]

12. A heavy spherical bob of diameter 10 cms. is suspended by a very fine wire. If the distance from the point of suspension to the centre of the bob be 1 metre, calculate the length of the equivalent simple pendulum. (Mad. U. 1947)

[*Ans.* : 100·1 cms.]

13. A uniform rod of length 100 cms. can rotate about a horizontal axis through one end. Find the angular velocity which will enable the rod just to make complete revolutions. (Mad. U. 1947)

[*Ans.* : 3·53 radians per sec.]

14. Define centre of suspension and centre of oscillation. Show that in a compound pendulum they are interchangeable.

What is the distance between the centre of suspension and the centre of oscillation of a uniform cylindrical metal bar used as a seconds pendulum ? (Diameter of the bar $=1$ cm., its density $=8$ gms./c.c. and $g=978$ cms./sec.^{2})

[*Ans.* : 99·19 cms.] (All. U. 1949)

15. Show that if the earth were not rotating about its own axis, the value of g at the equator would exceed its present value by 3·36 cms./sec.^{2} [Radius of the earth $=6{\cdot}37\times10^8$ cms. and angular velocity of earth $=7{\cdot}27\times10^{-5}$ rads /sec.]

[*Hints* : See Art. 117(c) *ii*.] (Agra U. 1959)

16. Show that when a rigid body is allowed to swing about a fixed horizontal axis, the motion is the same as that of a simple pendulum of a suitable length.

A uniform square lamina of side $2a$ is hung up by one corner and swings in its plane, which is vertical. Find the length of the equivalent simple pendulum.

[*Ans.* : $4\sqrt{2}a/3$]

17. A thin uniform bar of length 120 cms. is made to oscillate about an axis through its end. Find the period of oscillation and other points about which it can oscillate with same period. (Punj. U. 1953)

[*Ans.* : 1·8 secs. ; other end and 80 cm. from each end]

18. Describe one of the most accurate experiments for determining the mean density of the Earth.

19. Outline a method for determining the mass of the earth.

Assuming that a sphere of mass 40 kilos is attracted to a second sphere of mass 80 kilos when their centres are 30 cms. apart, with a force equal to the weight of $\frac{1}{4}$ mgm., calculate the constant of gravitation. ($g = 981$ cms./sec.^{2})

[*Ans.* : $6{\cdot}898 \times 10^{-8}$ C.G.S. units]

20. Find an expression for the gravitational potential at a point on the axis of a uniform disc of radius a cm. and of surface density ρ gms./cm.^{2} and hence find the gravitational field along the axis.

21. State Kepler's laws of planetary motion and deduce the third law from the law of gravitation assuming the planetary orbits to be nearly circular.

22. Define the Newtonian constant of gravitation. Assuming the earth to be a sphere of uniform density, explain how the intensity of gravity varies in passing from the surface of the earth to the centre.

Describe one laboratory method by which the constant of gravitation has been accurately measured. [Lond. B. Sc.]

23. Give an account of any experiment for finding the value of gravitational constant G and show how from this and other known quantities, the mean density of the earth can be found out. (*cf.* Utk. U. 1970. S ; C.U. 1943)

24. Give statement of Newton's law of gravitation and define Newtonian gravitational constant. Show that the gravitational intensity due to a uniform spherical shell of matter at a point inside is zero and write down a simple expression for the intensity at a point outside. [Lond. B. Sc.]

25. (*a*) What is gravitational constant ? Describe an experiment to find the mass of the earth. [Utk. U. 1951 ; Pat. U. 1949]

(*b*) Describe an accurate experiment for determining the Newtonian constant of gravitation. (C.U.H. 1968)

26. What are meant by the intensity of gravity and gravitational potential ? Deduce an expression for the gravitational potential at a point (*i* inside a homogeneous solid sphere [Pat. U. 1948]. (*ii*) outside the sphere. [C.U. 1942]

27. Assuming the earth to be a homogeneous sphere of radius $6{\cdot}37 \times 10^6$ metres and the acceleration due to gravity at its surface 981 cms./sec.^{2}, calculate the mass of the earth. (Burd. U. 1963)

[*Ans.* : $59{\cdot}67 \times 10^{26}$ gms.]

28. Given $G = 6{\cdot}7 \times 10^{-8}$, the radius of the earth $= 6{\cdot}4 \times 10^8$ cms. and its mean density is 5·5 gms./c.c., calculate the acceleration of gravity at the earth's surface.

[*Ans.* : 988·3 cms./sec.^{2}] (Pat.U. 1947)

29. Define gravitational potential. Calculate the value of potential due to a closed spherical shell at a point, (*i*) outside, (*ii*) inside the shell.

[Luck. U. 1951 ; *cf.* C.U. 1947]

30. Mention different methods for determining the constant universal gravitation, and describe one which you consider to be the most accurate.

[Punj. U. 1940, '44]

31. Explain how Cavendish determined the value of the gravitational constant. Indicate how from the knowledge of the gravitational constant it is possible to calculate the mass of the earth. (Saug U. 1948)

32. Give the theory of Cavendish's experiment explaining how the density of

the earth is determined. Explain why and how Boys modified the Cavendish's method. [Madras U. 1950]

33. What is meant by the gravitational constant? What are its dimensions? Give an account of the experiments of Cavendish and Boys to determine this constant. [B.H.U. 1945]

34. A small satellite revolves round a planet of mean density 10 gms./c.c., the radius of its orbit being slightly greater than that of the planet. Calculate the time of revolution of the satellite. ($G = 6{\cdot}66 \times 10^{-8}$ C.G.S. units) (Bombay U. 1940)

[*Ans.* : 3761 secs.]

Hints : Here $G\dfrac{Mm}{r^2} = m\omega^2 r$; or, $\omega^2 = \dfrac{GM}{r^3} = G.\dfrac{\frac{4}{3}\pi r^3\rho}{r^3}$;

or, $\dfrac{4\pi^2}{T^2} = G\frac{4}{3}\pi\rho.$ $\therefore$ $T^2 = 3\pi/G\rho.$]

35 Find the gravitational force of attraction exerted between two spheres of mass 1 kg. placed with their centres 20 cms. apart. ($g = 980$ cms./sec.^{2}, mass of earth $= 4 \times 10^{27}$ gms., radius of earth $= 6 \times 10^8$ cms.) (C.U.H. 1955)

[*Ans.* : $1{\cdot}47 \times 10^{-4}$ dynes]

36. Assuming the gravitational constant to be $6{\cdot}6 \times 10^{-8}$ c.g.s., the radius of the earth to be $6{\cdot}37 \times 10^8$ cms. and g to be 981 cms./sec.^{2}, calculate (*a*) the mean density of the earth, (*b*) the time of revolution of a satellite describing a circular orbit, close to the earth's surface. [Burd. U. (modified) 1963]

[*Ans.* : (*a*) 5·57 gms./c.c. ; (*b*) 5064/secs.] (*cf.* C.U. 1963]

37. A sputnik revolves round the earth in a circular orbit of radius 4900 miles under the attraction of the earth alone. Find its speed in miles per hour.

(Radius of earth = 4000 miles, $g = 32$ ft /sec.^{2}.) (Pat. U. 1959)

[*Ans.* : $1{\cdot}602 \times 10^4$ m.p.h.]

38. Calculate the gravitational potential due to a solid sphere of uniform density both at an internal and at an external point. (*cf.* Utk. U. '62 ; C.U.H. 1967)

39. What should be the velocity in Kms. per hour of an earth satellite revolving round the earth at an altitude of 1600 Kms. above the earth along a circular trajectory with the centre of the circle coinciding with the centre of the earth, which may be assumed spherical?

(Given radius of earth = 6400 kms.; mass of earth $= 6 \times 10^{27}$ gms. and $G = 6{\cdot}7 \times 10^{-8}$ c.g.s. units)

[*Ans.* : $2{\cdot}5 \times 10^4$ kms./hr.]

CHAPTER VIII

ELASTICITY

144. Introduction :—The bodies we see around us may be divided into two broad categories namely solids and fluids. All solid bodies are not similar in their external physical behaviours. Some of them are called rigid only within a limited sense of the term, because the physical definition of a rigid body is never satisfied by any substance in existence. "A *rigid body* is an extended piece of matter which can move as a whole with reference to the surrounding objects but whose component particles are incapable of any displacement relative to one another." As a matter of fact, nobody is rigid according to the above definition, but substances such as steel, glass etc. are nearly so as long as small forces act on them.

Liquids and gases are included under the general classification of fluids. A *perfect fluid* is defined as a deformable body which possesses no definite shape, so that alterations in form unaccompanied by any changes in volume require no work. Thus such a body can stand no tangential stress and, for it, the pressure is also independent of direction.

A perfectly rigid body and a perfect fluid have only theoretical existence and can never be realised in practice. According to definition the former should not yield to any deforming force and the latter should be deformed by the slightest tangential force.

As perfectly rigid bodies are never realised in practice, the various material bodies we see around us are always deformed under the action of external forces. This is because the constituent particles undergo a kind of relative displacement with respect to one another and as a result of this relative displacement a kind of deformation sets in, altering the very shape or the size or sometimes the both of the body subjected to this deforming force. According to their behaviour both when this deforming force acts as well as when it is removed, the various bodies may be divided into two groups : (i) A body is said to be **perfectly elastic** if it completely recovers its original shape and size when the forces causing the alterations are withdrawn. Bodies perfectly elastic within a certain limiting value of the deforming force, are not so when subjected to larger deformations and as such the meaning of the term 'perfectly elastic' is not absolute. Thus every material body is elastic but never perfectly so. When bodies are described as perfectly elastic, they are meant to be so within a particular limit only, called the **elastic limit** of the material for the particular nature of the deforming force used. (ii) A **perfectly plastic** body is one which can wholly retain its altered shape and size even when the forces producing the alterations are removed. No material body is found to be perfectly plastic.

145. Strain and Stress

Strain.—When a force or a system of forces acting upon a body produces a relative displacement between its parts, a change in size or shape or in both may take place. The body is then said to be under *strain*. The strain produced in a body is measured in terms of the *change* in some measure of the body, such as its length, or volume and so on, divided by the total measure. Strain is thus the ratio of two like quantities, and is a pure number without dimensions and has no *unit* for it.

Stress.—When a body is strained, internal forces of reaction are automatically set up within the body, which act in the opposite direction, due to which the body tends to return to its original size and shape on withdrawal of the deforming forces. This restoring force is called the *stress*. It is numerically equal to the deforming force, according to Newton's law of reaction, so long as the strain produced is within the elastic limit.

The stress or a component of it, which acts normally to any section of a body, is called **normal stress** to that section and that stress or a component which acts parallel to any section of the body, is called a **tangential stress** on that section.

The **stress intensity** or simply the **stress** *is measured by the force per unit area of a section and, when uniform, is obtained by dividing the total force by the total area over which it acts.* Hence stress $p=\frac{P}{\alpha}$ where P is the total force acting on a surface of total area α. The stress is expressed in dynes per cm.^{2} in c.g.s. units. Its dimensional formula is $ML^{-1}T^{-2}$.

146. Different Kinds of Strain :—

(i) **Longitudinal** (or **Tensile**) **Strain.**—When a body is acted on by a stretching (or compressive) force, the **fractional** *increase (or decrease) in the length* in the direction of the force is called longitudinal or tensile strain. The corresponding stress is called longitudinal or tensile stress. Thus, if due to stretching or compression, l = the change in length of a body of length L,

$$\text{the longitudinal (or tensile) strain}=\frac{l}{L}.$$

It being a ratio of two lengths is a pure number having no unit for its measurement. *Only solids can have such strains.*

Poisson's Ratio.—When a body is acted on by a stretching force, the extension in the direction of the applied force is always accompanied by a lateral contraction in all directions at right angles to the direction of applied force. It is found that this lateral strain is proportional to the direct strain,

i.e. $\dfrac{\text{lateral strain}}{\text{longitudinal strain}} = \sigma$, a constant, called the Poisson's Ratio,

whose value depends only on the nature of the material in question and not at all on the stress applied provided it is within the elastic limit. Poisson's Ratio for a stretching force is the same as that for a compressive force in which case there is *lateral expansion.*

(ii) **Volume (or Bulk) Strain.**—In such strains there is a change in volume only without any change in shape. This takes place when a body is subject to a uniform pressure acting normally at every point on its surface. The corresponding fractional change in volume is called the volume strain. If V be the original volume of a body and v, the change produced in the volume, then,

$$\text{volume strain} = \frac{v}{V}.$$

It is a pure number and has no unit for its measurement. Volume strain, even for very large deforming forces, is small for solids and liquids, while in the case of gases even a very small force produces a very large volume strain.

(iii) **Shearing Strain (or Shear).**—When the strain produced in a body is such that there is only change in shape or form of it but no change in volume, it is said to be a shearing strain or simply a *shear.* It is a *special property of solids* only because they only have a definite shape of their own.

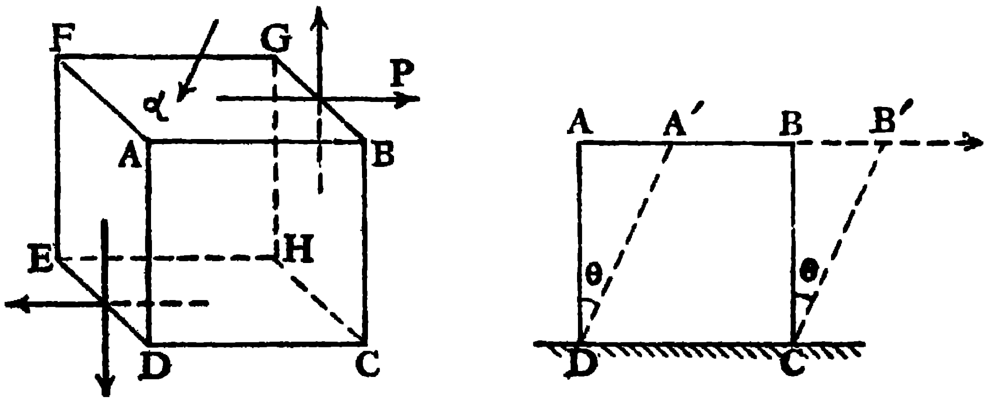


Fig. 155

Suppose of rectangular block $ABCDEFGH$ [Fig. 155 left], of a solid has its bottom face $CDEH$ fixed to a horizontal platform. If a force P be now applied so as to act uniformly and tangentially over the face (area $= \alpha$) in the direction shown, this face (section AB) will be displaced, suppose, to the position represented by the section $A'B'$ in Fig. 155.

right, relative to the face $CDEH$ represented by the section CD, the block assuming a rhombic form. The material of the block suffers a change in shape only without any change in volume. The strain produced in this case is a case of *shear* and is measured by the angle ADA' ($=\theta=$the angle BCB'), which is called the **angle of shear**. Let AA' be x and $AD=b$, then ($\because$ θ is small),

$$\textit{shearing strain} = \theta = \tan\theta = \frac{x}{b}$$

$$= \frac{\text{relative displacement of two planes of the body}}{\text{distance of separation of the two planes}}$$

$$= \text{relative displacement for planes at unit distance apart}$$

$$= \textit{displacement gradient.}$$

The corresponding stress, which is tangential to the surface is called the shearing stress and is given by P/α.

In addition to the above three general types of strain, a body may suffer changes of shape due to

(*a*) *Flexure*—(or bending) as is found in a rod clamped horizontally at one end and loaded at the other ;

(*b*) *Torsion or twist*—as is found in a rod or wire rigidly fixed at one end and twisted at the other.

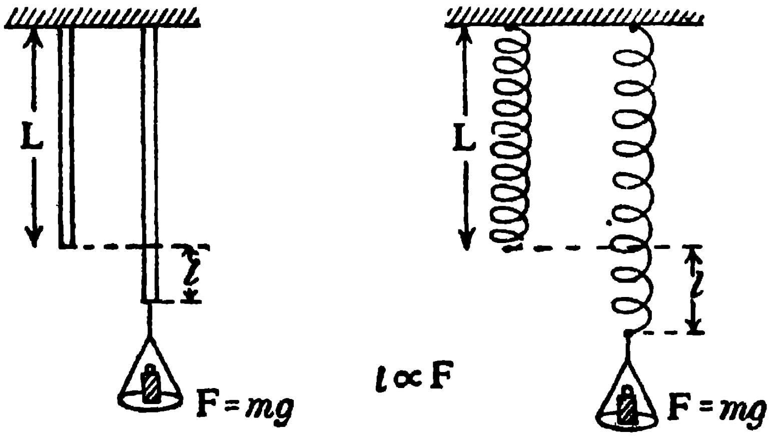
Note : As a variation of temperature causes a change in dimension of a body, the body may very well be supposed to be under a temperature strain and a temperature stress is also developed. So for a true knowledge of the elasticity the temperature effect is to be taken into account.

147. Hooke's Law :—This is the *basic* law of elasticity. It was established in 1678 by Robert Hooke of England.

In the original language the law was stated as '*ut tensio sic vis*', which means that the stretching is proportional to the force producing it. The law is true for all cases of elastic deformations, provided the deformations are small. Some elastic deformations of different kinds in the case of solids, such as stretching, compressing, bending, twisting, etc. are illustrated below.

In **stretching**, two cases of extension, (*i*) of a straight wire (Fig. 156, left), and (*ii*) of a spring (Fig. 156, right) have been depicted under a load m, where for the stretching force, $F=mg$, a length

L has been shown to be extended through l. According to Hooke's law, $l \propto F$.

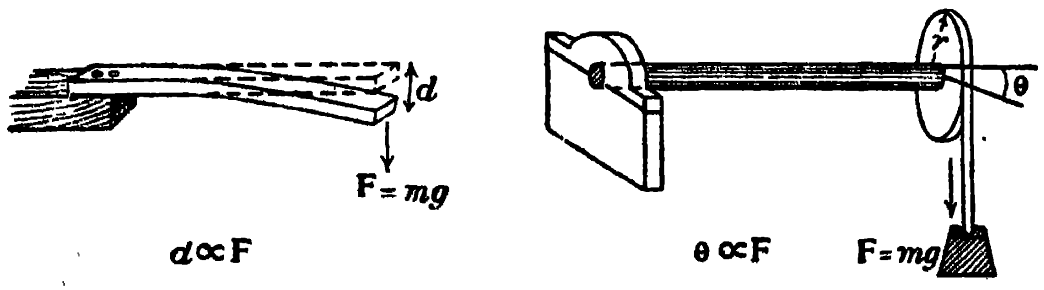


Stretching

Fig. 156

In **bending**, a straight bar (Fig. 157, left) fixed at an end has been shown to be deflected downwards at the other end through a vertical distance a from its original horizontal position due to vertical load exerting a force, F. The fibres of the bar along the top surface are extended while those at the bottom surface are compressed. Somewhere in the bar there is a layer along which there is neither any extension nor compression of the fibres. This layer is usually called the *neutral* layer ; in the unstrained position, the neutral layer is horizontal. According to Hooke's law $d \propto F$.

In **twisting or torsion,** a horizontal rod (Fig. 157, right), clamped at one end and a force applied tangentially at the other producing an angular displacement (*i.e.* twist), θ, has been depicted. According to Hooke's law, the twist θ, is proportional to the torque,



$d \propto F$ *Bending*

$\theta \propto F$ *Twisting*

Fig. 157

or, $\theta \propto (F \times r)$, where $F(=mg)$ is a vertical downward force acting tangentially at a distance r from the axis of the rod. Since r is a constant, $\theta \propto F \propto mg$.

Thomas Young modified the statement of the Hooke's law and it is his definition which is accepted at present as Hooke's law, the meaning of the law remaining the same.

Young's Definition of Hooke's Law.—*Within the elastic limit, stress is proportional to strain.*

That is, within the elastic limit, $\frac{\text{stress}}{\text{strain}} = a\ constant = E$ (say).

This *elastic constant*, E, is called the **Modulus** (or **coefficient**) **of elasticity** of the material of a body. As *strain* is a pure number, the coefficient of elasticity is measured in the same unit as that of stress and is expressed in dynes/cm^2. or lbs. or tons per sq. inch. The dimensional formula for the modulus of elasticity is given by $ML^{-1}T^{-2}$.

The law, as was pointed out before, holds good for all cases of strain, such as tensile, volume, twisting, bending, etc. But for different cases of strain the modulus has a different value and is differently named. Thus,

(*i*) Young's modulus $= \frac{\text{tensile stress}}{\text{tensile strain}} = \frac{P/\alpha}{l/L} = \frac{PL}{\alpha l}$

(*ii*) Bulk modulus $= \frac{\text{volume stress}}{\text{volume strain}} = \frac{p}{v/V} = \frac{pV}{v}$

(*iii*) Rigidity modulus $= \frac{\text{shearing stress}}{\text{shearing strain}} = \frac{P/\alpha}{\theta} = \frac{P}{\alpha\theta}$

(*iv*) **Axial modulus** or the modulus of simple longitudinal extension is defined exactly in the same way as Young's modulus with the proviso that no lateral change can occur to the sides of the specimen.

Here P, α, l, L, p, v, V, and θ indicate the quantities explained in Arts. 145 and 146. The first three elastic constants are generally designated by the letters Y, K and η respectively and the axial modulus by χ.

148. Elastic limit :—A body behaves as a perfectly elastic one only as long as the deforming force acting on it does not exceed a certain maximum value depending on the nature of the substance and the nature of the stress. This limiting value of the stress is called the *elastic limit* of the material of the body for that type of stress.* A high elastic limit is possessed by steel and a low one by lead.

149. General Elastic Behaviour of a solid : Load-Extension graph :—The elastic behaviour of a soild, particularly that of a metal, such as mild steel, when subjected to deforming forces ranging from low values to high values exceeding the elastic limit is well

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*** Various theories have been suggested as to when the elastic limit is reached by a body. Is it when the stress developed attains a limiting value or the strain a definite value for the substance, etc. ?**

illustrated by what is known as a *load-extension graph* as shown in Fig. 158. In obtaining the experimental values for such a graph, a wire of the material may be taken and elongated by hanging *weights* (called the *load*) from it.

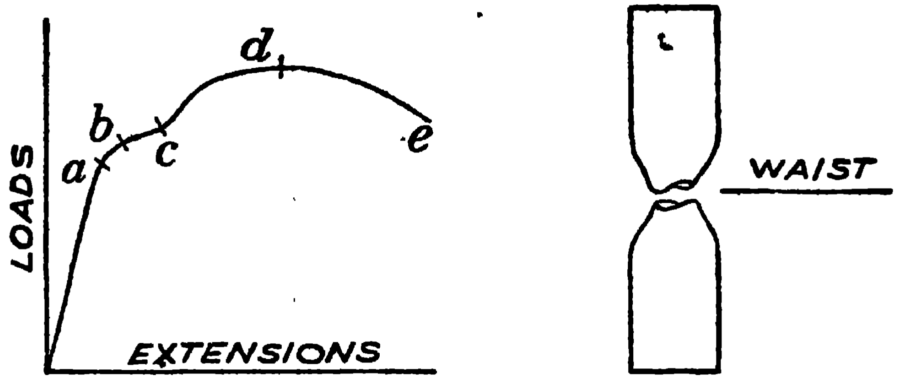

Fig 158

On commencing to load the specimen and noting for each increment of load the extension (change in length from the original) produced, it is found that a straight line is obtained when the loads and extensions are plotted. This straight line continues up to a point a [Fig. 158 left], but beyond a the graph becomes slightly curved. Up to the point a, the load is proportional to the extension. The next point b marks the **elastic limit**[*] for the material, and is so called because if we do not exceed b, the material shrinks back to its original length if the load is taken off and the material has not lost its elastic properties even in the least. The points a and b may, for all practical purposes, be regarded as one and the same point, as indeed they sometimes are, unless very accurate tests are intended. Very soon after the point b, the elastic limit is exceeded and a considerable amount of extension of the material takes place even though the increase of the load on the specimen is quite small. This extension is of a permanent nature and the body at this state is said to acquire a *permanent set* because even after the removal of the deforming stress the body does not regain its original condition. This point c which marks the limit of this stage of the specimen is termed the **yield point**. It is often named the *commercial elastic limit* in commercial testing of materials. After passing the point c, the material seems to regain its strength somewhat, as it is found that further additions of load are required for further extensions to be produced. The **maximum or ultimate load**, from which the *ultimate*

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[*] *N.B.* When a body is strained by an external force *continuously for a long interval of time* it is found that on removal of the external force, the body does not readily regain its original condition even though the applied stress may be within the elastic limit. Sufficient time is required by the body to return to its original state. This lagging property in elastic bodies is known as the *elastic after-effect.*

stress is calculated, is reached at the point *d*, and beyond this the specimen relieves itself of load by rapid stretching, whereby a '*waist*' (right diagram, Fig. 158) or *local contraction* develops at some part of the material where finally the fracture occurs. The position of the material where it will occur, is however, unpredictable. The waist is quite pronounced in ductile materials, and small for brittle ones.

The part of the extension up to the elastic limit *b* is termed elastic deformation, and the remaining part from *b* to *c* is termed non-elastic or *plastic* deformation.

150. Factor of Safety :—Materials with which machines and structures are made are often subjected to stresses, other than normal stresses, which cannot be always predetermined. To avoid failure of structure, designers therefore choose, as a measure of safety, a ***working stress*** for the material, which is much below the ***ultimate stress*** for the material, for design purposes. The ratio $\dfrac{\textit{ultimate stress}}{\textit{working stress}}$ is termed the **factor of safety**. The working stress so taken must also be less than the elastic limit stress so that no permanent deformation may take place in any part of the material.

151. Relations between Elastic constants :—The various elastic constants namely Young's modulus (Y), bulk modulus (K), the modulus of rigidity (η) and the Poisson's ratio (σ) are actually inter-related with one another. Hence it is possible to find out expressions connecting these different constants of elasticity. But before proceeding on to deduce these relations, we should first prove an important theorem on shear which will be helpful in subsequent deductions.

(a) A shear is equivalent to a compression and an extension at right angles to each other.—

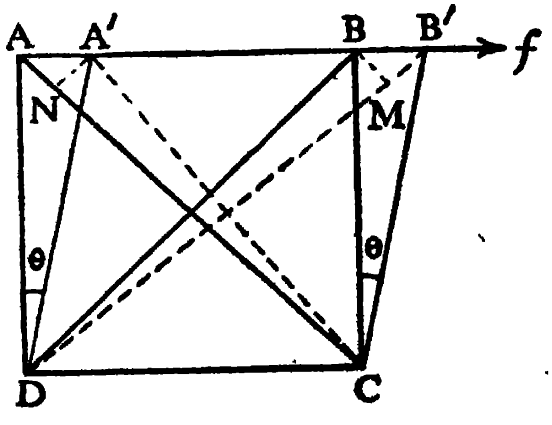


Fig. 159

Consider a cube *ABCD* (Fig. 159) whose face *DC* is fixed to a plane. Let a force *f* be applied along the face *AB*. *ABCD* is then distorted into the form *A'B'CD*. The angle θ through which the face *AD* or *BC* has been displaced is evidently the shearing strain. In this displaced position the diagonals *AC* and *BD* take their new positions *A'C* and *B'D*. *BM* is drawn perpendicular to *B'D*.

∴ The fractional extension along *DB* is given by

$\frac{MB'}{DM}=\frac{MB'}{DB}$(approximately) because $DB \approx DM$.

$$\therefore \quad \frac{MB'}{DM}=\frac{BB' \cos BB'M}{BC/\cos DBC}=\frac{BB'/\sqrt{2}}{BC \times \sqrt{2}}$$

$$(\because \quad \angle DBC = 45^\circ \approx \angle BB'M)$$

$$=\frac{1}{2}\cdot\frac{BB'}{BC}=\frac{1}{2}\tan\theta=\frac{\theta}{2} (\because \theta \text{ is small}).$$

Similarly, the fractional contraction of the other diagonal AC is $\frac{AN}{NC}=\theta/2$. Both these extension and contraction are at 45° to AB or CD and hence they are mutually at right angles.

So the shear θ may be considered to be a combination of an extension $\theta/2$ and a contraction $\theta/2$ perpendicular to the extension.

The converse statement is also equally true.

(b) Relation between Y, K, η and σ.—

Let a force P, acting normally outwards, be applied to each face of a unit cube (Fig. 160). If the force P acting along the X-axis be considered and if e be the extension produced along that direction, then as a unit cube is taken, P and e respectively indicate the stress and the strain along that axis. Hence if Y be the Young's modulus $Y=\frac{P}{e}$; or, $e=\frac{P}{Y}$. Again if σ be the Poisson's ratio, then contractions in the other two directions along y and z-axes are $\sigma . e=\frac{\sigma P}{Y}$, which can be taken to be equivalent to an extension $-\frac{\sigma P}{Y}$.

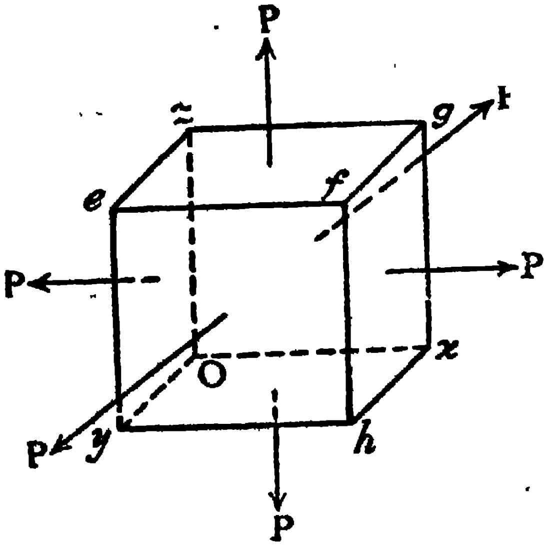


Fig. 160

Thus the force P acting parallel to x-axis produces extensions $\frac{P}{Y}$, $\frac{-\sigma P}{Y}$ (contraction) and $\frac{-\sigma P}{Y}$ (also contraction) along x, y and z-axes respectively. Similarly, the force P acting parallel to Oy produces along the above respective axes extensions $\frac{-\sigma P}{Y}$, $\frac{P}{Y}$ and $\frac{-\sigma P}{Y}$ and the force parallel to Oz, extensions, $\frac{-\sigma P}{Y}$, $\frac{-\sigma P}{Y}$ and $\frac{P}{Y}$.

Thus the total extension along each of the axes $=\frac{P}{Y}-\frac{2\sigma P}{Y}=\frac{P}{Y}(1-2\sigma)$ when the unit cube is simultaneously acted on by tensile force on all its faces. All these forces acting together produce a volume stress of magnitude P which produces a volume strain $=\frac{P}{K}$, where K is the bulk modulus. Now this volume strain is evidently equal to three times the longitudinal strain along each direction.

$\therefore\ \frac{P}{K}=3\times$ longitudinal strain $=3\times$ extension along each of the axes $=\frac{3P}{Y}(1-2\sigma)$.

$$\therefore\quad Y=3K(1-2\sigma) \qquad \ldots \qquad \ldots \qquad . \qquad \ldots \qquad (1)$$

Now if it is supposed that a tensile stress P acts along the y-axis *i.e.* on faces *efhy* and *oxgz*, and that a compressing stress P (along x-axis) acts on faces *fgxh* and *ezoy*, the net extension along x-, y- and z-axes will be respectively,

$$\left(\frac{-P}{Y}-\frac{\sigma P}{Y}\right),\ \left(\frac{P}{Y}+\frac{\sigma P}{Y}\right)\text{and}\left(\frac{-\sigma P}{Y}+\frac{\sigma P}{Y}\right).$$

The net result of applying such tensile and compressional forces together is a contraction $\left\{-\frac{P}{Y}\left(1+\sigma\right)\right\}$ along x-axis and an elongation $\left\{\frac{P}{Y}\left(1+\sigma\right)\right\}$ along y-axis.

This elongation and contraction together may be looked upon as equivalent to a shear θ in the xy-plane. The magnitude of this shear is equal to,

$$\theta=\frac{2P}{Y}\left(1+\sigma\right).$$

$$\text{Hence}\quad \eta=\frac{P}{\theta}=\frac{P}{2P(1+\sigma)/Y}=\frac{Y}{2(1+\sigma)} \qquad \ldots \qquad \ldots \qquad (2)$$

$$\text{or,}\quad Y=2\eta(1+\sigma) \qquad \ldots \qquad \ldots \qquad \ldots \qquad \ldots \qquad (3)$$

Now combining (1) & (3),

$$\frac{9}{Y}=\frac{3}{\eta}+\frac{1}{K}\ ;\ \text{or, } Y=\frac{9\eta K}{3K+\eta}\ \text{and}\ \sigma=\frac{3K-2\eta}{6K+2\eta}.$$

Note :

Since $2\eta\,(1+\sigma)=3K\,(1-2\sigma)$.

$$\frac{1+\sigma}{1-2\sigma}=\frac{3K}{2\eta}=\text{is a+ve quantity since both } K \text{ and } \eta \text{ are positive.}$$

$$\therefore\ \frac{1+\sigma}{1-2\sigma} \text{ is always positive.}$$

But when $\sigma<-1$, $\frac{1+\sigma}{1-2\sigma}$ becomes $-$ve and also when $\sigma>\frac{1}{2}$, $\frac{1+\sigma}{1-2\sigma}$ is negative. But these negative values are not acceptable as shown above. Hence σ must lie between $+\frac{1}{2}$ and -1.

152. Work done in Strain :—When a body is under a strain, the stress developed in the body produces a kind of displacement. So work is done in the process of straining. It is fully recoverable in the case of a perfectly elastic body when the corresponding stress is removed. But if strained beyond the elastic limit, the energy cannot be fully recovered—a part of it being lost in the form of thermal energy.

The work done in different types of strains are calculated as follows :—

(a) **Tensile Strain** (*i.e.* stretching).—

Since in a wire undergoing a tensile strain l/L, the Young's modulus, $Y=\frac{P/\alpha}{l/L}$, where P and α are the force and the cross-sectional area respectively, the stretching force, $P=Y\alpha.\frac{l}{L}$. The stretch l increases from 0 to the final value l by infinitesimally small steps of displacement dl due to applied force.

$\therefore$ Work done in the process is

$$W=\int_0^l P.dl=\frac{Y\alpha}{L}\int_0^l l.dl=\frac{1}{2}\frac{Y\alpha l^2}{L} \qquad \ldots \qquad (1)$$

Eqn. (1) can be put as $W=\frac{1}{2}\left[Y\frac{l}{L}\right]\left[\frac{l}{L}\right].\ (L\alpha)$.

where $Y\frac{l}{L}=$ stress (P/α), $\frac{l}{L}=$ strain and $L\alpha=$ volume of the wire,

$\therefore$ $W = \frac{1}{2}$ stress × strain × volume.

Hence work done per unit volume $= (\frac{1}{2}$ stress × strain) ergs per c.c.

(b) Volume Strain.—

If the volume V of a substance be subjected to an external pressure p and the corresponding volume change be v, the bulk modulus is given by,

$$K = \frac{p}{v/V} \qquad \therefore \quad p = \frac{Kv}{V}.$$

Due to the application of this pressure the volume change v takes place by steps of dv.

The work done in each such step

$$= dW = p.\,dv\,;$$

$\therefore$ The total work done for the change in volume by v is

$$W = \int_0^v p\,dv = \frac{K}{V}\int_0^v v\,dv = \frac{1}{2}\frac{K}{V}v^2 \qquad \ldots \qquad \ldots \quad (2)$$

$$\therefore \quad W = \frac{1}{2}.\left(K.\frac{v}{V}\right).\left(\frac{v}{V}\right).\,V = \frac{1}{2} \text{ stress} \times \text{strain} \times \text{volume}.$$

Work done per unit volume $= (\frac{1}{2}$ stress × strain) ergs/c.c.

(c) Shearing Strain.—

In case a body undergoes a shearing strain, the modulus of rigidity, $\eta = \frac{P/\alpha}{\theta}$, where $P =$ the tangential force applied, $\alpha =$ the cross-sectional area (*i.e.* $P/\alpha =$ the stress developed) and $\theta =$ the angle of shear.

Hence $P = \eta\theta\alpha$.

This shear θ, may be supposed to have occurred by steps of $d\theta$, so that the total amount of work done in the process can be represented as $W = \int_0^\theta P.b d\theta$, where $b d\theta =$ linear displacement corresponding to the elementary shear $d\theta$, b being the linear dimension of the body perpendicular to the applied force.

$$\text{Hence, } W = \eta\alpha b\int_0^\theta \theta.d\theta = \tfrac{1}{2}\eta\alpha b.\theta^2$$
$$= \tfrac{1}{2}(\eta\theta).\theta \times (\alpha b)$$
$$= \tfrac{1}{2} \text{ stress} \times \text{strain} \times \text{volume}.$$

$\therefore$ Work done per unit volume $= (\frac{1}{2}$ stress × strain) ergs/c.c.

153. Determination of Young's Modulus of a Wire :—

(i) Vernier Method.—The extension of a wire, when stretched by a load, being very small, a straight vernier may, with advantage, be used as follows, to measure it in determining the Young's modulus.

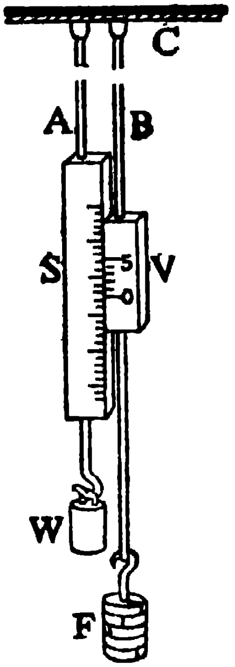


Fig. 161

The equal lengths (Fig. 161) are cut from the same sample of the given wire and are suspended close to each other from the same rigid support *C*. One of them, *A*, called the auxiliary wire, carrying a fixed scale *S* has got a heavy load *W* attached to it to keep it straight and free from kinks. The second wire, *B* (the exprimental wire), carries a hanger on which any desired weight *F* may be placed. A vernier, *V*, attached to the wire *B* can slide along the scale *S*.

First calculate theoretically the ***breaking weight*** of the wire. This is obtained by multiplying the area of cross-section of the wire with the breaking stress for the metal of which the wire is made. In practice, care must be taken never to place on the hanger more than half the 'breaking weight'. This is the maximum permissible weight and stands for the elastic limit.

Now place on the hanger this maximum permissible weight and for a few minutes allow the wire to remain stretched with it. Then remove the greater part of the weight leaving only just enough to keep the wire taut and free from kinks. This may be called the initial load. Take the scale and the vernier readings at this load. This is the initial reading.

Then increase the load by $\frac{1}{2}$ kgm. and again note the reading. Go on increasing the load by steps of $\frac{1}{2}$ kgm., and note the reading for each load up to the maximum permissible load. This is the point beyond which Hooke's law does not hold. Now reduce the load by equal amounts (*i.e.* by $\frac{1}{2}$ kgm.) till the initial load is reached, noting the reading in each case. Take the mean of the readings for increasing and decreasing load for each load. This mean gives the probable real reading corresponding to that load. The two sets of readings should closely agree, but if they differ appreciably, it is possible that the wire has been stretched beyond the elastic limit, in which case the experiment should be repeated with a new wire.

Measure the diameter of the wire very accurately with a micrometer screw gauge at several places along the length of the wire (from the point of support to the zero of the vernier) and in doing so reading should be taken twice in right-angled directions at each place. Find out the mean radius r from the above.

Tabulate the readings against the corresponding loads and find out the elongations for the various loads by subtracting the reading corresponding to each load from the initial reading. Then plot a curve with loads as abscissæ and the corresponding elongations as ordinates (Fig. 162). The graph should be a straight line passing through the origin, meaning thereby that the elongation is zero for zero load. Hooke's law is verified, if the graph is a straight line.

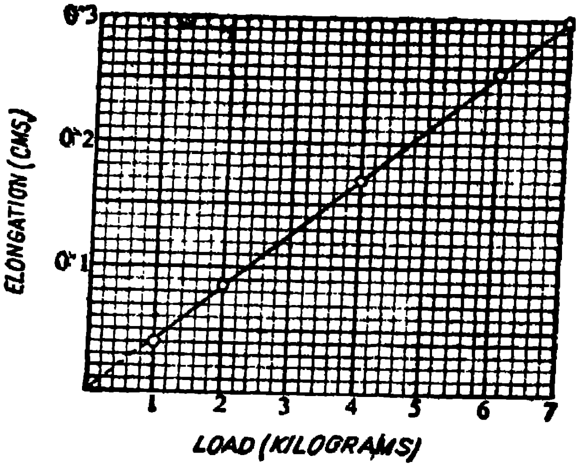


Fig. 162

From the graph find out the elongation l corresponding to any suitable load, say, m grams. Measure the length L of the unstretched experimental wire from the point of suspension up to the point where the vernier is attached. Then,

$$\textbf{Young's Modulus} = \frac{F/\pi r^2}{l/L} = \frac{mg/\pi r^2}{l/L} \text{ dynes/cm}^2.$$

Note.—(*a*) As the wires hang from the same support, any yield of the support will affect both the wires similarly and so there will be no relative motion of V over S due to this cause.

(*b*) As the two wires are made of the same material, any variation of temperature will affect both the wires by equal amounts and so the readings will not be affected.

(ii) Searle's Method.—This method and the Vernier Method are exactly identical except in the process of measurement of the extension in length. A straight vernier is used in the Vernier Method to measure the extension of the wire when stretched, while in Searle's apparatus a micrometer it adapted for the same purpose. The accuracy reached in this latter method is greater since a micrometer screw gauge is more accurate than a straight vernier.

In Searle's apparatus (Fig. 163) each of the two wires, the ***comparison wire*** A and the ***experimental wire*** B, carries a brass rectangle from the lower ends of which weights can be hung. A spirit level L is put across from one rectangle to the other. It turns freely round a hinge G at one end and the other end rests on the point of an accurately cut vertical screw C of small pitch, working in the same vertical line as the experimental wire B. The screw carries at its lower end a cylindrical head H whose circular edge is uniformly divided and forms a circular scale. The pitch of the screw is usually 1 *mm* and the circular scale is divided into 100 divisions. As the head of the screw is turned, the circular scale moves across a short vertical scale S. Thus, if the head is turned through 1 circular scale division, the point of the screw moves upwards or downwards through 0·01 *mm*.

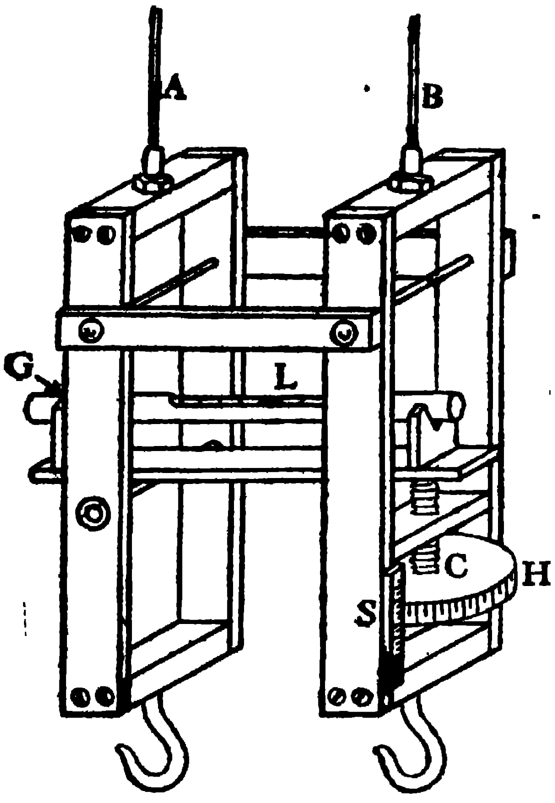


Fig. 163—The Searle's Apparatus.

The method of use of the apparatus is as follows. Initially some dead loads are to be hung from both the rectangles so as to keep the wires tout and free from kinks. The air bubble in the spirit level is brought to the middle of it by turning the cylindrical head and a reading is taken by the help of the linear and the circular scales. Now, when a given weight is hung from the experimental wire B, it extends downwards and the setting of the spirit level will be disturbed. Slowly and carefully the screw is turned to raise its point till finally the bubble is brought to the middle again. The number of circular scale divisions through which the head of the screw is turned, is used to calculate the extension of the wire under a given load.

Examples

(1) *If the Young's modulus of steel be 21×10^{11} dynes/cm.^{2}, what load is required to elongate a wire 10 metres long by 2·5 cms., the diameter of the wire being 2 mms.?*

Ans.: Young's modulus, $Y=\dfrac{P/\pi r^2}{l/L}$

Here $Y=21\times10^{11}$ dynes/cm^2.; P ?; $r=\cdot2/2=\cdot1$ cm.; $l=2\cdot5$ cms. and $L=10$ metres $=1000$ cms.

Hence $P=\dfrac{21\times10^{11}\times2\cdot5\times3\cdot14\times\cdot01}{10^3}$ $1\cdot648\times10^8$ dynes $=168$ kg.-wt.

(2) *Calculate the work done in producing an extension 2 mms. in a steel wire of length 3 metres and diameter 1 mm. Y for steel* $= 2\times10^{12}$ *dynes/cm.*^{2}].

Ans. : Work done, $W=\frac{1}{2}\frac{Y\alpha l^2}{L}$.

Here $Y=2\times10^{12}$ dynes/cm^2 ; $\alpha=\pi\times(\cdot05)^2$ sq. cm. ; $l=\cdot2$ cm. and $L=300$ cms.

$$\therefore\quad W=\frac{\frac{1}{2}\, 2\times10^{12}\times3\cdot14\times(\cdot05)^2\times\cdot04}{2\times300}=1\cdot047\times10^6 \text{ ergs.}$$

(3) *A metal rod having coefficient of linear expansion* $=1\cdot2\times10^{-5}$ *per °C has its temperature raised by 10° C. Calculate the linear compressive stress required to prevent expansion of the rod assuming* $Y=2\cdot0\times10^{12}$ *dynes/cm.*^{2} (C.U. 1955)

Ans. : Let l be the original length of the rod.

Increase in l due to rise in temperature is $dl=l\alpha t=l\times1\cdot2\times10^{-5}\times10$.

$$\therefore\quad \text{Longitudinal strain developed}=\frac{dl}{l}=1\cdot2\times10^{-4}.$$

Since $Y=\frac{\text{longitudinal stress}}{\text{longitudinal strain}}$, the compressive stress required to prevent the above expansion $=Y\times\text{strain}=2\cdot0\times10^{12}\times1\cdot2\times10^{-4}=2\cdot4\times10^8$ dynes/cm.^{2}

154. Torsion of a Cylinder (or a wire) :—When one end of a metallic cylinder (or wire of circular cross-section) is fixed and the other end is twisted through an angle about its axis the cylinder undergoes a torsional strain.

Consider a metallic wire of cylindrical shape (Fig. 164) fixed at the upper end O and twisted by means of a couple (shown by the arrow G) whose axis coincides with the axis of the wire. Due to the shearing strain produced in the wire, shearing stresses will be set up within it and these will give rise to a restoring couple, whose axis also coincides with the axis of the cylinder.

To calculate the moment of this restoring couple, consider an annular ring at O_1 distant l from O and having a radius x and width dx. If now due to the twist, an element on this ring at P is displaced to P_1 through an angle β, the linear displacement is $PP_1=x\beta$.

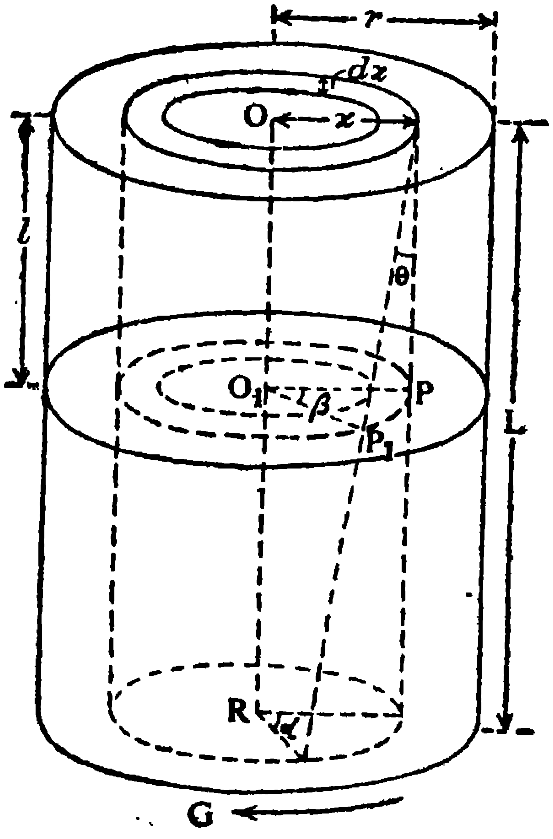


Fig. 164

$\therefore$ Shearing strain $= \theta = \tan\theta = \frac{PP_1}{l} = \frac{x\beta}{l}$. ($\because$ θ is very small).

$\therefore$ The shearing stress developed $= \eta\frac{x\beta}{l}$ where $\eta =$ modulus of rigidity.

So the shearing force for the whole annular ring

$= \eta\frac{x\beta}{l} \times$ area of the ring $= \eta\frac{r\beta}{l}$, $2\pi x.dx = 2\pi\eta\beta\,\frac{x^2dx}{l}$.

The moment of this shearing force about the axis OO_1 $= 2\pi\eta\frac{\beta x^2 dx}{l} \times x =$ the moment of the restoring couple about OO_1 at O_1.

$\therefore$ The moment of the restoring couple for the cylinder of length l and radius $r = \int_0^r \frac{2\pi\eta\beta}{l}x^3dx = \frac{\pi\eta\beta r^4}{2l}$.

This is a case of pure shearing strain and as such the angular displacement α at the distance L from O is given by $\alpha/L = \beta/l$ where α and L refer to the lowest end of the wire.

$\therefore$ At the lowest point of the wire, the moment of the restoring couple about OR at the point R is given by, $G = \pi\eta\frac{\alpha r^4}{2L}$, where α is in radians.

155. Determination of the Modulus of Rigidity of a wire :—

(a) Statical Method.—(By Barton's Apparatus).—

Barton determined the modulus of rigidity of a wire by applying a known couple at the lower end of a vertical wire. In this apparatus (Fig. 165) the wire under test is fixed to a rigid support O at the top and carries a massive metallic cylinder M at the bottom. This heavy cylinder acts as a flywheel and keeps the wire taut. Two strings ss attached to a pin c on the cylinder are wound on M in the same direction and are then made to pass over two pulleys PP fixed on the frame at diametrically opposite points with respect to M. Two scale pans are attached to the free ends ss such that when equal weights (each equal to mg) are placed on both, the strings would unwind and rotate the cylinder in the same sense. The external *terminal couple* applied by this device $= mg \times d$ where $d =$ the diameter of the cylinder. Due to this couple the wire will be twisted, the amount of this twist α at a distance L from the top being measured accurately by a pointer T and scale C arrangement at the point R of the wire.

Since in the equilibrium position, the moment of the restoring couple at R = the moment of the applied couple, the working formula can be easily derived by equating these two couples,

$$\therefore \quad \frac{\pi\eta\alpha r^4}{2L} = mgd, \text{(where } \alpha \text{ is in radians).}$$

Hence, if α be in degrees,

$$\eta = \frac{360\ gdL}{\pi^2 r^4}\left(\frac{m}{\alpha}\right).$$

In actual measurements d and r are measured respectively by a slide callipers and a screw gauge, weights (mg) are evidently the standard weights put on the pan, and L is measured by a beam compass.

In practice different value of α are plotted along y-axis against the corresponding values of m and from the straight line graph so obtained, any arbitrary value of (m/α) is chosen for the computation of η.

Note.—When the metallic cylinder is not of the form of a wire but is a stout cylindrical rod, a horizontal type of the torsion apparatus should be used. The apparatus is, in principle, similar to the vertical type described above. Here the rod is fitted in a horizontal plane with one of its ends firmly secured in a metal block and the other attached to a massive fly-wheel. A string (or better a metal ribbon) is wound round this fly-wheel and to the free end of the string is attached the scale pan and the weights. Thus a terminal twisting couple can be applied. The twist at any position of the rod (*i.e.* at any distance from the fixed end) can be read out by a pointer and a vertical circular scale fitted at that position. The working formula will be similar to that deduced excepting that here the arm of couple will be equal to the radius of the fly-wheel and not its diameter.

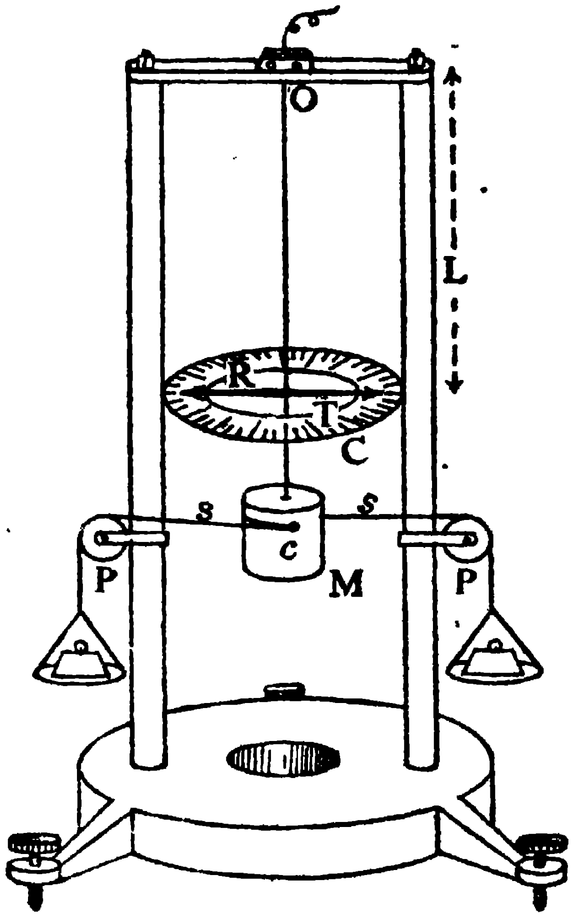


Fig. 165

(b) Dynamical method (Torsional Pendulum).—

The wire under test is attached at the top to a rigid support O (Fig. 166) and is attached to a massive metallic cylinder M at the

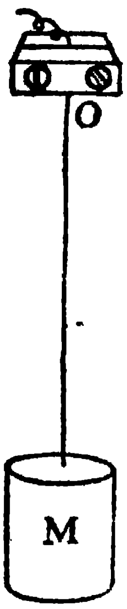


Fig. 166

bottom. The cylinder keeps the wire taut. Giving a slight rotation to M the system is made to oscillate about the wire as the axis.

If α be the angle of twist imparted to the wire, then the restoring couple set up within the wire $=\eta\frac{\pi r^4 \alpha}{2L}$, where L is the length of the wire, r=radius of the wire and η=modulus of rigidity.

The angular acceleration of the oscillating system $=\frac{d^2\alpha}{dt^2}$. Now if I be the moment of inertia of the cylinder M about the wire, the deflecting couple $=I\frac{d^2\alpha}{dt^2}$.

Hence $I\frac{d^2\alpha}{dt^2}+\eta\frac{\pi r^4}{2L}\alpha=0$; or, $I\frac{d^2\alpha}{dt^2}+\mu\alpha=0$,

(where $\mu=\eta\frac{\pi r^4}{2L}$).

It is an equation of a simple harmonic motion whose time period is given by,

$$T=2\pi\sqrt{I/\mu}\ ;\ \text{or, } \mu=4\pi^2 I/T^2.$$

$$\therefore\quad \eta\frac{\pi r^4}{2L}=4\pi^2 I/T^2\ ;\quad \text{or, } \eta=\frac{8\pi L I}{T^2 r^4}.$$

In actual determination of η by this method, T is determined with the help of a stop watch in the usual way. The diameter of the wire is *very carefully* determined by a screw gauge at different positions, the measurements being taken at each place twice at right angled directions. From the mean value of the diameter, r is obtained. I is determined from the mass and dimensions of the cylinder M. L is measured by a beam compass or a metre scale. Thus all quantities in the expression of η being known, its value can be calculated.

N.B. As r occurs in the expression of η as r^4, a slight error in its measurement will cause a comparatively larger error in the final result. So measurement of r should be very carefully done.

Examples

1. In a particular experiment with Barton's apparatus the following data [illegible] obtained :—

(i) Length of the wire=25·4 cms.; (ii) radius of the wire=0·144 cm. (iii) angle of twist=9°; (iv) diameter of the cylinder=4·38 cms.; (v) mass of each pan=800 gms.

Calculate the modulus of rigidity ($g=980$ cms./sec².) *(Pat. U. 1953)*

Ans.: From Art. 155 (*a*) $\eta=\dfrac{360\,gdL}{\pi^2r^4}\left(\dfrac{m}{\alpha}\right)$

$$=\frac{360\times980\times4{\cdot}38\times25{\cdot}4}{\pi^2\times(0{\cdot}144)^4}\cdot\frac{800}{9}$$

$$=2{\cdot}06\times10^{11}\text{ dynes/cms.}^2$$

2. *A circular disc of 10 cms. radius and mass 1 kgm. is suspended in a horizontal plane by a vertical wire attached to its centre. If the diameter of the wire is 1 mm., its length 1·5 metres and the period of torsional vibration of the disc is 5 secs., find the rigidity modulus of the material of the wire.* *(Bombay U. 1931)*

Ans.: The time period of torsional vibration is,

$$T=2\pi\sqrt{\frac{I}{\mu}},\text{ where }\mu=\frac{\eta\pi r^4}{2L}$$

Hence, $\eta=\dfrac{8\pi LI}{T^2r^4}$. Here the moment of inertia, I, of the disc about the axis of suspension (about an axis passing through the centre and perp. to the plane of the disc) is given by $MR^2/2$, where M=its mass and R=its radius.

$$\text{Thus }\eta=\frac{4\pi LMR^2}{T^2r^4}.$$

Here L=1·5 metres=150 cms., M=1 kgm.=1000 gms., R=10 cms., T=5 secs. and r=·05 cm.

$$\text{So, }\eta=\frac{4\times3{\cdot}14\times150\times10^3\times10^2}{5^2\times(\cdot05)^4}=1{\cdot}206\times10^{12}\text{ dynes/cm.}^2.$$

156. Bending of beams:—The bending of beam under loads is of immense importance in engineering. Moreover, a careful observation of the sag or depression produced in a beam under a known load provides an accurate laboratory method for determining some of the elastic constants of material of the beam.

When a beam (preferably thin) of solid material is bent by an applied couple G, the convex surface AB is stretched while the concave surface CD will suffer a compression (Fig. 167). In this strained condition of the beam, internal reaction of the material sets up forces which tend to straighten it. As the upper filaments of the beam are lengthened and lower ones shortened as shown in Fig. 167 there must exist some intermediate filament NN_1 which would retain its original length. This unstrained filament or layer is known as the ***neutral layer or axis.*** If ρ be the radius of curvature of this neutral layer, then $HO=\rho$. Let EF be another filament at a distance of z from the neutral axis. Then the radius of this filament is $EO=(\rho+z)$. If an elementary strip HK of the neutral layer subtends an angle $\delta\theta$ at the centre of curvature and if E and F in the upper layer be the corresponding points of H and K, then $HK=\rho\delta\theta$ and $EF=(\rho+z)\delta\theta$. So the extension of the upper filament is $EF-HK=z.\ \delta\theta$. Since the initial length of EF in the unstrained state was also equal to $HK=\rho\delta\theta$ the tensile strain produced is evidently $z\delta\theta/\rho\delta\theta=z/\rho$.

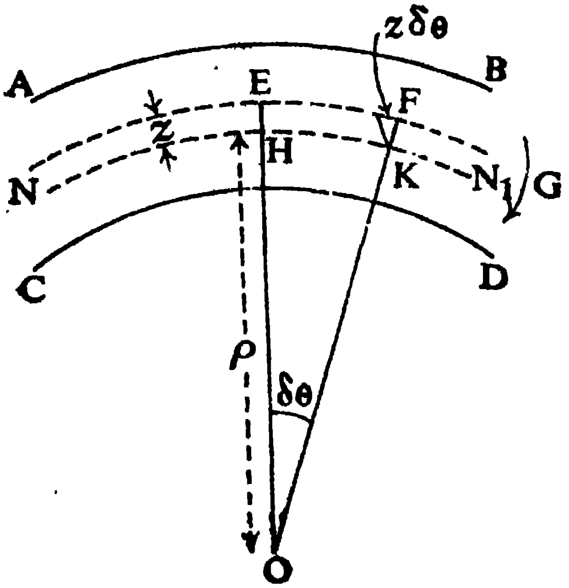


Fig. 167

If Y be the Young's modulus of the material of the beam, then $Y=\dfrac{\text{stress}}{\text{tensile strain}}=\dfrac{\text{stress}}{z/\rho}$. Stress on $EF=Y\frac{z}{\rho}$

Now consider a cross-section of the beam perpendicular to the neutral axis as shown in Fig. 168, nn_1 represents the line of intersection of the neutral layer and the section.

If an elementary area $d\alpha$ be considered on the layer ef (Fig 168) at a distance z from the neutral axis, the force acting on this element $=$ stress $\times\ d\alpha$ $=Y\frac{z}{\rho}d\alpha$. Such forces acting on the different elements of the section try to restore the original state of beam and they combine to form a couple which in

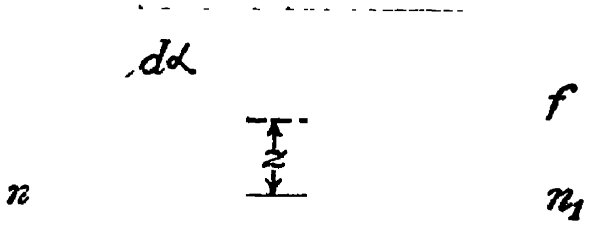


Fig. 168

the equilibrium state balances the applied couple. The magnitude of the internal couple (or internal bending moment) is found by summing up the moments of the elements of forces like $Y\frac{z}{\rho}d\alpha$ about the axis nn_1 of the section. Due to the force acting on $d\alpha$, the couple $=Y.\frac{z}{\rho}d\alpha.z=\frac{Y}{\rho}d\alpha z^2$.

So the total couple or internal bending moment exerted by the stresses developed in all the filaments of the beam is

$$M=\frac{Y}{\rho}\int d\alpha z^2 \qquad \ldots \qquad \ldots \qquad \ldots \qquad (1)$$

This M, as referred to above, is equal to the applied torque G in equilibrium position.

In the expression (1), the integral $\int d\alpha z^2$ is analogous to moment of inertia, dm in the expression of the latter being here replaced by an element of area $d\alpha$. It is generally known as the *geometrical moment of inertia* of the cross-section of the beam. As the $M.I.$ of a body is expressed as MK^2, M and K being the mass and the radius of gyration respectively, so is the geometrical moment of inertia written as AK^2 where $A=$the total cross-sectional area.

Hence we can write $M=\dfrac{YAK^2}{\rho}=G \qquad \ldots \qquad \ldots \qquad (2)$

In the above treatment it has been supposed that the beam is bent into an arc of a circle of radius ρ ; but the theory can be equally applied to general types of bending provided ρ is considered to be the radius of curvature of the neutral layer at the position of the section across which the stress is considered.

157. Cantilever :—A cantilever is a horizontal beam one end of which is rigidly fixed. Suppose AO is a beam whose end A is rigidly fixed or clamped (Fig. 169). It forms a cantilever. For simplicity of calculations let the weight of the beam itself be neglected in comparison with the load, W, applied at the free end and that the thickness of the beam is also very small compared to the radius of curvature of the bent beam.

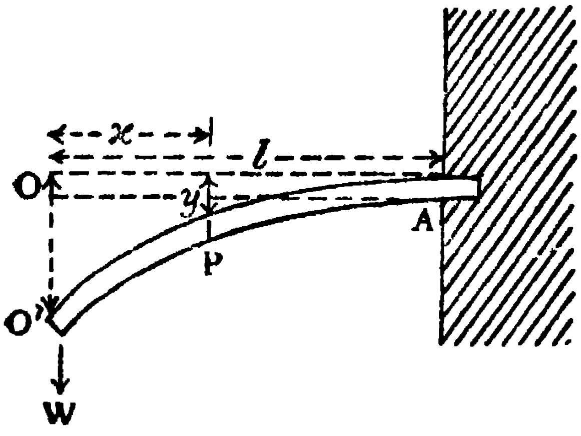


Fig. 169

The beam is actually depressed for two causes. By the shearing strain a part of the depression be accounted for , secondly, due to the tensile strain the bending occurs which causes a further depression. But as the beam is supposed to be of small thickness, the depression due to the shear is negligibly small (see "Note" below). So the whole depression here is supposed to be due to bending.

To calculate this depression, let us consider the undeflected position of AO to be along X-axis, the point O as the origin and the direction of application of the load, W, to be along Y-axis. If P be any point in the bent beam, its co-ordinates may be taken as x and y. From a well-known theorem in differential calculus, we know that the radius of curvature at P is given by,

$$\rho = \frac{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}.$$

Since in a bent metallic beam the slope, $\left(\frac{dy}{dx}\right)$, at P is naturally very small, its square can be neglected.

Hence, $\frac{1}{\rho} = \frac{d^2y}{dx^2}$.

The section PO' of the bent beam is in equilibrium under the action of the applied couple, $G = W \times x$ and the internal bending moment $M = \frac{YAK^2}{\rho} = YAK^2 . \frac{d^2y}{dx^2}$ acting over the section at P.

Hence, $$YAK^2 . \frac{d^2y}{dx^2} = Wx \qquad \ldots \qquad (1)$$

Integrating (1) we get,

$$YAK^2\frac{dy}{dx} = \tfrac{1}{2}Wx^2 + C_1 \qquad \ldots \qquad \ldots \qquad (2)$$

where C_1 is the constant of integration.

At A, where $x = l$, $\frac{dy}{dx} = 0$, ($\because$ the beam is clamped there).

$$\therefore \quad C_1 = -\tfrac{1}{2}Wl^2.$$

$$\therefore \quad YAK^2\frac{dy}{dx} = \tfrac{1}{2}W(x^2 - l^2) \qquad \ldots \qquad \ldots \qquad (2)$$

At the loaded end $x = 0$; so $\frac{dy}{dx} = -\frac{Wl^2}{2YAK^2}$ ... ... 2(a)

Integrating (2) we get,

$$YAK^2y = \tfrac{1}{6}Wx^3 - \tfrac{1}{2}Wl^2x + C_2.$$

Again from the condition of the problem, at A where $x = l$, $y = 0$.

$$\therefore \quad C_2 = \tfrac{1}{2}Wl^3 - \tfrac{1}{6}Wl^3 = \tfrac{1}{3}Wl^3,$$

$$\therefore \quad YAK^2y = \tfrac{1}{6}Wx^3 - \tfrac{1}{2}Wl^2x + \tfrac{1}{3}Wl^3 \qquad \ldots \qquad (3)$$

y in the expression (3) gives the value of the depression due to bending at any point of the beam.

When the depression y_0 at the end is required, since $x = 0$ at O, we can write,

$$YAK^2y_0 = \tfrac{1}{3}Wl^3;$$

$$\text{or,} \quad y_0 = \tfrac{1}{3}\frac{Wl^3}{YAK^2} \qquad \ldots \qquad \ldots \qquad \ldots \qquad (4)$$

In a rectangular beam, the area $A = bd$ (b = breadth and d = thickness) and $K^2 = \frac{d^2}{12}$.

$$\text{Hence,} \quad y_0 = \frac{4Wl^3}{Ybd^3} \qquad \ldots \qquad \ldots \qquad \ldots \qquad \ldots \qquad (5)$$

In the above treatment the weight of the beam has been neglected. If, however, the weight of the beam is taken into consideration, an additional couple is applied due to this weight. It can be shown that if W_1 be the weight of the beam, then

$$Y.AK^2y_0 = \frac{l^3}{3}(W + \tfrac{3}{8}W_1).$$

Notes.—(1) **Effect of Shear**:—If there were no tensile strain and hence no bending and the depression y_s were solely due to shear (in which case the beam in the displaced position would also have been straight), the shearing strain would be y_s/l. The stress is evidently W/A. Hence the modulus of rigidity,

$$\eta = \frac{W}{A}\Big/\frac{y_s}{l} = \frac{Wl}{Ay_s}\,; \qquad \text{or, } y_s = \frac{Wl}{A\eta}.$$

Thus the ratio of the depression (y_s) due to shear and that (y_0) due to bending would be,

$$y_s/y_0 = \frac{Wl}{A\eta} \times \frac{3YAK^2}{Wl^3} = \frac{3Y}{\eta}\cdot\frac{K^2}{l^2}.$$

For thin beams K^2 is small and the ratio $\frac{K^2}{l^2}$ becomes so small that y_s may be easily neglected in comparison with y_0 for such beams.

(2) Interchangeability of the positions of load and depression.—

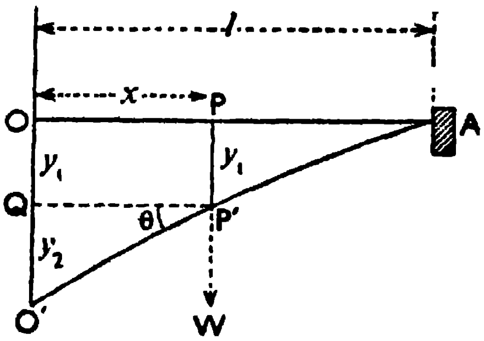


Fig. 169(a)

It can be shown that the position of the load and depression at an arbitrary point in a cantilever are interchangeable, that is to say that if the load is applied at the end and the depression observed at an arbitrary point P at a distance x from the end, then this depression will be identical with the depression at the end when the cantilever is loaded at P.

When the load is applied at the end, the depression at P at a distance x from the free end is given, from (3), as,

$$y_P = \frac{W}{YAK^2}\left[\tfrac{1}{6}x^3 - \tfrac{1}{2}l^2x + \tfrac{1}{3}l^3\right].$$

Again when the load is applied at P, the portion $P'O'$ of the beam does not undergo any bending which continues only upto P'. Since the length $AP = (l-x)$, the depression at P due to the load applied at that point is given from relation (4) as,

$$y_1 = \frac{W}{YAK^2}\left[\tfrac{1}{3}(l-x)^3\right].$$

At P the inclination of the beam is $\theta = \frac{dy}{dx} = \frac{W(l-x)^2}{2YAK^2}$,

from relation 2(a), the negative sign being neglected as it simply gives the sense of inclination. Due to this inclination the free end further sags through a distance $QO' = y_2$ as shown in fig. 169(a), where,

$$y_2 = \theta \times x = \frac{W}{YAK^2}\left[\tfrac{1}{2}(l-x)^2x\right]$$

∴ The total depression at the free end due to the load at P is

$$y_0 = OO' = OQ + QO' = y_1 + y_2$$

$$=\frac{W}{YAK^2}\left[\tfrac{1}{3}(l-x)^3+\tfrac{1}{2}(l-x)^2.x\right]$$

$$=\frac{W}{YAK^2}\left[\tfrac{1}{6}x^3-\tfrac{1}{2}l^2x+\tfrac{1}{3}l^3\right]$$

Hence $y_0=y_p$, that is, the depression at P when the beam is loaded at O is identical with the depression at O when it is loaded at P.

158. Vibration of a loaded Cantilever :—If a cantilever be loaded with a mass M, then $W=Mg$ and if under this load the beam undergoes a displacement y_1, then,

$$y_1=\frac{4Mgl^3}{Ybd^3}\ ;\quad \text{or,}\quad Mg=\frac{Ybd^3}{4l^3}.\,y_1 \qquad \ldots \qquad \ldots \quad (1)$$

Thus the force on the beam is proportional to the displacement y_1. If now the beam be slightly displaced through dy_1 from the equilibrium position, the above force Mg will try to restore it to its initial equilibrium position.

Hence the motion will be simple harmonic in nature and the equation of motion will be,

$$M\frac{d^2y_1}{dt^2}+Mg=0\ ;$$

$$\text{or,}\quad M\frac{d^2y_1}{dt^2}+\frac{Ybd^3}{4l^3}\,y_1=0\ ;$$

$$\text{or,}\quad \frac{d^2y_1}{dt^2}+\frac{Ybd^3}{4Ml^3}\,y_1=0\ ;$$

$$T=2\pi\sqrt{\frac{4Ml^3}{Ybd^3}}=2\pi\sqrt{\frac{y_1}{g}}.$$

Thus from a knowledge of the period of vibration of a cantilever under a known load, Young's modulus of the material of the beam can be determined.

159. Theory of the determination of *Y* by the method of Flexure :—

[Beam supported at two ends and loaded in the middle.]

When a beam is supported on two knife-edges at the ends and a load, W, is applied in the middle (as shown in Fig 170) each half of the beam may be looked upon as a cantilever of length $l=\frac{L}{2}$ where L=length of the beam between the knife-edges. This assumption is justified because the middle portion of the beam is horizontal. Taking this middle point as the rigidly clamped end of the cantilever, each half may easily be considered as an inverted cantilever. Since the load W is applied in the middle, the reaction at each end is $\frac{W}{2}$. So the effective load of each cantilever is W

Thus putting $l=\frac{L}{2}$ and replacing W by $\frac{W}{2}$ in the expression of depression of a cantilever, we get the depression y in the present case as,

$$y=\frac{1}{3}\cdot\frac{W}{2}\cdot\frac{L^3}{8}\cdot\frac{1}{YAK^2}=\frac{WL^3}{48YAK^2}.$$

If the beam be rectangular, $AK^2=\frac{bd^3}{12}$. Hence,

$$y=\frac{WL^3}{4Ybd^3}.$$

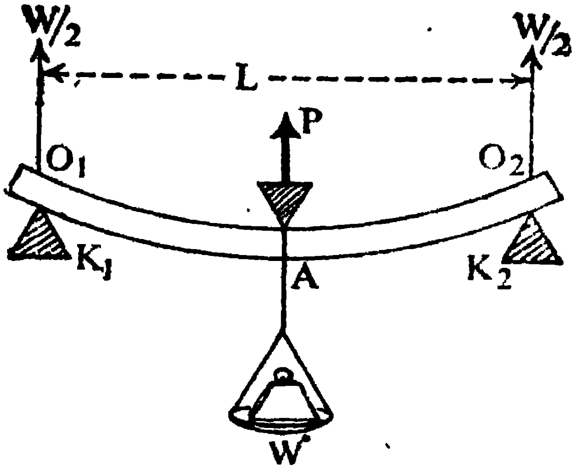


Fig. 170

This expression for depression provides us with an easy and accurate method for determining the Young's modulus of a material available in the form of a beam. If the beam be of rectangular cross-section the above relation can be directly used. In the cases of beams of circular or elliptical cross-sections the value of AK^2 should be changed accordingly.

The knife-edges mounted on separate stands are set exactly in the same horizontal plane. They are separated by a known distance L. Now the beam is placed on the knife-edges and the horizontality of the beam is ensured by a spirit level and, if necessary, adjusted by the levelling screws at the bases of the knife-edge stands. Now another inverted knife-edge provided with a scale pan hanger is placed in the middle of the beam. A pointer P is rigidly fitted on to the inverted knife-edge piece. The position of the tip of P is now read out by means of a travelling microscope or a cathetometer. Now weights are put on the scale pan by instalments and the corresponding readings of the pointer tip are recorded from the microscope or the cathetometer scale. From these readings the depressions corresponding to different loads are evaluated. Now a load-depression curve is drawn. From this curve any arbitrary value of load and its corresponding depression are chosen as W and y for evaluation of Y. The values L, b and d

are obtained by direct measurements with a metre scale, a slide callipers or a screw gauge as the case may be.

The above experiment may be made even more sensitive by means of a lamp and scale arrangement.

160. Poisson's Ratio in the case of a bent beam :—If symmetrical couple acting on a beam of rectangular cross-section bends the same in a curvature of radius ρ in the plane of the paper, there is simultaneously developed an *anticlastic curvature* of radius ρ' in a perpendicular plane. Due to the lengthwise extension of the upper filaments, a lateral contraction will occur to them. The filaments below the neutral axis on the other hand laterally expand because of their longitudinal contraction. So the anticlastic curvature will be concave above as shown in Fig. 171. The longitudinal strain in the bent beam $=\frac{z}{\rho}$ (elongation) as has been shown above. In a similar manner it may be shown that the lateral strain $=\frac{z}{\rho'}$ (contraction). Hence Poisson's ratio, $\sigma=\frac{z}{\rho'}\Big/\frac{z}{\rho}=\frac{\rho}{\rho'}$,(by definition). Thus knowing the two radii of curvatures, *synclastic* (ρ) and *anticlastic* (ρ'), by actual measurements, σ can be determined.

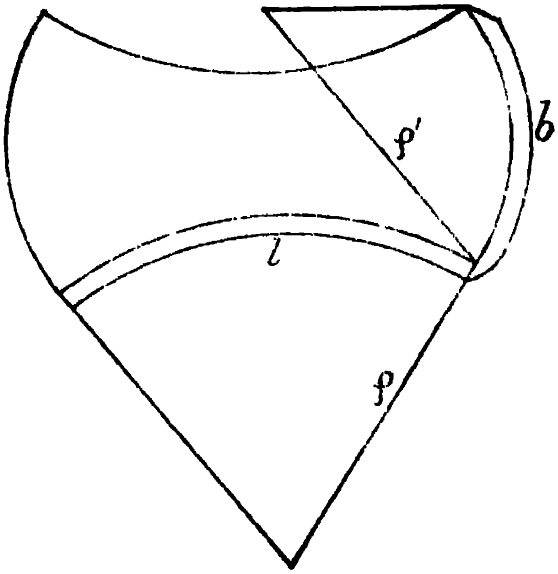


Fig. 171

Example

When a rubber cord is stretched, the change in volume due to change in linear dimensions is negligible. Obtain value of Poisson's ratio for rubber.

(Bihar U. 1959)

Ans. : **Let l = length of rubber cord and r = its radius.**

So its volume $=\pi r^2 l$.

Let α and β be the longitudinal and lateral strains after stretching. The increased length $=l+\alpha l=l(1+\alpha)$, and the changed radius $=r-\beta r=r(1-\beta)$.

$\therefore$ New volume $=\pi r^2(1-\beta)^2\times l(1+\alpha)\doteq\pi r^2 l(1-\beta)^2\,(1+\alpha)$.

But by the condition of the problem, as the volume does not appreciably change,

$$\pi r^2 l = \pi r^2 l\,(1-\beta)^2\,(1+\alpha).$$

$$1+\alpha = \frac{1}{(1-\beta)^2} = (1-\beta)^{-2}$$

As β is very small, $1+\alpha = 1+2\beta$;

or, $2\beta = \alpha$.

Hence Poisson's ratio, $\sigma = \dfrac{\beta}{\alpha} = \frac{1}{2}$.

Determination of the Poisson's ratio of a material.—

If the material is available in the form of a rectangular bar, its Poisson's ratio is determined by finding out the radii of curvature, ρ and ρ' (clastic and anticlastic) in the two mutually perpendicular directions when the rod is supported at the ends on two knife-edges and loaded in the middle. The radius of the clastic curvature is determined by observing the sag of the mid-point with a microscope If y be this sag or depression in the middle and l the length of the bar between the knife-edges, then from the property of the circle,

$$(2\rho - y)y = \left(\frac{l}{2}\right)^2 ;$$

$$\text{or, } 2\rho y - y^2 = \frac{l^2}{4}.$$

Hence knowing l and y, ρ can be determined.

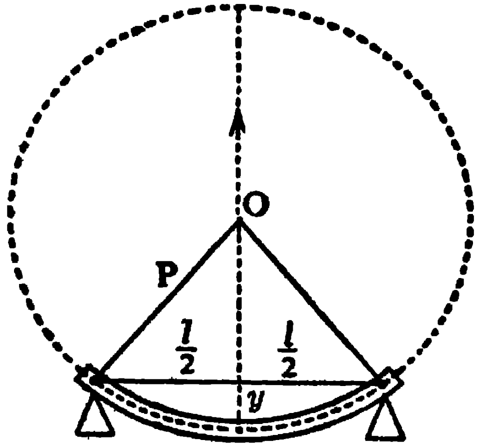


Fig. 172

In order to determine the anticlastic curvature two needles (about 40 cms. long) are rigidly mounted on two clamps soldered to the edges of the bar at their mid-points. One of these needles carries a mm. scale parallel to the breadth of the bar at its free end and the free end of the second needle just rests against this scale. If the length of the needle above the neutral axis of the bar be a and x be the shift of the second needle against the scale when the bar is loaded, the angle of sideways deflection of the bar is $\theta = x/a$. Now if b be the breadth of the bar and ρ' the radius of the anticlastic curvature, then $\theta = b/\rho' = \dfrac{x}{a}$.

$\therefore\ \rho' = \dfrac{ab}{x}$. Thus knowing ρ and ρ', σ can be calculated.

161. Determination of Elastic constants by Searle's method :—

The material whose elastic constants are to be evaluated is taken in the form of a *wire of short length l*. This wire is rigidly fixed to the centres of two identical bars at the two ends. The system is suspended by two vertical strings as shown in Fig. 173(*i*). In this position the bars would vibrate due to the bending of the wire. When the wire is bent into an arc of a circle of radius r, it exerts a torque on each rod equal to $\frac{YAK^2}{r}$ $=\frac{Y\pi a^4}{4r}$ ($\because$ In the case of wire, $A=\pi a^2$ and $K^2=\frac{a^2}{4}$, where $a=$ radius of the wire).

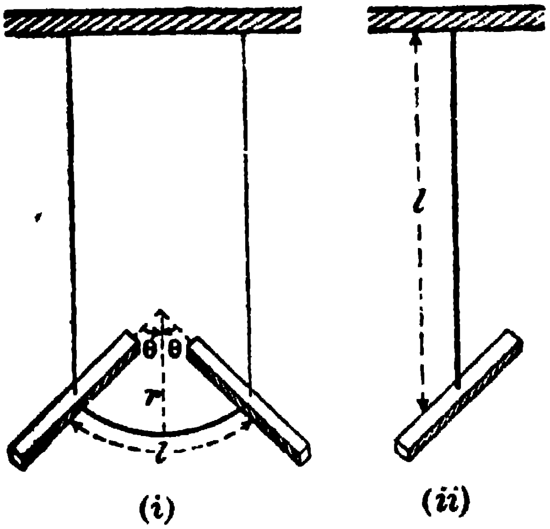


Fig. 173

When the experimental wire is straight, the rods were parallel. So, if θ be the rotation of each rod during the oscillation, then $2\theta=\frac{l}{r}$ or, $r=\frac{l}{2\theta}$.

The couple acting $=\frac{\pi a^4 Y}{4l/2\theta}=\frac{\pi a^4 Y}{2l}.\theta$. This shows that the couple is proportional to the angular displacement.

So the couple per unit angle of rotation $=\frac{\pi a^4 Y}{2l}=\tau$, (say). The time period of this rotational vibration is

$$T_1=2\pi\sqrt{I/\tau}=2\pi\sqrt{\frac{2Il}{Y\pi a^4}} \qquad (1)$$

Again, if one of the rods be fixed horizontally and the other set to rotational vibration as arranged in Fig. 173(*ii*), the time period of vibration is

$$T_2=2\pi\sqrt{\frac{2lI}{\pi\eta a^4}} \quad \ldots \quad \ldots \qquad (2)$$

Thus from relations (1) and (2), the values of Y and η can be calculated because l, I, T and a are all measurable quantities.

Moreover, knowing Y and η, other elastic constants can also be computed from the following relations :

$$2(1+\sigma)=\frac{Y}{\eta}=T_2^2/T_1^2 \qquad (3)$$

$$\text{and } \frac{1}{K} = \frac{9}{Y} - \frac{3}{\eta} \qquad \dots (4)$$

The elastic constants of glass have been also determined by Cornu by studying the interference fringes formed between a plane cover-glass and a rectangular plate of glass loaded in the middle.

***162. The Extension of a Flat Spiral Spring** :—When the coils of a spring are almost horizontal, *i.e.* when the pitch of the spring is small compared to its radius, it is known as a *flat spiral spring*. If such a spring be stretched by means of a small weight so that the horizontality of the coils is not appreciably disturbed, an expression for the extension of the spring can be easily found in terms of the above weight (*i.e.* stretching force). If the wire of the spring be bent inwards at the top P and at the bottom Q the weight mg extending it may be applied along the axis of the sprial (Fig. 174). Considering the equilibrium of the section of the wire at any point R, the moment of the force mg is given by $mg \times r$ where r is the radius of the spiral and this is balanced by the couple due to twist in the wire at R, when the section at R is vertical, *i.e.* when the spiral is almost flat. From Art. 154 we know that the torsional couple is given by $\frac{\pi\eta\theta a^4}{2}$ where θ is the twist per unit length, a = radius of the wire and η = modulus of rigidity.

∴ For equilibrium of the section,

$$mgr = \frac{\pi\eta\theta a^4}{2}.$$

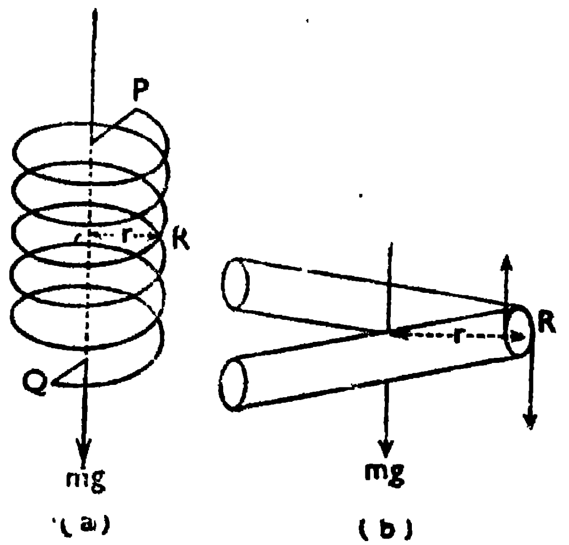


Fig. 174

Due to this twist θ per unit length every portion of the wire below the section will extend downwards by the amount $\theta \times r$. So considering the twist of the whole length of the wire constituting the spring, the total downward extension of the spring is $r \times \theta \times l$, where l = length of the wire. Let this total extension be denoted by e. Then,

$$e = rl \times \frac{2mgr}{\pi\eta a^4} = \frac{2mglr^2}{\pi\eta a^4} \qquad \dots (1)$$

Thus by measuring e for different weights, η may be experimentally found out.

Again if the above spring be set to vertical oscillation, it will execute S.H.M. and according to Art. 119(c) its period of oscillation is given by $T=2\pi\sqrt{m/\text{force per unit extension}}$ $=2\pi\sqrt{m\Big/\frac{mg}{e}}=2\pi\sqrt{\frac{2mlr^2}{\pi\eta a^4}}$, [from (1)].

Hence noting this time period also, η can be evaluated.

163. (a) Determination of the bulk modulus of a solid :-

In order to determine the bulk modulus of a body—generally taken in the form of a bar—it is to be immersed in a liquid subjected to a measured hydrostatic pressure. The pressure is usually applied by means of a piston acting in a cylinder which is to be connected to the vessel V, in which the liquid is contained [Fig. 175].

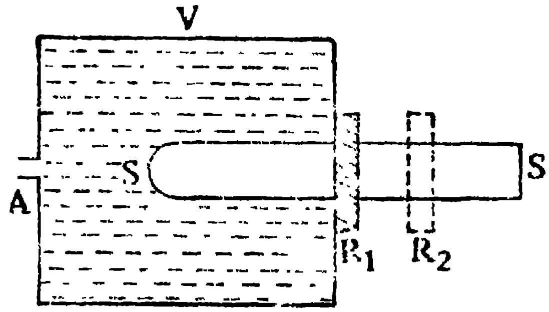


Fig. 175

In the figure the specimen is represented by SS_1. A part of SS_1 is enclosed in a steel vessel V containing the liquid. V is connected to the pressure applying device at A. The applied pressure is directly measured from such loads placed on the piston of another vertical cylinder also connected to V, as may maintain equilibrium with the applied pressure. The contraction of the rod SS_1 relative to the steel vessel V is measured by the shift of a fairly loose-fitting ring from R_1 to R_2. Correction for the increase in length of V should also be introduced. The linear contraction C_l due to the hydrostatic pressure is thus determined. Hence the volume strain is $3C_l$ approximately.

∴ If P=applied pressure, the bulk modulus, $K=\frac{P}{3C_l/V_0}$, where V_0 is the volume of the specimen subjected to hydrostatic pressure.

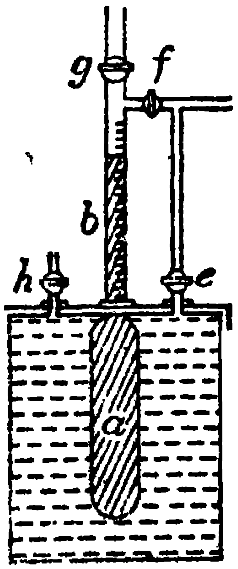


Fig. 176

(b) Bulk modulus of a liquid :—

We know that the liquids are highly incompressible. But Canton in 1762 demonstrated with the help of an apparatus, known as *piezometer* [Fig. 176] that under high pressures the liquids decrease in volume. So they must possess volume elasticity. The bulk modulus of liquids has been successfully determined by a piezometer. Here the experimental liquid is contained in the bulb a, which extends above into a graduated capillary tube b, which again may be attached to a compression pump and a manometer. The pressure applied to the liquid in a may also be communicated to the liquid in the vessel V in which a is immersed. This is done along the route fe.

Let dv=decrease in volume when both inside and outside of a are subjected to a pressure, P, *i.e.* the effect of contraction of vessel due to pressure cancel out.

If V be the volume of the liquid taken in a, then bulk modulus $K=P\Big/\frac{dv}{V}$. All the terms on the right-hand side of the expression for K being measurable, K can be determined.

164. Elasticity and temperature :—The elastic constants generally decrease with rise in temperature. For small temperature ranges, the variation is approximately linear. Schaefer showed that for temperatures from 15°C to that of liquid air, the order of ascending temperature coefficients for Young's modulus is of the order of increasing thermal expansion and decreasing melting point. Shave and Lee, and Andrews showed that at temperatures within 15°C of melting points of the materials, the relation between T and Y was of the form,

$$Y=Y_1\, e^{-b_1T}$$

where b_1 is a constant whose value is different for different temperature ranges. Y for quartz changes only slightly over the range of temperature 0°C to 800°C.

Kohlrausch and Loomis showed that the rigidity modulus changes according to following relation over the range 15°C to 100°C,

$$\eta_T=\eta_0(1-aT-bT^2).$$

Horton showed that for pure copper and steel, the relation was linear and for silver it was also nearly so. He showed that η depends on the previous heat treatment of the specimen.

Bridgeman showed that the bulk modulus of liquids in general increases with temperatures according to relation,

$$K=A+BT-CT^2$$

which has a maximum value at $T=B/C$ (nearly 50°C for water).

165. Boyle's Law :—Robert Boyle (1627-1691) first established the exact relationship between the pressure of a confined mass of gas and its volume when they are varied at a constant temperature and the law named after him may be stated as follows—

Temperature remaining constant, the volume of a given mass of gas varies inversely as the pressure.

Thus, if P be the pressure and V the volume of a gas, we have, $P\propto\frac{1}{V}$; or, $P=K.\frac{1}{V}$, where K is a constant whose value depends on the mass of the gas taken and its temperature.

$$\text{Thus, } PV = K.$$

If the pressure P be changed to P_1 at constant temperature, and the corresponding volume becomes V_1, we have, $P_1V_1 = K$. But $PV = K$.

$$P_1V_1 = PV.$$

Thus, if a given mass of a gas at constant temperature has volumes V_1, V_2, V_3, etc. under pressures P_1, P_2, P_3, etc. respectively then by Boyle's Law, we have, $P_1V_1 = P_2V_2 = P_3V_3 = \cdots\cdots\cdots = \text{a constant}$.

Pressure and density.—Boyle's Law may also be expressed in terms of the pressure and density of a gas. Thus, let a given mass of a gas at a given temperature have a volume V_1 and density ρ_1 under a pressure P_1, and a volume V_2 and density ρ_2 under a pressure P_2; then, since the mass m of the gas remains the same, we have,

$$m = \rho_1V_1 = \rho_2V_2 \text{ ; or, } \rho_1/\rho_2 = V_2/V_1 \quad \ldots \quad \ldots \quad (1)$$

Thus, *the density of a given mass of a gas is inversely proportional to its volume.* But by Boyle's Law $P_1V_1 = P_2V_2$;

$$\text{or, } \frac{V_2}{V_1} = \frac{P_1}{P} \therefore \frac{\rho_1}{\rho_2} = \frac{P_1}{P_2} \quad .. \text{ from (1).}$$

That is, the density of a gas at constant temperature is directly proportional to its pressure.

$\therefore$ P/ρ, *i.e.* the ratio of pressure to density of a gas is constant at a constant temperature.

Verification of Boyle's Law.—

Boyle's Law can be verified by a *Boyle's Law tube* apparatus (Fig. 177), which consists of a glass tube AB of uniform bore, connected by means of flexible rubber tubing P to one end of a sliding glass tube T of somewhat wider bore and open at the top. The closed tube is partly filled with dry air, and the rubber tubing with some portion of both the glass tubes is filled with clean mercury. The volume of the air in AB is increased or decreased by sliding the tube T downwards or upwards. The tubes are mounted on a vertical adjustable wooden board provided with a graduated scale fixed between the two tubes for the reading of the heights of the mercury levels in the two tubes.

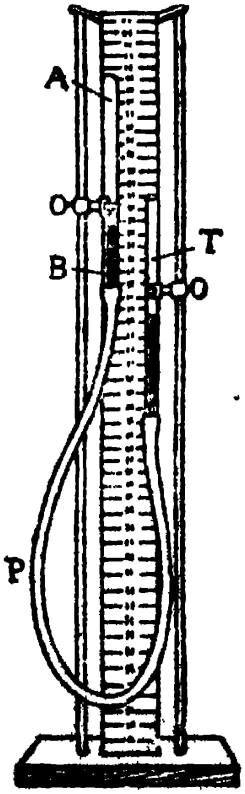


Fig. 177

Usually the tube AB is graduated from the top downwards to give the volume of the enclosed air directly. Some tubes again are not graduated. In their case, the volume of the air enclosed is given by the length of the air column above the mercury level multiplied by the cross-section of the tube which is uniform. That is, volume is proportional to the length of the enclosed air column. This length is determined by means of the attached vertical scale from the difference of the readings corresponding to the top of the tube and to the level of the mercury in the tube.

When the mercury levels in both the tubes are the same, the pressure of the enclosed air is *atmospheric*. If the level of mercury in the wider tube is **higher** than that in the other tube, the *pressure* of the enclosed air is *atmospheric pressure* **plus** *the difference of the two levels of mercury* ; and if it is **lower**, then the pressure is *atmospheric pressure* **minus** *the difference of the two levels*. The atmospheric pressure during the experiment is directly read from a barometer.

In an experiment for the verification of the law, the volume of the enclosed air and the corresponding pressure are determined for different heights of the mercury column in the tube T.

To verify the law *for pressure greater than the atmospheric pressure*, all the readings for pressure are taken keeping the *level* of mercury in the wider tube *higher* than that in the other tube ; and *for pressures lower than the atmospheric pressure*, the *level* of mercury in the wider tube should always be *lower* than that in the other.

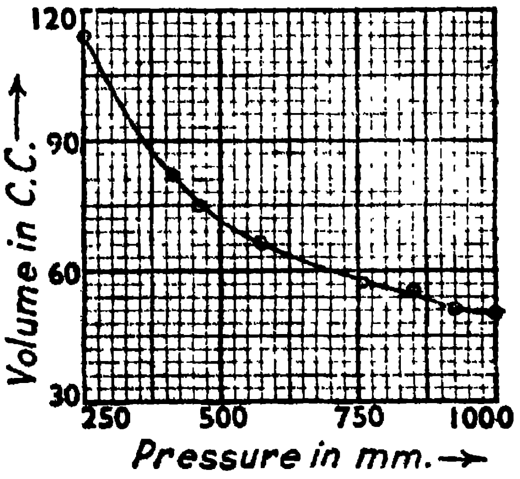


Fig. 178

Plot the results on a squared paper where the abscissæ represent the pressures, and the ordinates the corresponding volumes. The curve will be a *rectangular hyperbola* [Fig. 178].

But as P is proportional to $1/V$, *a straight line* is obtained if P is plotted against $1/V$.

Precautions.—For the success of the experiment with a Boyle's Law tube apparatus, the following precautions should be taken—

(*a*) The air enclosed in the tube AB must be perfectly dry.

(*b*) The scale by which the mercury levels are read must be accurately vertical.

(*c*) The volume of the enclosed gas must be changed *slowly* so [illegible] the temperature of the gas may remain constant during com[illegible]r expansion.

on the m

(*d*) In view of the variation in the atmospheric pressure during the experiment, the barometer should be read at the beginning and also at the end of an experiment, and the mean of the two readings should be taken.

Verification by Graph.—If a straight line results on plotting P against $1/V$, this is a verification of the law. It should be known that straight line graph is a far more conclusive evidence than a curved line graph for the verification of some law.

The law can be verified graphically even without the knowledge of the atmospheric pressure. To do this, plot the excess of pressure, *i.e.* pressure above the atmospheric pressure, against $1/V$ when again, a straight line will be obtained [Fig. 179]. As we have $PV = K$, a constant, $(H+X)V = K$, or $H + X = K/V = KY$ (1) where H represents the atmospheric pressure, X the excess pressure, and Y stands for $1/V$.

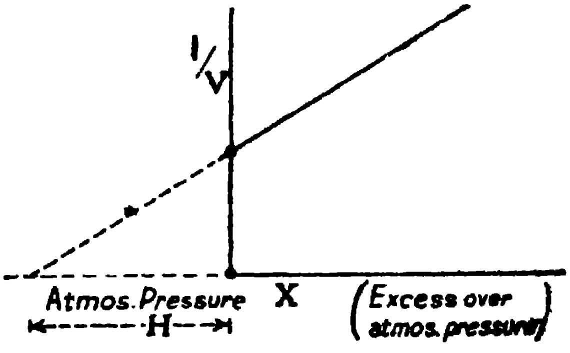


Fig. 179

This is an equation of a straight line. So if the graph of X, the excess of pressure, and Y, *i.e.* $1/V$, gives a straight line, the law is verified.

166. Bulk modulus of gases (Perfect) :—Gases, as we know, are highly compressible and as such they will suffer a change in volume with the application of pressure. So they must possess volume elasticity. Extensive experiments have been carried out by Andrews, Amagat, Despretz, Regnault and others in order to study the elasticity of gases (*see* Heat, Vol. II).

Though all gases must have volume elasticity, its magnitude will be solely dependent on the external physical conditions under which the strain is developed. This is because in gases along with the pressure and volume, another factor, *i.e.* temperature, plays an equally important part in controlling the behaviour of the gases. So if the strain is developed in a gas under an isothermal condition, the corresponding isothermal bulk modulus will evidently differ from the bulk modulus of the gas under an adiabatic condition.

(i) Coefficient of Isothermal Volume Elasticity.—

At constant temperature, a perfect gas obeys Boyle's law. Hence if P and v represent the pressure and volume of a perfect gas, then

$$Pv = \text{constant} \qquad \ldots \qquad \ldots \qquad \ldots \quad (1)$$

Differentiating (1) with respect to v, we get,

$$\frac{d}{dv}(Pv) = 0\,;$$

$$\text{or,}\quad P + v\frac{dP}{dv} = 0\,;$$

$$\text{or,}\quad -\frac{dP}{\frac{dv}{v}} = P.$$

But as due to increase in pressure, dP, the volume v ***diminishes*** by dv, the volume strain is negative and is given by $-\frac{dv}{v}$.

Hence the coeff. of volume elasticity of the gas under the isothermal state (*i.e.* when temp. t is constant) is given by $E_t = \frac{dP}{-\frac{dv}{v}} = P$.

Thus the isothermal bulk modulus of a perfect gas is equal to the pressure of the gas.

(ii) Adiabatic Bulk Modulus of a Perfect Gas.—

Under the adiabatic conditions, a perfect gas obeys the equation, $Pv^{\gamma} = \text{constant}$, where P=pressure, v=volume, and γ is the ratio of the sp. heat at constant pressure and that at constant volume of the gas.

$$\therefore \quad \frac{d}{dv}(Pv^{\gamma}) = 0.$$

$$\text{or,}\quad P.\ \gamma v^{\gamma-1} + v^{\gamma}\frac{dP}{dv} = 0.$$

Dividing both sides by $v^{\gamma-1}$, we get $\gamma P = -v\frac{dP}{dv} = -\frac{dP}{dv/v}$.

But $\frac{dP}{-\frac{dv}{v}}$ is the bulk modulus of elasticity under adiabatic conditions, which is represented by E_H.

$$\therefore \quad E_H = \frac{dP}{-\frac{dv}{v}} = \gamma P.$$

Thus the adiabatic bulk modulus of a perfect gas is γ times the pressure of the gas.

Examples

(1) *A square metal bar of 2·51 cms. sides, 37·95 cms. long and weighing 826 gms. is suspended by a 37·95 cms. long wire of radius 0·0501 cm. It makes 50 complete oscillations in 335·7 secs. Calculate the modulus of rigidity of the wire.*

Ans. : Time period of oscillation, $t = 2\pi\sqrt{\frac{I}{c}} = 2\pi\sqrt{\frac{2lI}{\eta\pi r^4}}$

Here, $\eta = ?$; $r = 0{\cdot}0501$ cm. ; $l = 37{\cdot}95$ cms. ;

$t = \frac{335{\cdot}7}{50} = 6{\cdot}714$ secs. and $I = m\left(\frac{l^2+b^2}{12}\right) = \frac{826}{12}\left[\left(37{\cdot}95\right)^2 + \left(2{\cdot}51\right)^2\right]$

$= 99540$ gms -cm.^{2}.

Hence, $\quad 6{\cdot}714 = 2\times 3{\cdot}14\sqrt{\frac{2\times 37{\cdot}95\times 99540}{\eta\times 3{\cdot}14\times(0{\cdot}0501)^4}}.$

Hence, $\quad \eta = 3{\cdot}34\times 10^{11}$ dynes/cm.^{2} nearly.

(2) *A metal bar 1 cm. square in cross-section is supported on two knife-edges 100 cms. apart. A load of 1 kgm. applied at the centre of the bar depresses the point of application by 2·51 mms Calculate the Young's modulus of the material of the bar.*

Ans. : The depression, $y = \frac{Wl^3}{48YAK^2}$.

Here $W = mg = 1000\times 981$ dynes ; $l = 100$ cms. ; $Y = ?$;

$AK^2 = \frac{bd^3}{12} = \frac{1}{12}$ cm.^{4}, $(\because \; b = d = 1$ cm.), $y = {\cdot}251$ cm.

Hence $Y = \frac{Wl^3}{48AK^2y} = \frac{981\times 10^3\times 10^6\times 12}{48\times{\cdot}251} = 9{\cdot}77\times 10^{11}$ dynes/cm.^{2}.

(3) *A vertical steel rod of circular section of radius 1 cm. is rigidly fixed in the earth. Its upper end is 3 metres from the earth. A thick string which can stand a maximum tension of 2 kgms. is tied to the upper end of the rod and pulled horizontally Find how much the top will be deflected before the string snaps. ($Y = 2\times 10^{12}$ c.g.s. units and $g = 1000$ c.g.s. units.)* (Saug. U. 1958)

Ans. : It is the case of a vertical cantilever whose lower end is fixed. 2 kgms. force is horizontally applied at its upper end. Hence the deflection is

$$\delta = Wl^3/3YAK^2$$

Here W = force applied $= 2000\times 1000$ dynes.

$l = 3$ metres $= 300$ cms.

$Y = 2\times 10^{12}$ dynes/cm^2.

$$AK^2=\pi r^2\times r^2/4=\frac{\pi\times(1)^4}{4}=\frac{\pi}{4}$$

$$\therefore\ \delta=\frac{2\times10^6\times(300)^3}{3\times2\times10^{12}\times\pi/4}=11{\cdot}47\text{ cms.}$$

(4) *A circular bar of length 40 cms. is horizontally supported on two knife-edges at the ends. When a load of 10 kg. is applied at the centre, the sag at the central point is found to be 4·88 mms. Calculate the Young's modulus of the bar. (Radius of the circular bar = 4 mms.)*

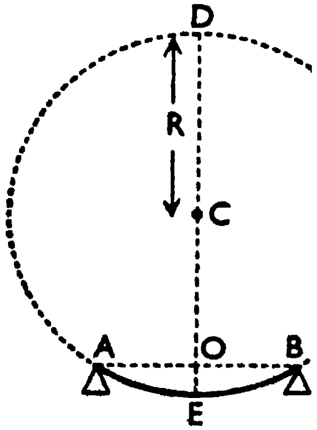


Fig. 179(*a*)

Ans. : Since load at the centre is 10 kg, its reaction at the knife-edge is 5 kg.

$\therefore$ Half the bar, *i.e.* EB [Fig. 179(*a*)] may be supposed to be an inverted cantilever with a terminal load of 5 kg.

Hence the external bending moment

$$=G=\frac{40}{2}\times5\times10^3\times981\text{ dyne-cm.}$$

$$=981\times10^5\text{ dyne-cm.}$$

Internal bending moment$=\frac{YAK^2}{R}$,

where $AK^2=\frac{\pi r^4}{4}=\frac{3{\cdot}14\times({\cdot}4)^4}{4}$.

As is evident from Fig. 179(*a*), $OB^2=DO\,.\,OE$.

Here $OB=20$ cms. and $DO=DE-OE=2R-\delta$, where R is the radius of curvature of the beam and δ, its depression.

$\therefore$ $20^2=(2R-\delta)\delta=2R\delta$, neglecting δ^2 which is very small.

$$\therefore\ R=\frac{400}{2\times{\cdot}488}.$$

From relation, $G=\frac{YAK^2}{R}$

$$Y=\frac{G\times R}{AK^2}=\frac{981\times10^5\times400\times4}{2\times{\cdot}488\times3{\cdot}14\times({\cdot}4)^4}$$

$$=\frac{981\times10^{12}}{2\times4{\cdot}88\times3{\cdot}14\times16}=20{\cdot}04\times10^{11}\text{ dynes/cm}^2.$$

(5) *Compare the density of water at the surface and the bottom of a lake 100 metres deep given that the compressibility of water is* $\frac{1}{22000}$ *per atmosphere and that the density of mercury is 13·6 gms./c.c.*

Ans. : Compressibility $C=\frac{1}{\text{bulk modulus}}=\frac{v_1-v_2}{v_1}\Big/P.$

where P=press. in atmos. at the bottom of the lake and v_1 and v_2 are the sp. volumes of water at the surface and at the bottom respectively.

$\therefore$ $C\times P=1-\frac{v_2}{v_1}=1-\frac{d_1}{d_2}$ where d_1 and d_2 are the densities of water at the surface and bottom respectively.

$$\therefore\ \frac{d_1}{d_2}=1-C\times P=1-\frac{1}{22000}\times\frac{10000}{13{\cdot}6\times76}=0{\cdot}99956.$$

(6) *You are given 200 c.c. of air at a pressure of 760 mm. of mercury. On increasing the pressure by 1 mm. of mercury, without change of temperature, the volume decreases by 0·263 c.c. Find the coefficient of volume elasticity of the gas.* (*C.U. 1952*)

Ans. : The coefficient of volume elasticity at constant temperature is given by $K = \dfrac{dP}{-\dfrac{dv}{v}}$.

Here $dP = 0{\cdot}1 \times 13{\cdot}6 \times 980$ dynes/cm.^{2} ; $-dv = 0{\cdot}263$ c.c. ; $v = 200$ c.c.

$$K = \frac{0{\cdot}1 \times 13{\cdot}6 \times 980 \times 200}{0{\cdot}263} = 1{\cdot}0146 \times 10^6 \text{ dynes/cm}^2.$$

Table of Elastic Constants

Substance	Young's Modulus (dynes/cm.^{2})	Rigidity Modulus (dynes/cm.^{2})	Bulk Modulus (dynes/cm.^{2})
Copper	$12{\cdot}5 \times 10^{11}$	4×10^{11}	$14{\cdot}3 \times 10^{11}$
Iron (wrought)	$19{\cdot}5 \times 10^{11}$	8×10^{11}	$14{\cdot}6 \times 10^{11}$
Steel	20×10^{11}	$8{\cdot}5 \times 10^{11}$	$18{\cdot}1 \times 10^{11}$
Glass (flint)	$5{\cdot}5 \times 10^{11}$	$2{\cdot}3 \times 10^{11}$	$3{\cdot}7 \times 10^{11}$
Water	...	...	$0{\cdot}205 \times 10^{11}$
Mercury	...	...	$2{\cdot}6 \times 10^{11}$

Questions (Chapter VIII)

1. Explain what are meant by *elastic limit, Hooke's law, yield point, modulus of elasticity*, and *Young's modulus*.

Describe how you would determine Young's modulus for a material in the form of a uniform wire.

2. What are meant by strain and elastic after-effect ?

A steel wire 2 mms. in diameter is just stretched between two fixed points at a temperature of 20°*C*. Determine the tension on it when the temperature falls to 10°*C*. (Coef. of linear exp. of steel$=0{\cdot}000011$ and Young's modulus for steel$=2{\cdot}1 \times 10^{12}$ dynes/cm.^{2}]. ($\log \pi = 0{\cdot}4972$)

[*Ans.* : $7{\cdot}257 \times 10^6$ dynes]

3. Show that for a homogeneous isotropic substance,

$$Y/\eta = 2(\sigma + 1)$$

where Y is Young's modulus, η the simple rigidity and σ, Poisson's ratio.

4. Find the energy stored in a stretched wire of area of cross-section 2 mm^2. and initial length 50 cms. if it be loaded with a mass of 1000 gms., and Young's modulus of the material of the wire is 10^{12} dynes/cm^2.

[*Ans.* : 1203 ergs]

5. How is the bulk modulus of a solid determined ? Calculate the bulk modulus for a specimen of steel, given that the Young's modulus and the rigidity modulus for the specimen are 21×10^{11} and 8×10^{11} dynes/cm.^{2}, respectively.

[*Ans.* : 18.7×10^{11} dynes/cm^2.]

6. Show that for an isotropic material Poisson's ratio must lie between $+\frac{1}{2}$ and -1. (Bihar U. 1956 ; *cf.* C.U. 1947, '62)

7. Obtain an expression for the internal bending moment acting on a uniform thin rod of neglible weight when it is bent into the form an arc of a circle of radius ρ, in terms of its geometrical moment of inertia and Young's modulus.

8. A sphere of mass 800 gms. and radius 3 cms. is suspended from a wire of length 100 cms. and radius 0.5 mm. If the period of torsional vibration is 1.23 secs., calculate the rigidity of the material of the wire. [Bom. U. 1936]

[*Ans.* : 7.65×10^{11} dynes/cm.^{2}]

9. Define Young's modulus, bulk modulus and modulus of rigidity. If E, K and n represent these moduli respectively, prove the relation

$$E = \frac{9nK}{3K+n}.$$ [*cf.* C.U. 1966 (SP.) ; Utk.U. 1962 ; All.U. 1943]

10. Define Young's modulus. Show that the shear is equivalent to a compression and extension.

Find an expression of the work done in stretching a wire and hence deduce an expression for the energy per unit volume of the wire. (C.U. 1966 ; Madr. 1947)

11.(*a*) Define modulus of rigidity. Explain clearly the dynamical method of determining this constant for a metal wire. Derive the necessary formula. (Punj. U. 1948)

(*b*) If a solid cylinder is twisted keeping one end fixed, calculate the restoring couple developed in it. (Utk. U. 1970.S)

12. Derive formula for the torsional oscillations of a heavy cylinder attached to the lower end of a wire suspended from a rigid support, and describe an experiment for determining the modulus of rigidity of the wire material by this method. (C.U. 1968)

13. A gold wire 0.32 mm. in diameter elongates by 1 mm. when stretched by a force of 330 gms. weight and twists through one radian when equal and opposite torques of 145 dynes-cm. are applied at its ends. Find the value of the Piosson's ratio for gold : $g = 981$ cms./sec.^{2} (Bihar U. 1954 ; Nag. U. 1953)

[*Ans.* : 0.429]

14. Find the work done in stretching a wire.

A wire 50 cms. long and 1 sq. mm. in cross-section has Young's modulus 1.24×10^{12} dynes/sq. cm. How much work is done in stretching it through 1 mm. ? (Alig. U. 1947)

[*Ans.* : 0.124 Joule]

15. In an experiment the diameter of the rod was 1.26 cms. and the distance between the knife-edges 70 cms. On putting a load of 900 gms. at the middle point the depression was 0.025 cm. Calculate the Young's modulus of the substance. ($\pi = 3.14$) (Agra U. 1948)

[*Ans.* : 20.39×10^{11} dynes/cm.^{2}]

16. Compare the loads required to produce equal depression for two beams, made of the same material and having the same length and weight

with the only difference that while one has a circular cross-section, the cross-section of the other is a square. (San. U. 1950)

[*Ans.* : $W_1 : W_2 = 3 : \pi$]

17. Find the force necessary to stretch by 1 mm. a rod of iron 1 metre long and 2 mm. in diameter. Also calculate the energy stored in the stretched rod. ($Y = 2 \times 10^{12}$ c.g.s. units) (Pat. U. 1949)

[*Ans.* : (*i*) 64 kg.-wt. (*ii*) $3 \cdot 14 \times 10^6$ ergs.]

18. A disc of mass 1 kg. and of radius 10 cms. is suspended horizontally by a vertical wire of length 60 cms. and radius 0·5 mm. If the period of torsional oscillation is 4 secs., calculate the modulus of rigidity of the wire. (Pat. U. 1957)

[*Ans.* : $7 \cdot 519 \times 10^{11}$ dynes/cm^2.]

19. Poisson's ratio of a material is 0·379, its rigidity is $2 \cdot 87 \times 10^{11}$ dynes/cm^2, find Young's-modulus. (Baroda, U. 1948)

[*Ans.* : $7 \cdot 916 \times 10^{11}$ dynes/cm^2.]

20. What are the various elastic constants and how they are related ?

A weightless straight elastic beam of rectangular cross-section is rigidly clamped horizontally at one end and loaded at the free end. Calculate the deflection at the free end, assuming the bending to be small By what factor the deflection of the free end changes, if the weight be moved to a distance '*d*' from the free end ?

(C U.H. 1967)

21. Briefly describe a method of comparing the mudulus of rigidity and the Young's modulus of a material in the form of a short wire and give the underlying theory. (C.U. 1968)

CHAPTER IX

SURFACE TENSION

167. Surface Tension and Molecular Theory :—When a small quantity of mercury is dropped on a flat piece of wood it is found that it breaks up into a number of differently sized globules—all approaching a sphere in shape. Similarly, any liquid taken in a small quantity and not subjected to any outside disturbance is found to assume a shape which is also nearly spherical. We know that a sphere has the smallest surface area for a given volume. So it appears that a liquid drop tends to contract so as to occupy the smallest area. This contractile property of a liquid surface is known as its ***surface tension***. The tension existing along the surface of a liquid can also be demonstrated by taking a wire ring to which are tied two slack threads *AA* and *BB* (Fig. 180). If this ring be dipped in a soap solution a continuous film will form on the ring and the slack threads will not interfere with the formation. They will still remain loose because of equal tension experienced on both sides of them. But if the space between the ring and *BB* be pierced by a straw, *BB* takes up the form of an arc of a circle due to the existence of the tension on the right side of thread that on the other side being absent due to the disappearance of the film in that region. This clearly shows that in the liquid film there is a contractile tendency. This tendency can be explained by the help of molecular theory in the following way :

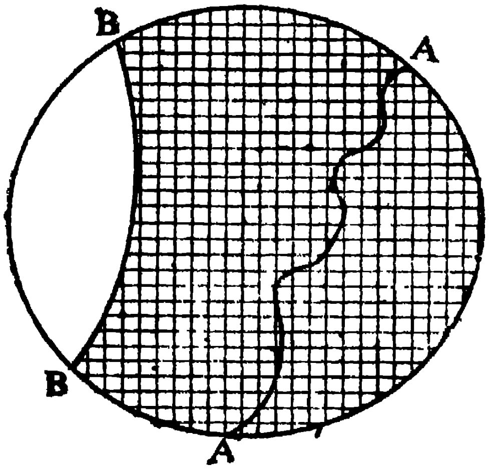


Fig. 180

Because of the closeness of the molecules in a liquid, they exert an attraction on each other. This attraction is somewhat analogous to the gravitational attraction but is actually electrical in nature, because gravitational forces are far too small to yield the experimentally observed effects. If it were possible to enunciate the exact law of force between the attracting molecules of a liquid, the behaviour of liquids could be mathematically calculated and predicted. But we are rather too ill-informed about the liquid state to have a definitely formulated law of attraction. Laplace, however, advanced a hypothesis based on the concept of a sphere of molecular attraction. Each molecule, according to this view, is supposed to attract other molecules surrounding it when they are within this sphere of attraction. At a distance greater than the radius of this sphere, the molecular attraction practically vanishes. This assumption helps us to explain at least qualitatively the cause of the tension that exists on the surface of a liquid. Let us suppose a molecule (1) to be situated at a distance greater than the radius of the molecular attraction below the surface of a liquid [Fig. 181(*a*)]. This molecule will be equally attracted by the adjacent molecules in all directions and as such will have no resultant force on it. Again if we consider two other molecules (2) and (3) one at a distance less than the radius of molecular attraction from the surface and the other on the surface of the liquid [Fig. 181 (*b*, *c*)], in the case of molecule (2), the number of molecules attracting it below will be greater than that above and in molecule (3) the whole attraction will be directed downwards. Evidently the attraction in molecule (3) is greater than that in (2) and is maximum since it is on the liquid surface. Here the hemisphere of force is completely unbalanced. Thus molecules like (2) and (3) will experience a downward force normal to the liquid surface tending to prevent them from escaping. This downward pull on the surface molecules will cause the surface to behave as a stretched membrane and is the ultimate cause of the phenomenon of surface tension. So we can say that ***the surface tension is a molecular phenomenon.***

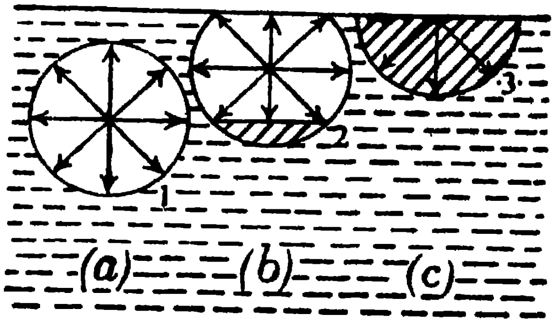


Fig. 181

Imagine a line drawn on the surface of a liquid. On account of the contractile tendency of the surface, the liquid molecules on the two opposite sides of the line are pulled apart from each other at right angles to the line as shown in Fig. 182. Rupture is, however, prevented by the cohesive forces which exist between such molecules and bind them together.

Such ***a force, which acts along the surface of a liquid, tending to contract its area to a minimum measured per unit length of any straight line imagined to be drawn on the surface of the liquid, is called the surface tension of the liquid. It is generally designated by T or S.***

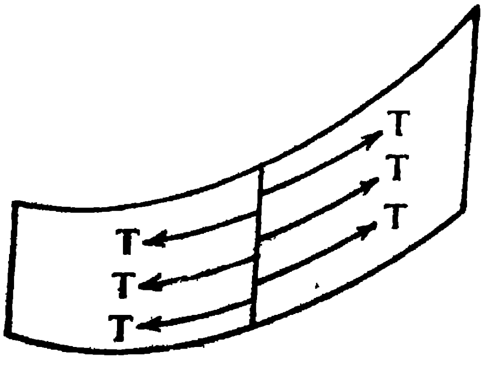


Fig. 182

Its unit is expressed in dynes per cm. and its dimensional equation is MT^{-2}.

168. Surface Energy of liquid :—A stretched membrane, on account of its configuration, has potential energy ; the surface of a liquid behaves like a stretched membrane and as such it has potential energy too. It may be shown that the surface tension of a liquid is numerically almost equal to the potential energy of the surface per unit area.

Consider a film of liquid produced in a rectangular frame *ABCD* (Fig. 183) of which the cross piece *BC* of length, *L*, can be slided along the arms *AB* and *DC*. Since a liquid surface behaves like a stretched membrane and is contractile, the force *F* which tends to diminish the area of surface $=2\times(T\times L)$, where $T=$the surface tension of the liquid and the factor 2 comes since the film has two surfaces, upper and lower. If the area *ABCD* is extended by pulling *BC* through x as shown in the figure, the work required to be done$=F.x=2TL.x$, and the increase in area$=2L\times x$, since the film has two surfaces. So the work required to be done in stretching the film through unit area, which is a measure of the potential energy per unit area$=2TLx/2Lx=T$, the surface tension of the liquid.

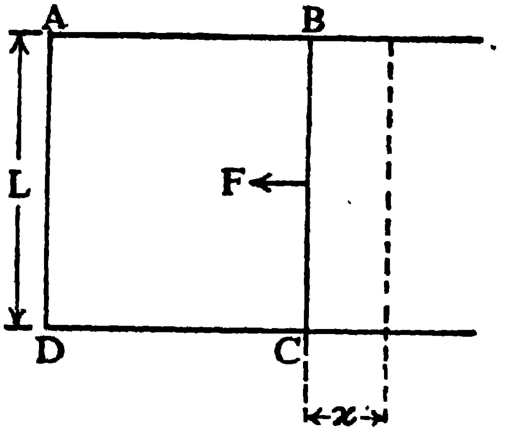


Fig. 183

This is only an approximate value for the energy per unit area ; for, if the film temperature is to remain constant, some more energy will be necessary to prevent the chilling of the film during the expansion. The surface tension of water at 0°C. is about 75 dynes/cm. ; for clean mercury, it is about 430 dynes/cm.

169. Effects of Different Factors on the Surface Tension of a liquid :—(1) The surface tension of a liquid depends on the nature of the media in contact with its surface. The effect is small if the contacting medium is a gas. (2) Substances like oil, grease, etc. reduce the surface tension of a liquid considerably. (3) It decreases with the increase of temperature, and probably vanishes at the critical temperature. The exact nature of the variation of S. T. with temperature can be represented by Eötvös-Ramsay-Shield's relation, namely.

$$T(Mvx)^{\frac{2}{3}}=K(\theta_c-\theta-d),$$

where T = S. T. of the liquid

M = Molecular weight of the liquid

v = Specific volume

x = the degree of association of the liquid

$$= \frac{\text{molecular wt. of liquid}}{\text{mol. wt. of unassociated liquid}}$$

K = a constant

θ_c = critical temperature

θ = temperature of the liquid

d = a constant very nearly equal to 7.

* (4) Surface tension decreases when a liquid is electrified. This is so because due to electrification the liquid surface experiences a kind of outward normal pressure tending to increase the surface area. This tendency being opposite to the contractile tendency of S.T., the latter effectively diminishes with the electrification of the liquid. (5) It varies with concentration too.

170. Experiments on Surface Tension :—

(i) **A Floating Needle.**—Take a slightly greased sewing needle and carefully place it horizontally on the surface of water kept in a vessel. The needle rests in a small depression (Fig. 184) in the same way as a heavy body would do when placed on a sheet of stretched rubber.

Neglecting adhesion between grease and water which is weak, the phenomenon may be explained as follows : For the needle to sink by breaking through the surface (iron being 7·8 times heavier than water), the molecules must be pulled apart. This cannot take place on account of the large cohesive forces which bind the molecules together. What happens then is that the surface becomes depressed until the resultant upward force W due to the surface tension T acting as shown in the figure is equal to the downward force W due to the weight of the needle.

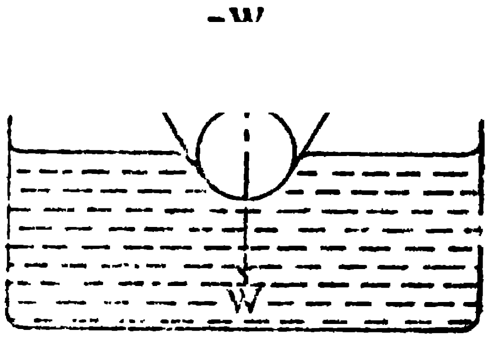


Fig 184—A Floating Needle.

The walking and running of insects on the surface of liquids are also possible due to similar reasons.

(ii) **Spreading of Oil on Water.**—Take a little quantity of oil, preferably kerosene and drop it on water. It is pulled in all directions until it spreads over the entire surface. This is because the surface tension of oil is much less than that of water ; the greater tension of the water stretches the oil in all directions.

Take some camphor shavings and simply put them on a water surface. They are smartly turned or moved hither and thither in different directions. The fact is that at each pointed end each flake readily goes into solution in the water and this reduces the surface tension at that end more than at any other, resulting in a motion of the flake.

(iii) **Soap bubble.**—Force air into a soap bubble carefully when it will expand. Remove the mouth from the pipe end, the bubble will contract forcing the gas out. This happens because due to surface tension the surface of a liquid behaves as a stretched membrane having a tendency to contract.

(iv) **Camel Hair Brush Expt.**—Dip a camel hair brush into a liquid. When the brush is taken out, the hairs are all found to be drawn together as if the hairs are now connected by a stretched membrane.

171. Spherical Shapes of Liquid Drops :—On account of surface tension the skin of liquid tends to contract in area and to attain a shape in which the exposed area is minimum for a given volume *i.e.* it takes on a spherical shape because a sphere has the least surface area for a given volume. The effect of gravity on a liquid is to make the liquid flat, as under this condition the centre of gravity of the liquid will be at the lowest. In small masses of liquids, usually the effect of the surface tension predominates over that of gravity, while in larger masses the effect is the reverse. The spherical shape of soap bubbles, rain drops, etc. illustrates the effects of surface tension in small masses of liquids, while in tanks and ponds the water assumes a flat surface illustrating the effect of gravity. Because of its large surface tension, mercury when spilt on a floor takes on the shape of small pellets in defiance of gravity.

172. Part played by Cohesion and Adhesion :—When the mutual attraction between the molecules of a liquid, (*cohesion*) contained in a vessel is less than their attraction to the sides (*adhesion*), the liquid **wets the side of the vessel** as in the case of water in a glass vessel ; but if the adhesion is less than that of cohesion, as for mercury in a glass vessel the liquid **does not wet glass** ; so mercury sprinkled on a glass surface separates out into spherical drops, whereas water or oil easily spreads over a glass surface.

173. The Angle of Contact : -If a plate is plunged vertically in a liquid, the liquid is drawn a little up the wall when the liquid wets it, as in the case of water, alcohol, copper sulphate solution, ether, etc. (Fig. 185, left), while the liquid is depressed a little when it does not wet the wall as in the case of mercury (Fig. 185, right). The section of the liquid surface near the plate is a continuous curve and is known as the *capillary curve*. Consider a point *C* where the capillary curve meets the solid surface. The angle *ABC* in the liquid, which *AC*, the tangent to the capillary curve at *C*, makes with the solid surface *BC*, is called the *angle of contact* between the liquid and the solid. It is an *acute angle* when the liquid wets the solid and is *obtuse* when the liquid does not wet it. The angle of contact of water with glass in air is very small and can be taken as zero.

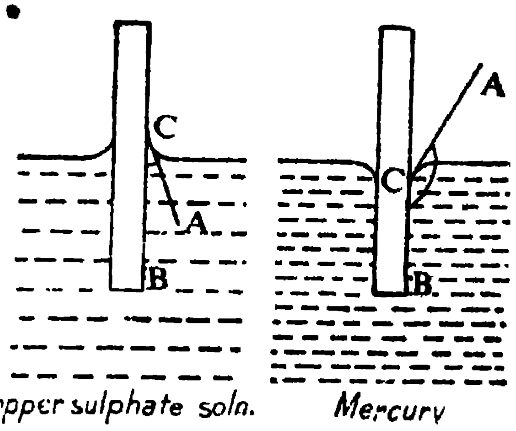


Fig. 185

The angle of contact between a liquid and a solid depends on (i) the nature of the liquid and the solid, (ii) the medium above the free surface of the liquid and is independent of the angle of inclination of the solid to the surface of the liquid.

It is actually found that the angle of contact between mercury and glass when air is above mercury is different from the corresponding angle of contact when a layer of water is spread over mercury.

174. Neumann's Triangle :—Whenever on the two sides of a surface the media are different, that is, whenever there is a discontinuity, the surface tension would show its existence and its value will depend on the medium in contact with the surface. The S. T. of mercury in contact with air is different from that in contact with water. So with three substances meeting along a line it may or may not be possible to obtain

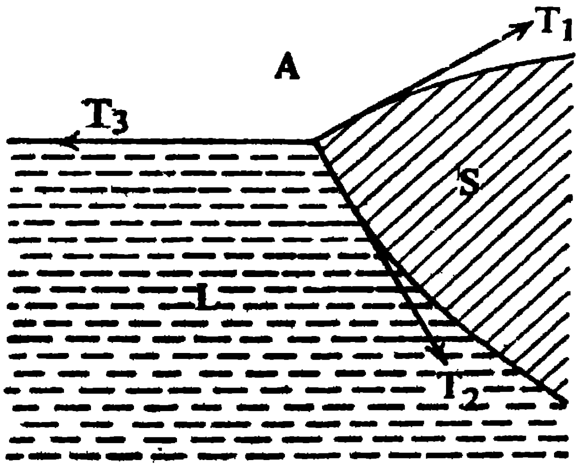


Fig. 186

equilibrium. Suppose A, S, L to be three different media of which A is air, S is a solid and L a liquid having a horizontal surface in contact with air (Fig. 186) The directions of surface tensions are obtained by drawing tangents at the lines of common contact. Let T_1, T_2 and T_3 be these tensions. If the vector sum of these three tensions meeting at a point be zero, then they must be represented by a triangle (of tension) generally known as Neumann's triangle, which determines the angles of contact of the surface. If any two tensions are not together greater than the third, this Neumann's triangle cannot be drawn and the arrangement would not be in equilibrium.

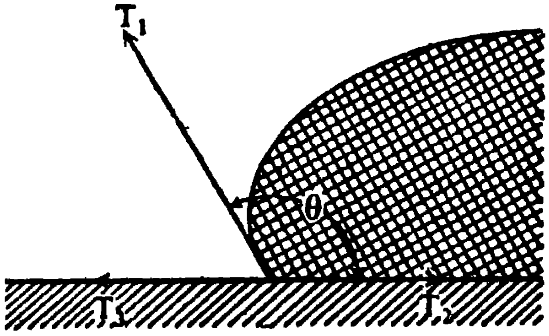


Fig. 187

When a liquid is poured on a horizontal solid surface let T_1, T_2, T_3, respectively be the surface tensions for air-liquid, liquid-solid and solid-air (Fig. 187). If now $T_3 > T_1 + T_2$, there cannot be any equilibrium and the liquid spreads over the solid, *i.e.* the liquid exhibits complete wetting.

If, however, $T_3 = T_2 + T_1 \cos \theta$, where θ = angle of contact of the liquid with the solid, a state of equilibrium is reached. Here

$$\cos \theta = \frac{T_3 - T_2}{T_1}.$$

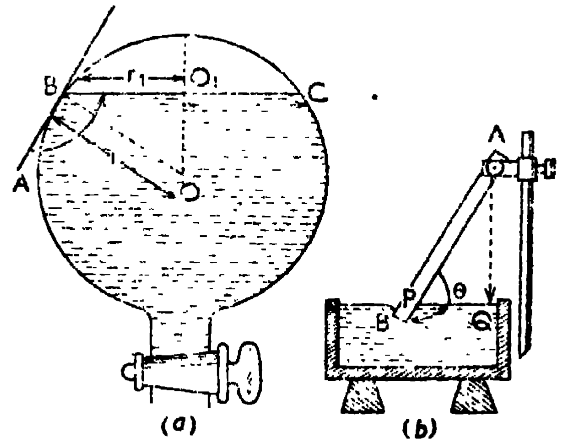
When $T_3 > T_2$, $\cos \theta$ is positive and θ is less than 90° as in the case of water. If $T_2 > T_3$, θ lies between 90° and 180° and this is the case with mercury for which θ is about 140°.

175. Measurement of the Angle of Contact of Mercury and Glass :—The angle of contact of mercury and glass can be conveniently measured by filling an inverted spherical glass bulb with clear mercury till mercury rises to such a level that no curved portion appears at the position where mercury meets the glass. This is to be optically tested by observing an image of a bright spot of light reflected from the surface of mercury. A bright band near the point of contact indicates a curvature at the position and a distinct image spot indicates the horizontality of the surface. When an exact horizontal surface has been ensured [Fig. 188(*a*)], the angle of contact is evidently the angle between the tangent at the point of contact and the liquid surface as measured within the liquid (*i.e.* $\angle ABC$ in the figure). When the bulb is of thin glass, the diameter and hence the radius of the circle at the position BC can be directly measured by slide callipers without appreciable error.

Let this be $r_1 = BO_1$. Now the diameter of the bulb is again measured by the same callipers and hence its radius $r = BO$ is determined.

Hence $\cos OBC = \frac{r_1}{r}$. So $\angle OBC$ is found. The angle of contact is evidently given by $\angle ABC = 90^\circ + \angle OBC$.

A simpler method for determining the angle of contact for mercury and glass is to dip a slanting glass plate AB partly into mercury [Fig. 188(b)]. The inclination of AB is to be so adjusted that the mercury and glass meets at P without any curvature. The angle of contact for glass and mercury is then given by $\angle BPQ = (180 - \theta)$ where $\angle \theta = \angle APQ$. θ is measured by dropping a plumb line through A so as to meet the surface of mercury at Q and then measuring AQ and PQ.



(a) (b)

Fig. 188

Evidently, $\tan \theta = \frac{AQ}{PQ}$; or, $\theta = \tan^{-1}\frac{AQ}{PQ}$. Thus the angle of contact $(180^\circ - \theta)$ is determined.

By a similar method Adam determined the angle of contact between glass and water also.

Surface Tensions at 20°C. and Angles of Contact (liquid—glass)

Liquid	S. T. (dynes/cm.)	Angle of contact
Water-air	78·0	8° to 9°
Soap solution-air	30 (approx.)	......
Paraffin oil-air	26·4	26°
Mercury-air	465·0	130° (approx.)

176. Pressure and Curvature of a Liquid Surface.—If the pressure on both sides of a liquid surface is the same, the surface becomes plane. But if the pressure on one side exceeds that on the other, the surface on the high pressure side becomes concave. Let ab (Fig. 189) represent a plane section of the liquid surface and

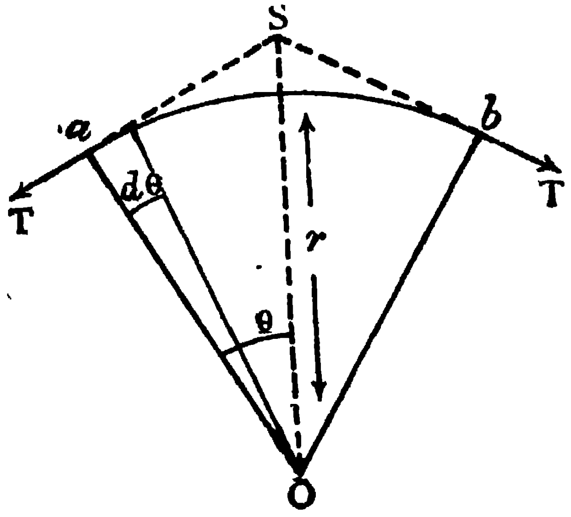


Fig. 189

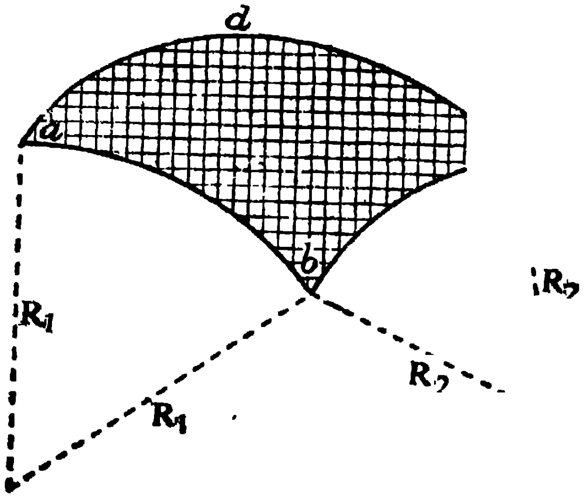


Fig. 190

let r be the radius of curvature of the surface in the plane of the section and O its centre of curvature. If a unit length of this surface be considered in a plane at right angles to the plane of the paper and if P_1 be the excess pressure on the concave side over the other, then the force acting on a small strip of area, $(rd\theta \times 1) = rd\theta$, is given by $P_1 rd\theta$ acting normally on the surface. Its resolved part parallel to OS is $P_1 rd\theta \cos\theta$.

$\therefore$ The total resolved force on ab acting parallel to OS is

$$\int_{-\theta}^{+\theta} P_1 r \cos\theta d\theta = 2P_1 r \sin\theta$$

Again the surface tension, T, at the point a resolved parallel to OS is $T \sin\theta$ and that at b resolved in the same direction is also $T \sin\theta$ ($\because$ length is unity).

Hence for equilibrium of ab,

$$2P_1 r \sin\theta = 2T \sin\theta ;$$

$$\text{or,} \quad P_1 = \frac{T}{r}.$$

If the curvature of the surface at right angles to the plane considered above be R_2; then due to this curvature the excess pressure P_2 is given by T/R_2 (Fig. 190). Now denoting the radius of curvature of ab by R_1, the total excess pressure,

$$P = T\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$$

For a spherical surface, $R_1 = R_2 = R$ (say), and $P = 2T/R$. If a spherical soap bubble is considered, as it has two surfaces, $P = 4T/R$.

For a cylindrical soap bubble, since $R_1 = \infty$ and $R_2 = R$, $p = \frac{2T}{R}$.

177. Excess Pressure within a Spherical Soap Bubble (Alternative proof) :—Let a soap bubble (Fig. 191) be of radius R. Imagine the bubble to be bisected by a horizontal plane BC. If P be the excess internal pressure, the plane horizontal section will be acted on by a force $= P \times \pi R^2$ which may, because of symmetry, be supposed to be acting at O along OA. Due to $S.T.$ the boundary of the plane section will be subjected to a force $2T$ per cm. (since the film has two surfaces), acting normally to the bounding line and tangentially to the suarface as shown in the figure. Since the length of the boundary line $= 2\pi R$, the total force due to $S.T.$ on the boundary line produces a resultant force of $2T \times 2\pi R$ which may be supposed to act at O opposite to OA. Hence for equilibrium of the section,

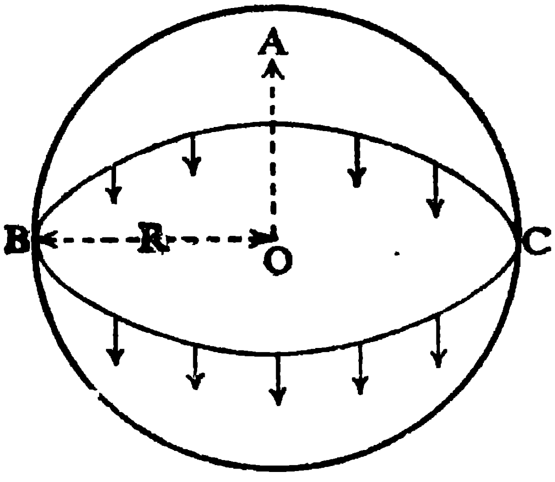


Fig. 191

$$P\pi R^2 = 4\pi RT \quad \text{or,} \quad P = \frac{4T}{R}.$$

Note : Energy of the bubble :—Neglecting the compression of the gas in the bubble, the total energy of the bubble

= surface area of the bubble × surface tension on both the surfaces taken together.

This is, because the surface tension is numerically equal to the energy per unit area approximately.

$\therefore$ The total energy $= 4\pi R^2 \times 2T = 8\pi R^2 T = 2\pi PR^3$ (approximately).

***178. Cylindrical Film** :—When two identical wire rings are brought in contact with two opposite sides of a spherical soap bubble, a cylindrical film of soap solution may be formed. The shape of the film may be different depending on the distance between the rings. For a particular distance of separation between the rings the surface of the film between them will be cylindrical whereas the portions bulging out of the rings remain spherical (Fig. 192).

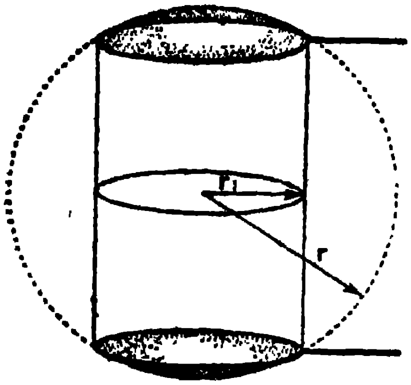


Fig. 192

For this separation the excess pressure inside the bubble is everywhere the same. Since there are two surfaces of the bubble, this excess pressure, considering the cylindrical surface, is $p=\frac{2T}{r_1}$ where r_1 is the radius of each ring or of the cylindrical bubble. Again considering the equilibrium of the spherical portion of the bubble, the same pressure excess is given by,

$p=\frac{4T}{r}$ where r is the radius of the sphere.

$$\frac{4T}{r}=\frac{2T}{r_1} \quad \text{or, } r=2r_1.$$

So under the above conditions, the spherical portions of the film must have a radius of curvature twice that of the cylindrical portion.

When the rings are further drawn apart, the value of p decreases and r increases and gradually the spherical caps are flattened. Due to this decrease in internal pressure the cylindrical surface will be drawn in and as such r_1 will diminish. The decrease in r_1 normally indicates an increase in the internal pressure. But as shown in Fig. 193 the longitudinal curvature, r_2, of the film being directed outwards introduces a negative term in the expression of excess pressure, which may be written as,

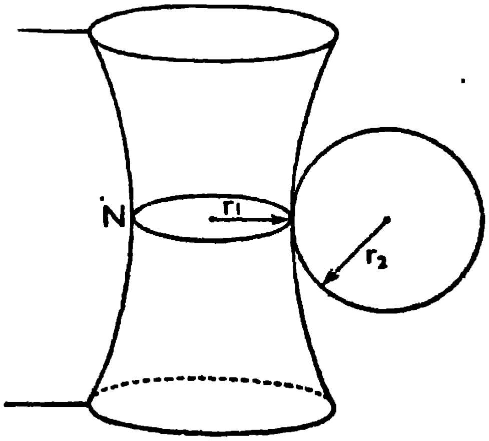


Fig. 193

$$p=2T\left(\frac{1}{r_1}-\frac{1}{r_2}\right).$$

The negative term opposes the expected increase in p due to decrease in r_1.

When the spherical caps become plane, the pressure inside is equal to the pressure outside.

Hence the excess pressure $p=0$. $\therefore$ $r_1=r_2$.

At this position even if the plane faces of the film be destroyed, the film between the rings remains as it is. If the rings are still further drawn apart without destroying the plane faces, the neck at *N* will be further drawn in due to fall in the excess pressure, and ultimately the film breaks at *N* giving rise to two separate bubbles on the two rings.

***179. Stability of Cylindrical Films :**—It can be shown from theoretical considerations that if the length of the cylindrical film is less than half of its circumference an outward bulge in the film necessitates an increase in the internal pressure while the formation of a neck or waist requires a decrease in the internal pressure. Again if the length of the cylindrical film be greater than half of its circumference, an increase in the internal pressure develops a waist whereas a decrease in the internal pressure causes an outward bulge.

These theoretical deductions were verified by Prof. Boys by the following arrangement.

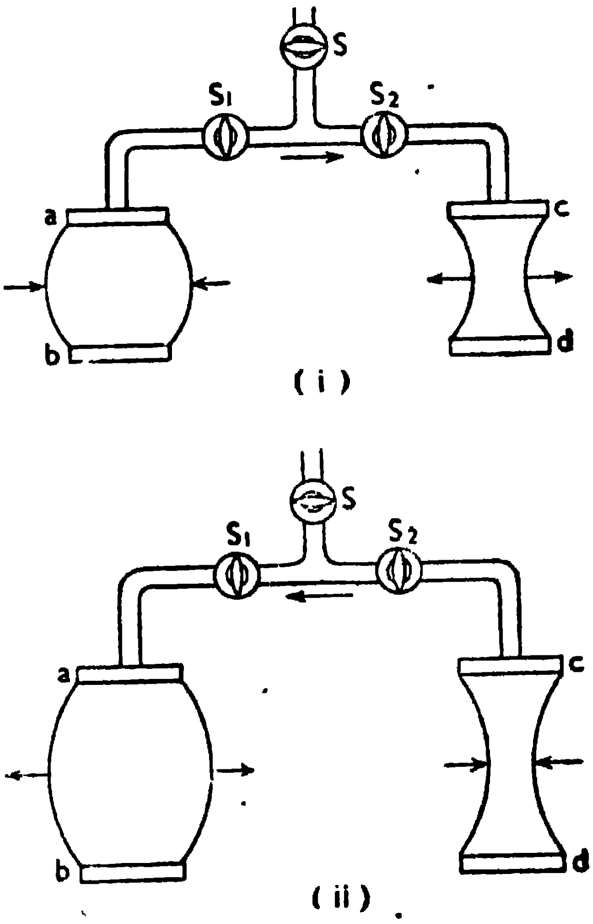


Fig. 194

A branched tube, provided with three stop-cocks S, S_1, S_2 for regulating the pressure within the bubble, is fitted with two identical cylindrical metal rings (*a*, *c*) at the ends of the two arms. These rings are placed on two other exactly similar short pieces of metallic cylinder just dipped in soap solution. Now raising the branched tube two cylindrical films may be formed on the two arms as ***ab*** and ***cd*** [Fig. 194(*i*)]. By adjusting the pressures in the bubbles with the help of the stop-cocks, an outward bulge is produced in ***ab*** and a neck in ***cd***. If the lengths of ***ab*** and ***cd*** be both less than the semi-circumference of cylinder, the pressure in ***ab*** is greater than that in *cd* [Fig. 194(*i*)]. If S be kept closed and S_1 and S_2 opened, air from ***ab*** will rush on to ***cd*** so as to equalise the internal pressure in both. Thus both the cylindrical films will assume a more cylindrical appearance and will be in a stable state of equilibrium. When put in communication, the two films may be supposed to form a single film in equilibrium. So a cylindrical film is in equilibrium when the total length of the film is less than the circumference of the film.

Again if the lengths of each of *ab* and *cd* be made greater than the semi-circumference of the cylinder and if a bulge is produced in *ab* and a neck in *cd* by properly adjusting the pressure [Fig. 194(*ii*)], the internal pressure in *cd* will be greater than in *ab*. So when *ab* and *cd* are put in communication, air from *cd* will gradually pass on to *ab* and there will be a gradual narrowing of the neck where the film will ultimately break. Thus the bubble becomes on the whole unstable. When connected together, *ab* and *cd* form a single cylindrical film of length greater than its circumference, which is thus found to be unstable.

So in the case of cylindrical film of length less than its circumference, the formation of a constriction or neck anywhere is attended with a decrease of pressure at that position and this decrease is compensated by the air blowing from the comparatively wider region of the bubble. Thus equality of pressure is established everywhere throughout the bubble, which thus attains a state of stability. Again in the case of a cylindrical film of length greater than its circumference a neck at any position is attended with a relative increase of pressure there and air from that position is gradually driven out on to the wider regions of the bubble until it breaks up at the position of the neck thereby indicating the unstable nature of the film.

***180. Stability of Liquid Cylinders: Liquid Jets**:—As indicated in the case of cylindrical bubbles, liquid cylinders are in stable equilibrium only when their lengths do not exceed the corresponding circumferences. If a cylinder of greater length be formed, it will break up into a series of drops of different sizes.

In the case of very viscous liquids, the viscosity itself may, however overcome the instability of the long cylinders. Thus the very viscous liquids like molten glass or quartz and syrup, etc., on account of their high viscosity can be drawn into long cylindrical threads without rupture.

When a jet of comparatively less viscous liquid like water issues out of a circular aperture, it forms a long liquid cylinder which being unstable soon develops waists and bulges and ultimately breaks into drops. When the larger drops break away, small droplets like *S* in Fig. 195 are produced at the position of the waists which are drawn out. These small droplets are known as **Plateau's spherules.** A jet generally breaks up into droplets of irregular sizes due to the small accidental disturbances imparted to the nozzle through which the liquid escapes. If it is possible to impart a regular series of disturbances to the nozzle, the necks developed in the issuing liquid cylinders will be equispaced and a series of equidistant and identical drops will follow

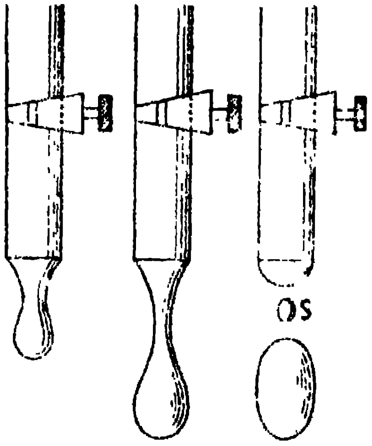


Fig. 195

*181. Bubbles and liquid Drops.

—It is a well-known proposition in Physics that a mechanical system in stable equilibrium possesses the minimum potential energy. So it follows that a liquid drop when not acted upon by external forces must take up a spherical shape because the sphere possesses the minimum surfaces area for a given volume and hence minimum energy due to S. T.

If some quantity of oil is placed in a mixture of water and alcohol of the same density, the potential energy due to gravity is not affected by the change in shape of the oil. So under the circumstances the oil will take up a spherical shape, wherein the area of the surface as also the potential energy due to surface tension are minimum.

When a liquid drop is acted on by external forces its shape is determined by the condition of minimum potential energy. So a sufficiently small drop of a liquid resting on a horizontal plate but not wetting the plate will be approximately spherical in shape for the potential energy due to gravity in this case is negligibly small compared to that due to surface tension. In a large drop, however, the potential energy due to gravity plays a predominant part and as such the drop takes the shape of a flat pool of liquid thereby lowering its centre of gravity.

When a drop of liquid falls freely in a vacuum it will assume a spherical form because the gravity no longer exerts its effect on its shape. Raindrops falling through air are also approximately spherical since the viscosity of air is very small.

182. Calming of Waves by Oil

:—A thin film of oil spread on the surface of sea is very effective in preventing the breakers which are generally caused by the wind. When oil is poured, the film of oil is carried forward due to the wind and behind this film is left a comparatively clean sheet of water. In the forward direction where the oil gathers the surface tension is decreased. So against the wind the surface tension increases and the back pull is also sufficiently increased than the forward pull. So the movement of the waves in the forward direction is retarded due to this greater surface tension and the waves are calmed.

183. Capillarity :—If a glass tube of small bore is dipped in a liquid, then, in cases where the liquid wets glass, as in the case with water, the internal level of the liquid will be higher than the level outside [Fig. 196(*a*)] but with mercury, which does not wet glass, the internal level will be *below* the outside level [Fig. 196(*b*)]. The surface in the case of water in glass is *concave* upwards, but for mercury in glass, it is *convex* upwards

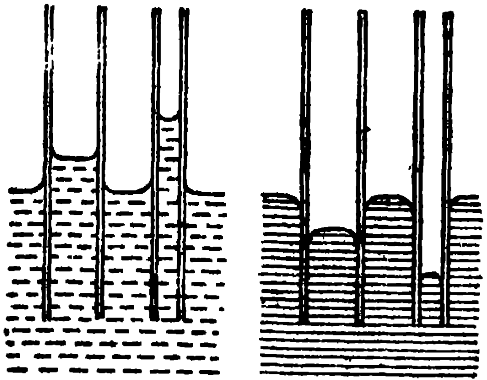
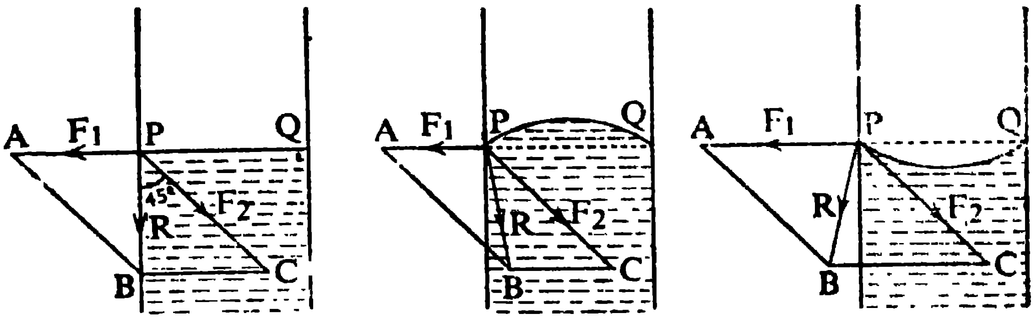

(a) (b)

Fig. 196

These results are said to be due to what is known as *capillarity*, (from Latin, *capillus*-hair) which is a consequence of surface tension of the liquid and smallness of the bore of the tube. The explanation of the rise or depression of a liquid in a capillary tube is based on the facts (*i*) that between the adjacent molecules of a liquid there is a force of cohesion and between the molecules of a liquid and the solid molecules there exists a force of adhesion, and, (*ii*) that the resultant force acting on the liquid surface is always directed at right angles to the surface,

Considering a small volume of the liquid near the point P where the liquid is in contact with the wall in a capillary tube, the adhesive force F_1 acts along PA [Fig. 197(*a*)] at right angles to the wall at P. The cohesive force F_2 acts along PC making an angle 45° with the wall at P, This is so because the molecule at P is acted on by identical cohesive forces in the vertical and horizontal directions, *i.e.* along PQ and PB due to the attraction of the liquid molecules. So the resultant cohesive force F_2 is inclined to the vertical at an angle of 45°. The resultant of F_1 and F_2 (*i.e.* R) evidently depends on the magnitudes of F_1 and F_2. (*a*) When $F_1 = F_2 \cos 45^\circ = F_2/\sqrt{2}$, R acts vertically out[illegible]wards. So the liquid surface is horizontal as in Fig. 197(*a*). [illegible]n $F_1 < F_2/\sqrt{2}$ as in Fig. 197(*b*), R lies inside the liquid normal surface at P and hence the liquid surface is convex. (*c*) Again When[illegible]$/\sqrt{2}$ as in Fig. 197(*c*), R lies outside the liquid and the out of [illegible]face becomes concave. In the case of water $F_1 > F_2/\sqrt{2}$ and [illegible]se of mercury $F_1 < F_2\sqrt{2}$ and that is why water has a concave meniscus while mercury has a convex meniscus. Once a liquid has a concave meniscus in a capillary tube, its angle of contact is acute and



(*a*) (*b*) (*c*)

Fig 197

as a consequence the unbalanced vertical component of the force of reaction on the tube due to surface tension causes the liquid to rise up the tube. Depression in the cases of some liquids may also be similarly explained. The elevation or depression of the liquid in the tube is inversely proportional to the diameter of the tube ; so the capillary effect can be clearly shown only in the case of very narrow tubes, called **capillary tubes**.

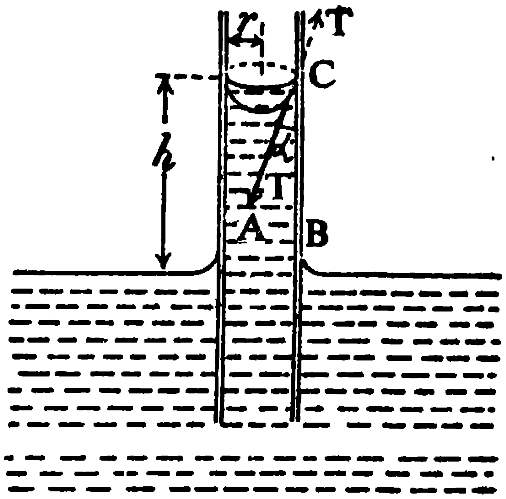


Fig. 198

The rise of oil in wicks of lamps, the soaking up of ink by blotting paper, the retaining of water in a piece of sponge, the rapid absorption of liquid by a lump of sugar, the wetting of a towel when one end of it is allowed to stand in water, are all ***instances of capillarity.***

184. Height of a Liquid in a Capillary Tube (Capillary Rise of the Liquid) :—Let a capillary tube of radius r be dipped into the liquid and the liquid rises in the tube until it stands at a height (Fig. 198). The surface of the column at the top assumes the shape of a spherical cup with its concavity turned upwards. Let the height of the column be h measured from the level of the surface of the liquid outside the tube up to the lower meniscus of the cup. Let the angle of contact ($\angle ACB$) between the liquid and the wall be α. A force T due to surface tension acts along the tangent to the liquid surface in the direction CA at each point of contact O of the liquid with the wall. According to Newton's third law, this force sets up an equal reaction in the opposite direction, as shown by the dotted line. The component of this reaction in the vertically upward direction is

$T\cos\alpha$. Since the liquid surface in the tube makes a circle of contact with the wall of the tube, the total vertical force upwards $= 2\pi r \times (T\cos\alpha)$. This force lifts up the liquid in the tube. The mass of the raised liquid in the position of equilibrium $= \left\{ (h+r)\pi r \quad \tfrac{4}{3}\pi r^3 \right\} \rho$. where ρ = density of the liquid. For equilibrium,

$$2\pi r T\cos\alpha = \left\{ (h+r)\pi r^2 - \frac{\tfrac{4}{3}\pi r^3}{2} \right\} \rho.g = \pi r^2\left(h+\frac{r}{3}\right) \rho.g.,$$ where g = acceleration due to gravity.

$$\therefore\quad T = \frac{r.\rho g.\left(h+\frac{r}{3}\right)}{2\cos\alpha} \qquad \ldots \qquad \ldots \qquad (1)$$

For water, alcohol, chloroform, etc. $\alpha = 0$, approximately. Neglecting $r/3$ compared to h,

$$T = \frac{r.\rho.g.h}{2} \text{ approximately} \qquad \ldots \qquad \ldots \qquad (2)$$

185. Jurin's Law :—The elevation or depression of a liquid in a capillary tube is inversely proportional to the radius of the tube at the place of contact. This is known as Jurin's Law of capillarity. This at once follows from equation (1) above, for T, ρ and g are constants for a given liquid at a given place, *i.e.* according to Jurin's Law, $\left(h+\frac{r}{3}\right) \times r =$ constant for a given liquid at a given place.

186. Determination of the S.T. of a liquid by the Capillary Ascent Method :—The method of capillary ascent may be utilised in determining the surface tension of a liquid which wets glass.

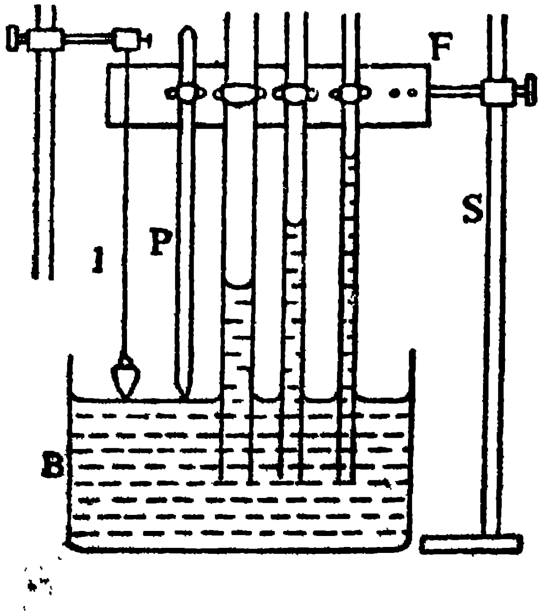


Fig. 199

Here a glass tube is first thoroughly cleaned by nitric acid and then washed first by caustic soda solution and then in a stream of tap water. It is then dried in hot air blast. The tube is then cut into pieces out of which capillary tubes are drawn by heating them in a fish-tail burner. A number of such capillary tubes of different diameters are chosen. These tubes together with an index rod of pointed ends P are fixed parallelly on a glass frame F

(Fig. 199) near their upper ends by means of wax. The lower tip of P should be above the lower ends of the tubes. Now the liquid is taken in a shallow cylindrical glass basin. F is vertically held by a clamp attached to a vertical stand S and the lower ends of the tubes are dipped in the liquid to a depth slightly greater than the desired depth in their fixed position. This ensures the proper wetting of the tubes. Now by means of a plumb line l, the tubes are adjusted in the vertical position and the depth is also fixed such that the lower tip of P just touches the liquid surface.

With the help of a travelling microscope the reading corresponding to the upper tip of P and those of the horizontal parts of the concave meniscus of the liquid in different tubes are taken. The positions of the meniscus in the tubes are marked by ink or any other suitable means. P is now taken out and its length is measured from the horizontal traverse of the travelling microscope. The difference between a meniscus reading and the reading to the upper tip of P subtracted from the length of P, gives the height of the liquid meniscus. In this way the heights of ascent in different tubes are determined. The capillary tubes are now cut at the marked positions of their liquid meniscus and their diameters (hence radii) are measured by the said microscope in two mutually perpendicular directions. The mean diameters (hence radii) are thus found out. A graph is then drawn relating the height of ascent, h, with radius, r, of the tubes. It is a rectangular hyperbola From the graph a suitable value of h and its corresponding r are chosen for calculation of S. T. from the relation,

$T=\frac{1}{2}r\rho g\left(h+\frac{r}{3}\right)$, where ρ=density of the liquid at the room temperature.

Precautions and Errors :—

(1) In order that the bore may be uniform the capillary tubes should be drawn with uniform pressure.

(2) In washing or in determining S. T. of water preferably electrically distilled water or tap water should be used as the ordinary distilled water may contain traces of grease.

(3) As dirt and grease may alter the value of S. T., the tubes are to be previously freed from such contaminations.

(4) The verticality of the tubes should be tested with a plumb line.

(5) Density of the liquid should be noted at the ambient temperature.

(6) The pointer should be of such a length that the upper tip lies within the vertical traverse of the microscope.

(7) The tubes once dipped in the liquid should not be taken out of it during the measurement of height as otherwise air bubbles may enter the tubes and vitiate the measurements.

187. Determination of the Surface Tension of a Soap Solution :—For determining the surface tension of soap solution, usually arrangements as shown in Fig. 200 are made. Here A is a metal box the two opposite faces of which are made of glass. The metal base is bored in the centre through which a metal stand carrying a holder H can be adjusted in any position. The experimental soap solution is kept in a flat vessel on this holder. A glass tube Cg of about 3 mm. diameter having a U-shaped end is inserted inside A in the manner shown in the figure. This tube can be rotated about a horizontal axis. The free end of the tube outside A is provided with a side tube and a stop-cock arrangement connected to a sensitive manometer. The horizontal portion is connected through a rubber tubing and a pinch cock (at C) to a low pressure air supply. To form the soap bubble the end g of Cg is made to point downwards and H is raised so as to dip the end g. H is then lowered when a film of soap solution is formed across the end g. By blowing air the bubble is now formed at g and Cg is then rotated through 180 to bring the bubble in the position shown in the figure. The slight excess liquid draining from the bubble collects at the bend of the tube Cg. Opening the stop cock S, the pressure excess inside the bubble can be directly obtained from the manometer. If h be the difference in the two columns of liquid in the manometer, this excess pressure $=p=h \rho g$.

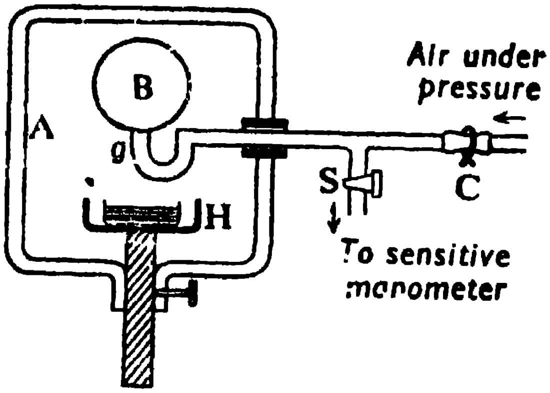


Fig. 200

But excess pressure in a soap bubble is given by $p=\frac{4T}{R}=\frac{8T}{D}$

where R and D represent the radius and the diameter of the bubble.

$$\therefore \quad \frac{8T}{D}=h \rho g \text{ ; or, } T=\frac{h\rho g D}{8}.$$

Thus knowing all quantities on the right-hand side T can be determined. The diameter D of the bubble is found by illuminating it and casting the image of the same on a screen at a known distance with the help of a convex lens of known focal length.

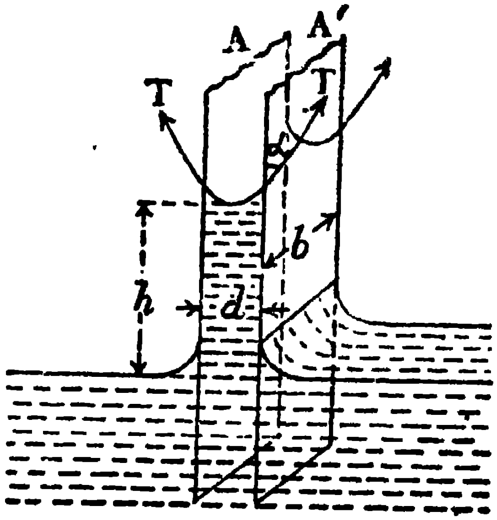


Fig. 201(a)

188. (a) Rise of Liquid between two Parallel Plates :— Let d be the distance between the plates A, A', immersed in a liquid [Fig. 201(a)] and h be the vertical ascent of the liquid whose density is ρ. Now suppose T and ∢ be the surface tension and angle of contact of the liquid respectively. Consider the equilibrium of the liquid raised between the plates of breadth b. The weight of the liquid is $h\,(db).\rho.\,g$, approximately (neglecting the curvature of the meniscus). This weight is supported by the vertical components of the force $T \times b$ acting on each plate (arising due to surface tension) *i.e.* by $2Tb \cos$ ∢. Hence for equilibrium, $hdb\rho g = 2Tb \cos$ ∢ ;

$$\text{or, } h = \frac{2T \cos \sphericalangle}{d\,\rho\,g}.$$

The expression gives us the magnitude of the vertical ascent of a liquid between two parallel plates When the angle of contact is obtuse (as in the case of mercury), cos ∢ becomes negative and hence h is also negative. So the liquid meniscus in between the plates will lie below the horizontal level of the liquid outside the plates.

In the case of *two glass plates partially dipped in water* since the pressure in between the plates, in the region where the liquid rises, is less than the atmospheric pressure acting on the outer surfaces of the plates, they will be pushed together.

This is also the reason why when two *small floating bodies wetted by the liquid are attracted when they are close to each other.*

In the case when *two glass plates are partially dipped in mercury* which does not wet the plates, the mercury level between the plates will be depressed. So the pressure at a point in between the plates above the depressed meniscus will be less than the pressure at points on the same horizontal level on the other sides of the plates because the former pressure equals the atmospheric pressure and the latter is the atmospheric pressure plus the hydrostatic pressure due to a column of mercury equal to the depth of the point below the free surface of mercury. So here also the plates are drawn together. Similarly, *attraction exists also between small floating bodies which are not wetted by the liquid.*

When one of the floating bodies is wet and the other is not wet, it is found that these two bodies are repelled when close to each other. This can also be explained from the theory of surface tension as above.

(b) Liquid drop between two plates.—When two plates of glass are put face to face, no difficulty is experienced in separating them. But if there is a liquid drop in between and it is squeezed into a thin film, it requires a considerable force to pull them apart.

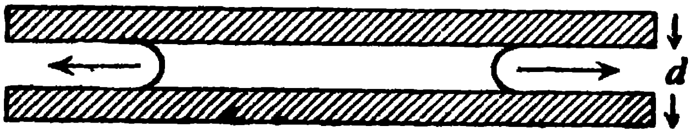


Fig. 201(*b*)

If the liquid wets the glass plates, the film spreads over an almost circular area. If d be the thickness of the liquid film between the plates which is concave outwards [Fig. 201 (*b*)], the radius of curvature of the concave edges of the film is nearly $d/2$. If r be the radius of the circular film of liquid, the excess pressure p inside the film over the outside pressure is given by,

$$p=\left(\frac{T}{r}-\frac{2T}{d}\right),$$ where T is the surface tension of the liquid.

As r is very large compared to d, we can write, $p=-2T/d$.

The negative sign indicates that the pressure inside the film is less than the atmospheric pressure. As d is very small p will be quite appreciable. So if A be the area of the plate wetted by the film, the total force pressing the plates together is given by $\frac{2TA}{d}$. Hence to separate these plates a force greater than this will be always necessary. As d becomes smaller, this force increases in magnitude.

189. Other methods of determining S. T. :—

(a) Jaeger's Method.—

This method is based on the fact that the pressure inside spherical bubble exceeds that outside by $p=\frac{2T}{r}$.

Here a reservoir *R* (Fig. 202) attached to a force pump is fitted with a vertical tube *CDAB* bent in the way shown in the figure. The tube is of fine bore and is fitted with a stop cock *S* and a manometer *M*. The lower end of *AB* dips into a liquid contained in a vessel.

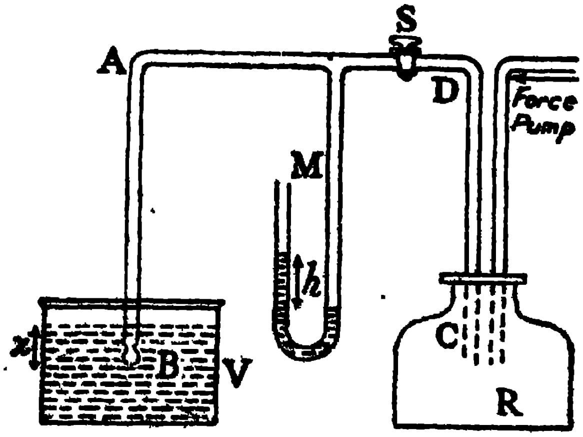


Fig. 202

V. Let x be the depth of the liquid column up to the end, B, of the tube AB and T be the S.T. of the liquid. Suppose r is the radius of the orifice at B.

If air is slowly forced into the tube by opening S slightly the surface of the liquid in AB sinks lower and lower until finally a hemispherical bubble of the same radius r as the orifice at B is formed and breaks off with sudden change in the indications of the manometer. Another bubble is gradually formed and breaks off as usual, and so on. The manometer indicates a pressure difference of h cms. of liquid which is maximum when the bubble is just about to break off. This h is measured by a vertical traverse of a travelling microscope, when the bubbles form slowly at the rate of one or two per second. The diameter (hence radius) of the orifice is also measured by a travelling microscope.

The maximum internal pressure in the bubble $= h\rho g + \pi$ where $\rho =$ density of the manometric liquid and $\pi =$ atmospheric pressure at the time of the experiment.

Again pressure just outside the bubble $= \pi + xdg$ where $d =$ density of the experimental liquid.

$\therefore$ Excess internal pressure $= h\rho g - xdg$

$$= g(h\rho - xd) = \frac{2T}{r}$$

(since the bubble here has only one surface).

$$\therefore T = \frac{gr}{2}(h\rho - xd).$$

All the terms on the right-hand side being known, T can be calculated.

The method is suitable for determining the S.T. of any liquid at different temperatures because the temp. of the vessel V can be easily controlled. It is also suitable for determining S.T. of a molten metal.

(b) U-tube method.—The surface tension of a liquid can conveniently be determined at different temperatures by the U-tube method which is nothing but a modified capillary tube method. A U-shaped

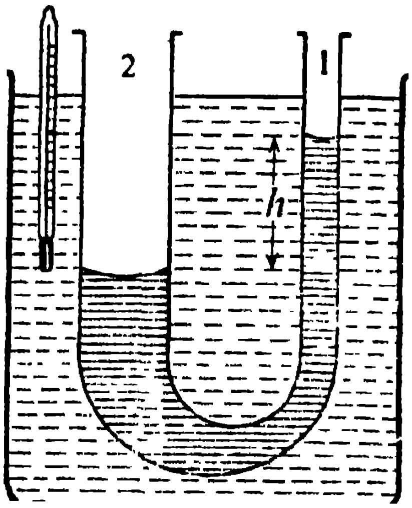


Fig. 208

glass tube is taken one limb of which is very narrow and the other broader. The tube is partly filled with the experimental liquid and placed inside a bath at a suitable temperature, measured by a thermometer inserted inside the bath. If the angle of contact of the liquid and glass be acute, the liquid in the narrow limb will stand at a higher level. Let h be the difference of levels of the liquid meniscus in the two limbs. If P be the atmospheric pressure and r_1 and r_2 the radii of the narrower and the broader limbs, the pressure just below the meniscus in the two limbs are respectively.

$$p_1=\left(P-\frac{2T\cos\theta}{r_1}\right) \text{ and } p_2=\left(P-\frac{2T\cos\theta}{r_2}\right),$$

T and θ being the surface tension and the angle of contact.

But p_1 p_2 = hydrostatic pressure due to the difference in levels $(h)=h\rho g$.

$$\therefore\ 2T\cos\theta\left[\frac{1}{r_1}-\frac{1}{r_2}\right]=h\rho g.$$

Thus knowing h, ρ, g, r_1, r_2 and θ, T can be determined.

It is to be noted that surface tension of all liquids decreases with the rise of temperature with the two exceptions of molten copper and cadmium.

(c) Quincke's method as modified by Edser (Drop method).—

When a quantity of liquid is poured on a horizontal solid surface which it does not wet, liquid drop is formed whose shape is partly determined by gravity and partly by surface tension. So when mercury is poured on a wooden platform droplets will be formed. When these droplets are smaller in size surface tension effects are predominant and the gravitation plays a minor part. So these droplets are spherical in shape. But as the size of the drops increases the gravitation effect gets the upper hand and S.T. effect becomes small. So the drop is flattened

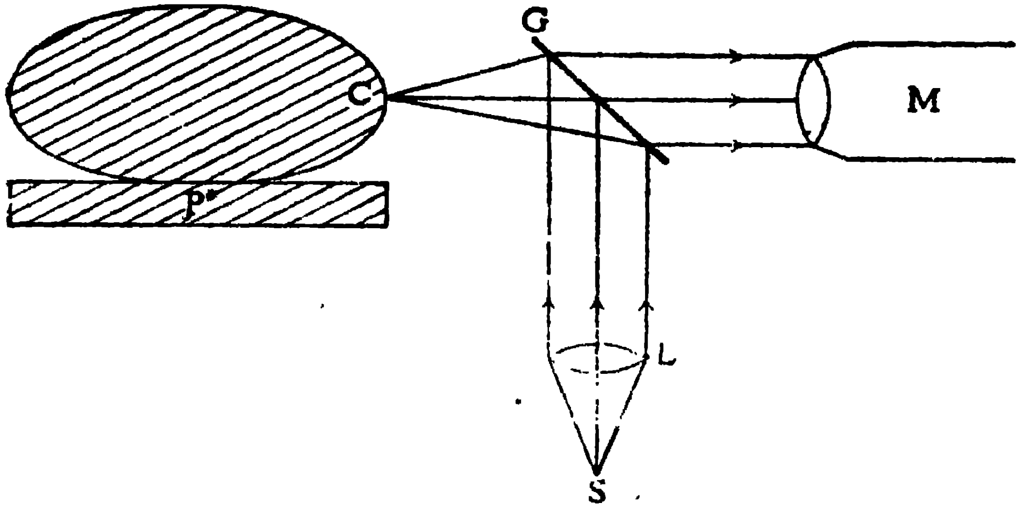


Fig. 204

and ultimately the upper surface becomes horizontal in the equilibrium position (Fig. 204). Whatever may be the shape at the edges of the drop, there is always a position C at the edge where the tangent to the surface is exactly vertical.

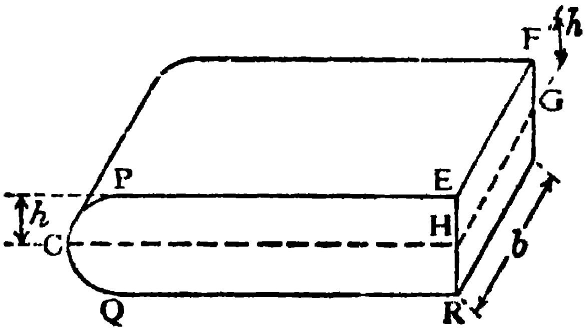


Fig. 204(a)

To establish a relation between the gravitational effects and S. T. in a large drop, let us consider a thin slice of the drop [Fig. 204(a)] of horizontal width b with the vertical faces parallel. The forces on the face $PQRE$ and its parallel and opposite pair are equal and opposite because of symmetry. If a horizontal plane CHG be considered through the point (where the liquid surface is vertical), the horizontal forces acting on the section above CHG cannot have any contribution due to S T. at C which is vertically directed. At the end of the plane vertical section $EFGH$, the horizontal forces arise out of the hydrostatic pressure on the section and the pull due to S.T. over EF.

The total horizontal hydrostatic thrust on $EFGH$

$$= \text{area} \times \text{average hydrostatic pressure}$$

$$= (b \times h) \times \tfrac{1}{2} h \rho g = \tfrac{1}{2} h^2 \rho g b.$$

Where $h = HE$ = distance between the horizontal surface of the drop and the point C.

The force of surface tension on $EF = T \times b$.

For equilibrium $Tb = \frac{1}{2} h^2 \rho g b$;

or, $T = \frac{1}{2} h^2 \rho g$.

This relation supplies us with a method of determining S.T. of a liquid without measuring its angle of contact with the solid plate.

The actual experimental arrangement as was adopted by Edser is shown in Fig. 204.

Light from an incandescent lamp S is focussed on the edge of the mercury drop with the help of a lens L and a glass plate G set at 45 to the horizontal. A travelling microscope M receives the light reflected from the edge as it passes straight through the glass plate G. When the microscope is focussed, a sharp horizontal line is found at C. By raising or lowering the microscope this line is made to coincide with the horizontal cross-wire of the eyepiece. Now specks of lycopodium powder are sprinkled on the upper horizontal surface of the drop and the microscope is raised and focussed on some of these specks. The vertical distance traversed by the microscope during the operation gives h.

Hence knowing h, T can be easily obtained from the above relation.

(d) Ferguson's method for a small quantity of liquid (Horizontal tube method).—

This method is really a modification of the capillary tube method. Here a narrow capillary tube C is mounted horizontally and a pressure is applied by any suitable device connected at the end E of the horizontal tube. The pressure excess p is measured by a manometer M connected to the tube. A small quantity of the liquid is introduced in the capillary tube and, by adjusting the excess pressure p inside, the surface of the liquid at the free end of the capillary is made plane. This is checked by the regular reflection taking place at the plane liquid surface, as shown in Fig. 205. When the liquid wets glass, the angle of contact may be supposed to be zero. Hence if r be the radius of the capillary tube,

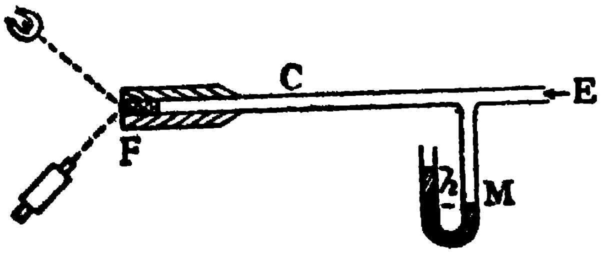


Fig. 205

$$2\pi r T = p \times \pi r^2 = h \rho g \pi r^2 \text{ ;}$$

or, $$T = \frac{rh\rho g}{2},$$

where T = surface tension, ρ = density of the manometric liquid, r is actually measured by drawing in the capillary tube a mercury pellet, finding its lengths at different positions and carefully determining its weight.

This method is suitable only for liquids wetting the glass and is applied particularly when a small quantity of it is available.

(e) Searle's balance method for determining surface tension of a liquid :—

This is one of the simplest and quickest of methods for the determination of surface tension of a liquid whose angle of contact is very nearly zero.

The apparatus (Fig. 206) consists of a metal strip AB fixed to a fine steel wire, w, which is horizontally stretched and rigidly attached to two supports R_1 and R_2 at the two ends. The end A of the strip serves as a pointer moving against a scale S. A scale pan P provided with a hook at the bottom is suspended from AB conveniently at a point in between the pointed end and the wire. From the hook is suspended a thin glass plate above the liquid contained in a vessel V whose height can be adjusted by a suitable arrangement. The strip AB is then made horizontal by adjusting the position of the movable weight W. The vessel of liquid is now raised so that G is dipped a little in the liquid. The vessel is then slowly lowered till a film of liquid just connects the lower edge of the glass plate with the liquid surface. In this position the glass plate is pulled down with a force due to surface tension acting along the perimeter of the lower edge of G. Due to this pull, the pointer A is also moved down. The reading of the pointer is noted. The vessel is now removed and G dried. When this dry glass plate is again suspended from P in air the horizontality of AB is regained. Now if by placing a weight mg on P the pointer again deflects downwards to the same extent, the force due to surface tension must be equal to mg.

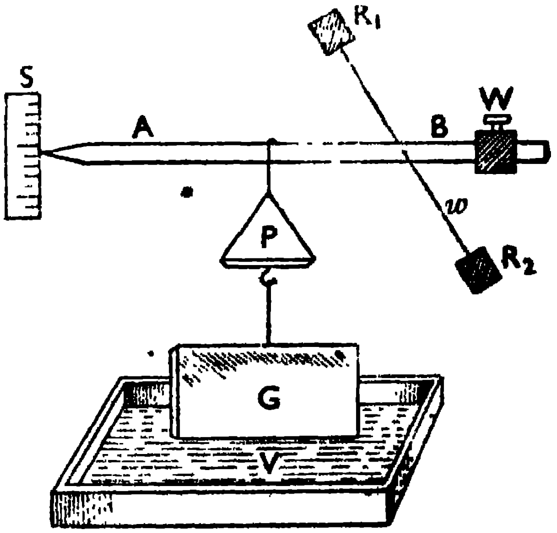


Fig. 206

The pull due to S.T. is evidently given by $2T(l+t)$ where l = length of the plate and t = its thickness, the angle of contact being zero.

$$\therefore \quad 2T(l+t)=mg\ ; \quad \text{or, } T=\frac{mg}{2(l+t)}.$$

Thus knowing l, t and mg, T can be found.

(f) **Drop-weight method.**—In this method, the liquid under test is taken in a burette to the nozzle of which is attached a piece of India-rubber tubing. Into the lower end of this tubing is inserted a vertical glass tube of circular cross-section (of 3 to 4 mm. bore). The flow of liquid through the glass tube is controlled by fitting a pinch-cock to the rubber tubing. The liquid is allowed to flow through the glass tube in such a way that the drops formed at the open end get detached at the rate of one per minute. The average mass of a falling drop is determined by collecting these drops in a previously weighed clean beaker, counting the number of drops so collected and weighing the beaker once again after collection. The radius of a drop is directly obtained from the radius of the bore which is measured by a travelling microscope. From these data the surface tension of the liquid can be roughly computed from the theoretical relation derived below.

When a liquid drop is on the point of detachment, it becomes cylindrical and the vertical forces on the drop must balance in the stable state. These vertical forces are :—(*a*) mg, the weight of the drop acting downwards, where m is the mass of the drop ; (*b*) $2\pi rT$ due to surface tension acting upwards where T is the surface tension and r, the radius of the cylindrical part which is supposed to be the same as that of the outer radius of the tube ; (*c*) $p\pi r^2$ acting downwards where p is the excess internal pressure in the drop.

$$\therefore \quad 2\pi rT=mg+p\pi r^2 \qquad \ldots \qquad \ldots \qquad \ldots \qquad (1)$$

But as the drop is cylindrical, $p=\dfrac{T}{r}$.

$\therefore$ Substituting in (1), $2\pi rT=mg+\pi rT$.

$$\text{Hence, } T=\frac{mg}{\pi r} \qquad \ldots \qquad \ldots \qquad \ldots \qquad \ldots \qquad (2)$$

Thus knowing m, r and g at the locality, T can be found out. Lord Rayleigh suggested a modified relation, $T=\dfrac{mg}{3\cdot8r}=0\cdot26\dfrac{mg}{r}$, which is supposed to yield better results. Harkins and Brown showed that Rayleigh's relation is true only when the ratio $\dfrac{V}{r^3}$ does not exceed 10, where V refers to the volume of the drop. The method is not a reliable one and gives only a rough value of the surface tension.

(g) Rayleigh's Jet Method :—

Rayleigh devised a method of determining the surface tension of a liquid by using the principle of the breaking up of a liquid into drops. When the drops first form, the various parts in them are in relative motion and their shape oscillates about the mean spherical form. This change can be observed just before the drops separate from each other. This is so because of the cyclic change in the lateral dimension of the jet in a particular direction. In Fig. 206(a), the jet is formed at the end of a glass tube attached to a funnel and the point A and B cover up the extent of one complete cyclic change of cross-section of the jet. The time period of vibration of a liquid drop about its mean spherical form under the influence of surface tension is given ifrom dimensional formula as,

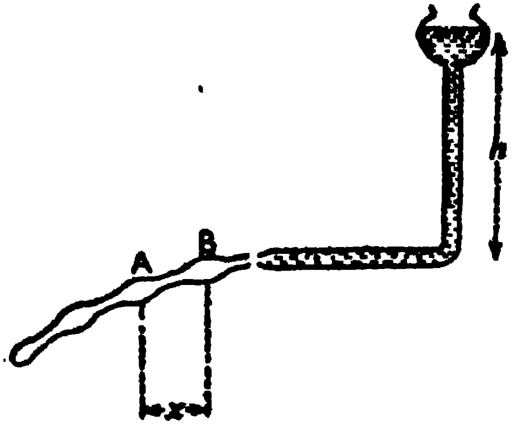


Fig. 206 (a)

$$t=k\sqrt{\frac{\rho r^3}{T}}.$$

(See Art. 269, Ex. 1)

where k is a constant, whose value from experimental results is found to be $\frac{\pi}{\sqrt{2}}$, ρ = the liquid density, T = the surface tension and r = the radius of the drop.

Hence $t=\pi\sqrt{\frac{\rho r^3}{2T}}$.

The horizontal velocity of the liquid jet if the delivery at the end of the glass tube is horizontal is given by,

$$v=\sqrt{2gh}\ \text{(vel. of efflux) (Vide, Art. 197)}$$

where h is the height of the liquid surface above the orifice. So the horizontal distance x between A and B is $x=vt$.

$$\therefore\quad \frac{x}{\sqrt{2gh}}=\pi\sqrt{\frac{\rho r^3}{2T}}$$

$$\text{or,}\quad T=\frac{\pi^2\rho g h r^3}{x^2}.$$

From this relation T can be determined by measuring x with a travelling microscope and taking r to be the same as the radius of the orifice, the other quantities ρ, g, h in the expression being also directly or indirectly determinable quantities.

(h) Ripple method.—

It can be shown theoretically that the velocity V of ripples on the surface of a liquid is given by $V^2=\dfrac{2\pi S}{\lambda\rho}+\dfrac{g\lambda}{2\pi}$,

where S = the surface tension of the liquid,
λ = the wavelength of the ripples,
g = acceleration due to gravity
and ρ = the density of the liquid.

Lord Rayleigh utilised the above relation in determining the surface tension of mercury from the wavelength and frequency of the ripples formed on mercury surface ($\because$ $V=n\lambda$). The method has been subsequently modified by many workers. The method due to R. C. Brown, which is supposed to be one of the most accurate modifications is described briefly below.

Here the experimental liquid was water contained in a shallow brass tank. The ripples were produced by a dipper D attached rigidly to a reed R_1 which was actuated by an alternating current passing through the loud speaker unit U_1. The natural frequency of the reed was adjusted in unison with that of the a.c. supply. To measure λ of the ripples a stroboscopic arrangement was made. Light from a pointolite lamp P was focussed on a horizontal slit S_1 by a lens. With the help

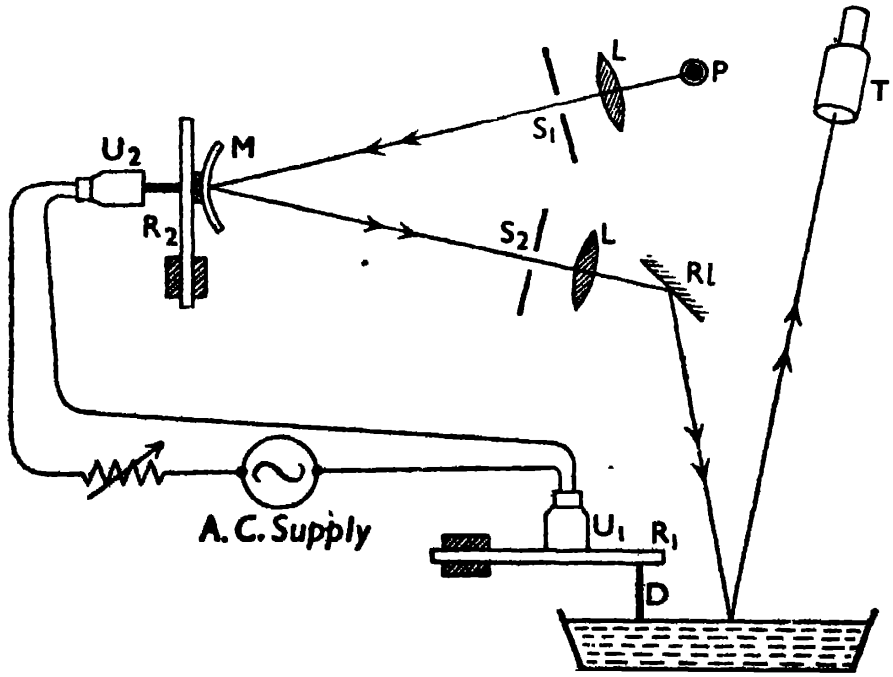


Fig. 207

of a concave mirror M, the light from the slit was focussed at a point S_2 in the focal plane of a second lens. The parallel beam of light coming out of this lens was reflected on to the disturbed surface of water by means of a plane reflector Rl. The concave mirror M was fitted to another reed R, also tuned to the a.c. supply and actuated by the current through another loud speaker unit U_2 which was connected in series with U_1. So for every ripple produced by the dipper, the light from M passed twice through S_2 and the intermittent illumination of the liquid surface had twice the frequency of the dipper. With a telescope T the light reflected from the liquid surface was observed. A series of straight line images parallel to the edge of the dipper was visible in the field of view. The distance between the successive lines, which was equal to $\lambda/2$, was measured. Hence knowing frequency of the reed, V was calculated from relation $V=n\lambda$. Then with the aid of the above relation S was computed.

*Ripples formed on Liquid Surfaces.—

During the propagation of waves on liquid surfaces the forces, that come into play, are gravity, surface tension and hydrostatic pressure, neglecting the damping effect due to viscosity. When the amplitude of the wave is moderate and the wavelengths long (*deep water waves or gravity waves*), the gravitational forces are far more predominant and the effect of surface tension is only negligible. In cases of small waves with relatively sharp curvatures (*capillary waves or shallow-water waves*), surface tension becomes the dominant force But in the cases of *ordinary ripples* which come under neither of the above categories, all the three types of forces will act conjointly. In such cases the motion is neither entirely transverse nor longitudinal. Each particle describes an orbit in the upper part of which it moves in the direction of the wave propagation. If the motions are S.H. in nature, these orbits are circular provided the depth of the liquid is large compared with the wavelength.

If the waves travel from left to right with velocity V, imagine a velocity $-V$ to be superposed on the whole system to bring the waves to rest. The liquid now streams with an average velocity V from right to left on which are superposed the orbital motions of the individual particle (*i.e.* the vortex velocity). If the velocity of the particle be v, then the relative velocity of the liquid flow at R will be $(V-v)$ and that at Q will be $(V+v)$. This

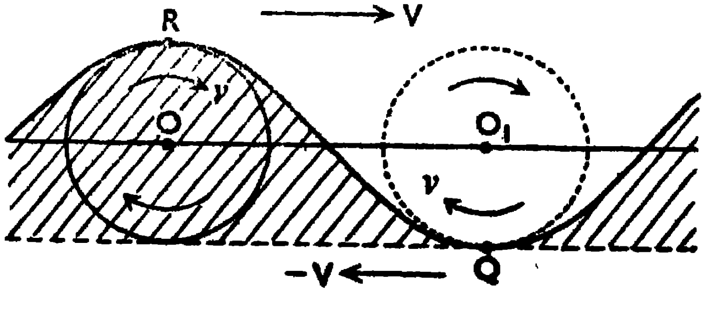


Fig. 207(a)

conclusion automatically follows from the equation of continuity of flow wherein the product of the velocity of flow and the corresponding sectional area must be constant. As the sectional area at Q is smaller, velocity will be greater there.

Supposing OO_1 to be the normal horizontal level, we can apply Bernoulli's theorem (Art. 196) at R and Q and get,

$$P_1 + h_1 \rho g + \tfrac{1}{2}\rho V_1^2 = P_2 + h_2 \rho g + \tfrac{1}{2}\rho V_2^2 ;$$

$$\text{or, } (P_1 - P_2) + (h_1 - h_2)\rho g = \tfrac{1}{2}\rho(V_2^2 - V_1^2) \qquad \ldots \qquad \ldots \quad (1)$$

where P_1, h_1, V_1 are the pressures on the liquid surface (due to S.T.), the height above the horizontal level and velocity of the liquid at R and P_2, h_2, V_2, the corresponding quantities at Q, ρ and g carrying their usual meaning.

If the wave be sinusoidal in nature, we have,

$y = a \sin\frac{2\pi}{T}\left(t - \frac{x}{V}\right)$, where T is the period and a, the amplitude of the wave which is identical with the radius of the circular orbit described by the particles of the liquid medium.

Hence $h_1 = a$ and $h_2 = -a$ in the present case.

$$\therefore \quad h_1 - h_2 = 2a.$$

Again $v = \omega a = \frac{2\pi}{T} a = 2\pi \frac{V}{\lambda} a$, ($\lambda$ = wavelength).

$$\therefore \quad V_2^2 - V_1^2 = (V+v)^2 - (V-v)^2 = 4Vv = \frac{8\pi V^2 a}{\lambda}.$$

Substituting in (1), we get,

$$P_1 - P_2 + 2a\rho g = \frac{4\pi a \rho V^2}{\lambda} \qquad \ldots \qquad \ldots \qquad \ldots \quad (2)$$

Now if r be the radius of curvature of the ripple at R and $-r$, that at Q, then,

$$P_1 - P_2 = \left(\frac{S}{r} - \frac{S}{-r}\right) = \frac{2S}{r}.$$

Since, $\frac{1}{r} = \frac{d^2y}{dx^2}$ approximately, we get from the wave equation above,

$$\frac{dy}{dx} = -\frac{2\pi a}{\lambda} \cos \frac{2\pi}{T}\left(t - \frac{x}{V}\right).$$

$$(\because \quad \lambda = VT)$$

and $$\frac{d^2y}{dx^2} = -\frac{4\pi^2 a}{\lambda^2} \sin \frac{2\pi}{T}\left(t - \frac{x}{V}\right).$$

At R and Q, $\frac{dy}{dx} = 0$. Hence, $\cos \frac{2\pi}{T}\left(t - \frac{x}{V}\right) = 0$;

$$\text{or,} \quad \sin \frac{2\pi}{T}\left(t - \frac{x}{V}\right) = \pm 1.$$

$$\therefore \quad \text{At } R \text{ and } Q, \; \frac{1}{r} = \frac{d^2y}{dx^2} = \frac{4\pi^2 a}{\lambda^2}.$$

$$\therefore \quad P_1 - P_2 = \frac{2S}{r} = \frac{8\pi^2 aS}{\lambda^2}.$$

Hence eqn. (2) reduces to,

$$\frac{8\pi^2 aS}{\lambda^2} + 2\rho g a = \frac{4\pi a\rho V^2}{\lambda} \qquad \ldots \qquad \ldots \qquad \ldots \quad (3)$$

When *deep water waves* are considered, $S \to 0$.

$$\therefore \quad \frac{4\pi a\rho V^2}{\lambda} = 2\rho a g \; ;$$

$$\text{or,} \quad V = \sqrt{\frac{g\lambda}{2\pi}}.$$

This is the *velocity of the gravity waves.*

For capillary waves, the second term in eqn. (3) is unimportant and can be neglected. So for the capillary waves,

$$\frac{8\pi^2 aS}{\lambda^2} = \frac{4\pi a\rho V^2}{\lambda} \; ;$$

$$\text{or,} \quad V = \sqrt{\frac{2\pi S}{\lambda\rho}}$$

This is the expression for the *velocity of the capillary waves.*

For **ordinary ripples**, eqn. (3) holds good and we get,

$$V^2 = \frac{8\pi^2 aS}{\lambda^2} \times \frac{\lambda}{4\pi a\rho} + 2\rho g a \times \frac{\lambda}{4\pi a\rho}$$

$$= \frac{2\pi S}{\lambda\rho} + \frac{g\lambda}{2\pi} \qquad \ldots \qquad \ldots \quad (4)$$

190. Molecular Monolayers (Mono-molecular layers) :—Some substances like oils and fats are practically insoluble in water But by taking a known quantity of such a substance, dissolving it in benzene and pouring it over the surface of water, it is possible to get a thin film of it formed on the surface. The solution spreads over a known area and as the benzene evaporates out, a film is formed whose thickness can be calculated from the area and the mass of the substance used. Langmuir produced, by this method, films or layers one molecule thick which are known as *monolayers.* These monolayers are found to obey certain laws similar to gas kinetic laws.

191. Vapour Pressure on a Curved Liquid Surface :—The process of evaporation is nothing but the escape of molecules from the surface of a liquid. So molecules, evaporating from a liquid surface, shall have to overcome the attraction exerted by other neighbouring molecules. Surface tension, as we know, also arises out of this attraction. So a definite relation should be expected between surface tension and the vapour pressure. Though the evaporation process is retarded by the molecular attraction, the process of condensation is not retarded. The number of vapour molecules striking a liquid surface is proportional to the vapour pressure and as each of these striking molecules is captured by the liquid surface, they all condense the rate of this condensation being, as a consequence, proportional to the vapour pressure of the liquid. If in an enclosed space, the pressure of the vapour is such that as many molecules escape from the liquid surface as will be captured by it due to condensation, the space will be saturated with the vapour and the vapour pressure will attain the maximum (or saturation) value.

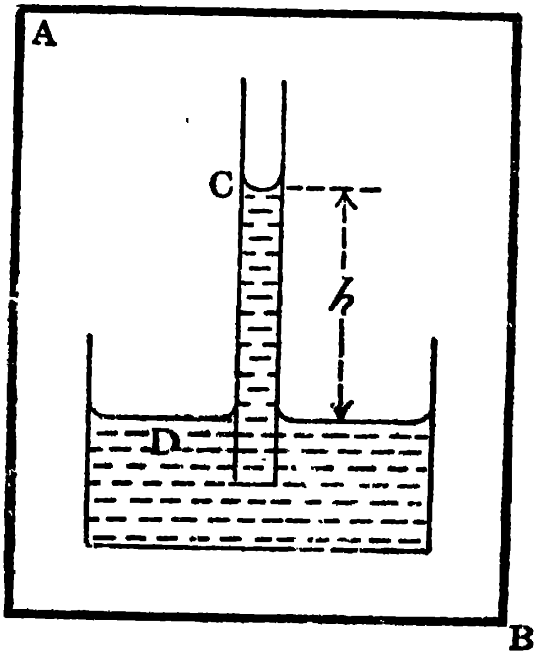


Fig. 208

In order to have a relation between the surface tension, T, of a liquid and its saturation vapour pressure let us consider a chamber AB which thermally and otherwise isolates the arrangements inside it. Inside AB, a capillary tube CD (Fig. 208) is dipped in liquid contained in a basin such that the capillary rise h is given by $T=h\rho gr/2$ where ρ= density of the liquid and r is the radius of the capillary tube. If σ be the density of the vapour inside AB, the difference of vapour pressure between C and D is given by $h\sigma g=dp$. Now if p be the maximum pressure of vapour in contact with horizontal liquid surface at D, the pressure of vapour in contact with the curved meniscus at C will be $(p-h\sigma g)$ and this is also the maximum pressure of the vapour at that position. This is so because otherwise an unbalanced movement of vapour continuously goes on inside AB without the relevant energy supply which goes against the fundamental physical principle of conservation of energy.

Hence the saturation vapour pressure over a flat surface exceeds that over a concave surface by $dp=h\sigma g$. Now as $2T=h\rho gr$ or $h=\frac{2T}{\rho gr}$, we can write $dp=h\sigma g=\frac{2T\sigma}{\rho r}$, neglecting the slight correction that may be introduced due to the pressure of the air column of length CD.

If the vapour is supposed to obey gas laws, we get, $pv=R\theta$, where θ is the absolute temperature. Again as $v=\frac{1}{\sigma}$, $p=\sigma R\theta$, or, $\sigma=p/R\theta$.

$$\therefore \quad dp=\frac{2Tp}{R\theta\rho r} \qquad \ldots \qquad \ldots \qquad \ldots \qquad \ldots \quad (1)$$

This dp gives the lowering or $\frac{dp}{p}$ the relative lowering of the vapour pressure over a concave liquid surface. Similarly, it can be shown that in the case of a convex liquid surface the rise of saturation vapour pressure is also given by an exactly identical expression as (1).

From above, we find that the saturation vapour pressure over a convex surface is greater than that over a plane liquid surface. So if a water drop be taken to a space saturated with water vapour, the corresponding vapour pressure is below the saturation vapour pressure over the surface of the drop and as a result the drop will evaporate gradually. As the drop grows smaller in size its radius decreases and from relation (1) it follows that, dp *increases* or in other words, the space over the drop is still more unsaturated. So the drop tends to diminish in size gradually and ultimately evaporates out. Thus saturation or even supersaturation may not necessarily help formation of droplets in the space which is perfectly dust-free because the droplets even if they be formed shall have a two-fold tendency to contract—one due to continuous evaporation and the other due to the contractile tendency of a liquid surface arising from S.T.

But if dust particles be present in the above space, they serve as the condensation nuclei and as such the droplets will be formed. If the size be sufficiently large then dp decreases thereby decreasing the tendency of evaporation. Again if charged ions are present they may also help formation of droplets by partly serving as condensation nuclei and partly balancing the contractile property of S. T. by the outward electric pressure acting normally on the surface of the charged droplets. This principle is utilised in the working of Wilson's cloud chamber.

***192. Clouds and Emulsions** :—It has been found in Art. 191 that the vapour pressure in equilibrium with a curved liquid surface increases with the convexity of the surface. So the smaller the convexity of the surface or the greater the radius of curvature of the surface, the more readily will the condensation take place. It can be shown by actual calculation that the rise in saturation vapour pressure is almost negligible unless the liquid surface has an extremely small radius of curvature.

To have an idea about the magnitude of the effect produced when the radius of curvature is of a different order, let us evaluate dp for water-drops of different sizes at 0°C. For a water-drop of radius 1 mm., since $T=75$ dynes per cm. at 0°C, $\rho=1$ and $\sigma=48\times10^{-7}$gms./c. c., from Art. 191, we have,

$$dp=\frac{2T\sigma}{r\rho}=\frac{2\times75\times48\times10^{-7}}{\cdot1\times1}=0\cdot0072 \text{ dynes/cm.}^2 \text{ (nearly).}$$

The saturated vapour pressure over a plane surface of water at 0°C. being about 4·6 mm. of Hg which is nearly 6150 dynes/cm.^{2}, the above change dp is practically negligible.

But when the radius of the drop is 100 mμ or 10^{-5}cm., $dp=72$ dynes/cm.^{2}, which is only about 1% of the initial pressure. Again if the radius of the drop is 1 mμ or 10^{-7}cm., $dp=7200$ dynes/cm.^{2}.

Hence in this case the pressure over the curved surface

$$\begin{aligned}&=\text{pressure over the plane surface}+dp\\&=6150+7200=13350 \text{ dynes/cm}^2.\end{aligned}$$

This is more than double the original saturation vapour pressure over a plane water surface at 0°C.

So it follows that minute drops of radius of the order of 10^{-7} cm. would readily evaporate even in a space normally saturated with aqueous vapour. Thus it is possible to cool water vapour sufficiently below its dew-point without any condensation occurring; for, in absence of any condensation nuclei the formation of very minute drop in the initial state of condensation is almost impossible because of ready evaporation.

Aitken showed that the presence of dust particles is generally necessary for condensation to take place. These particles serve as the *condensation nuclei* round which initial deposition of water molecules may take place so that even at the very beginning the water-drops formed may have a considerable dimension which may not favour ready evaporation.

Wilson in 1897 showed that the gaseous ions can serve as the condensation nuclei at the dew-point even without the presence of dust particles. Condensation may, however, occur in absence of both the dust particles and the gaseous ions provided a high degree of supersaturation is attained. In Wilson's experiments with cloud chambers it has been found that a dense mist is formed when dust-free saturated air is suddenly allowed to expand to about 1·4 times its original volume. Here due to the adiabatic expansion the supersaturation is sufficient for the formation of drops of very minute size.

The dense fog in industrial cities is mainly due to the presence of sufficiently big condensation nuclei in the atmosphere in the form of smoke and dust particles. At higher altitudes, where air is generally very clean, the formation of clouds is largely facilitated by the gaseous ions present in the air. When clouds are formed the constituent water droplets fall in the air with a terminal velocity obeying Stokes' law [Art. 207 (*c*)]. If an upward current of air is present, the cloud may rise relative to the earth. If the water particles in cloud fall through a warmer region of dry air, they evaporate and the cloud slowly disappears. If the particle falls through a hotter region saturated with water vapour further condensation occurs on their surfaces and so they will grow in size and finally fall as rain drops. In thunder-storms the larger drops are due to the union of smaller droplets under the influence of the electrical action, the effect of which is to make the droplets attract one another so that when they come closer they may coalesce

Just as the condensed droplets of a liquid in the atmosphere of its saturated vapour are stable when proper condensation nuclei are present, so does the stability of many colloidal solutions or *emulsions* depend on the presence of small amounts of a third substance which keeps the particles of the colloids dispersed. This third substance is known as the *protector*. Probably the protectors make the emulsion stable by forming a film over the colloid particles. Milk is an emulsion of butter droplets in water. Casein and calcium salts serve as the protectors or stabilisers in this case. Rubber latex or the emulsion of rubber globules are generally stabilised by an alkali as the protector.

Examples

1. A sphere of water of radius 1 mm. is sprayed into a million drops all of the same size. Find the energy expended in so doing S.T. of water $=72$ dynes/cm.
(Punj. U, 1954)

Ans. : Radius of the sphere $=1$ mm. $=\cdot 1$ cm.

$\therefore$ Area of the sphere $=4\pi(\cdot 1)^2$ sq. cm.

and its volume $=\frac{4}{3}\pi(\cdot 1)^3$ cm.^{3},

Let the radius of each droplet be r cm.

$\therefore \frac{4}{3}\pi r^3 \times 10^6 = \frac{4}{3}\pi(\cdot 1)^3$;

or, $r\times10^2=\cdot1$; or, $r=\cdot001$ cm.

Hence the total surface area of the droplets formed

$=4\pi\,(\cdot001)^2\times10^6=4\pi$ sq. cm.

∴ Due to spraying the surface area increases by

$4\pi-4\pi(\cdot1)^2=4\pi(1-\cdot01)=4\pi\times\cdot99.$

So the energy expended $=72\times4\pi\times\cdot99=896$ ergs.

2. A U-tube is supported with its limbs vertical and is partly filled with water. If the internal diameter of the limbs are 1 cm. and ·01 cm. respectively, what will be the difference in height of water at the two limbs? S. T. of water = 70 dynes/cm. (*Cf.* C. U. 1955 ; Utkal U. 1951)

Ans. : We know, $2T\cos\theta\left(\frac{1}{r_1}-\frac{1}{r_2}\right)=h\rho g$

Here $T=70$ dynes/cm., $\cos\theta=1$ (for water).

$r_1=\cdot005$ cm. and $r_2=\cdot5$ cm., $\rho=1$, $g=980$ cms./sec².

∴ $h\times980=2\times70\times\left(\frac{1}{\cdot005}-\frac{1}{0\cdot5}\right)$;

or, $h=28\cdot28$ cm.

3. A small hollow sphere which has a small hole in it is immersed in water to a depth of 40 cms before any water penetrates into it. If the surface tension of water is 73 dynes/cm., find the radius of the hole.

Ans. : When water just enters the sphere through the hole an air bubble must escape whose radius is equal to that of the hole and pressure excess inside which is equal to the hydrostatic pressure at the depth.

∴ Excess pressure inside the bubble (with single surface) is $\frac{2T}{r}=h\rho g.$

∴ $r=\frac{2T}{h\rho g}=\frac{2\times73}{(40\times980)}=\cdot0037$ cm.

4. Calculate the radius of the largest drop of water that might evaporate at 0°C without heat being supplied to it. The surface energy of water at 0°C = 117 ergs/cm.² and the latent heat of evaporation of water at 0°C = 606 cals./gm. One calorie equals 4·2 × 10⁷ ergs. (Bihar U. 1958 : C. U. '49)

Ans. : Let r be the radius of the largest drop so that its area is $A=4\pi r^2$ and its volume is $V=\frac{4}{3}\pi r^3$.

The condition of the problem demands that as the area A changes by dA, the corresponding release of surface potential energy supplies the latent heat necessary for evaporation of a volume dv of the liquid.

Here $dA=8\pi r dr$ and hence potential energy released $=8\pi r dr\times117$ ergs.

Again $dv=4\pi r^2 dr$. Its mass $=4\pi r^2 dr\times\rho=4\pi r^2 dr$. ($\rho$ being unity)

∴ Latent heat required for evaporation $=4\pi r^2 dr\times606$ calories.

$=4\cdot2\times10^7\times4\pi r^2 dr\times606$ ergs.

Since this *L. H.* is equal to the surface energy supplied, we have,

$4\cdot2\times10^7\times4\pi r^2.dr\times606=8\pi r dr\times117$

∴ $r=\frac{2\times117}{4\cdot2\times10^7\times606}=9\cdot19\times10^{-9}$ cm.

5. The pressure of air in a soap-bubble of 0·7 cm. diameter is 8 mms. of water above the atmospheric pressure. Calculate the surface tension of the soap solution. (D.U. 1944)

Ans. : The excess pressure inside the soap bubble is

$$p=\frac{4T}{r}\ (T \text{ surface tension };\ r=\text{radius of bubble}).$$

Here, $p=8$ mm. of water$=0.8$ cm. of water.

In absolute units, $p=0.8\times1\times981$ dynes/cm.^{2} ;

$r=0.7/2=0.35$ cm. :

$$\therefore\ T=\frac{pr}{4}=\frac{0.8\times981\times0.35}{4}=68.67 \text{ dynes/cm.}$$

6. *Two glass plates are separated by water. If the area of each plate wetted be 8 sq. cm. and distance between the plates be 0·0012 mm., what is the force urging the plates together ? ($T=75$ dynes/cm.)*

Ans. : Force$=\dfrac{2TA}{d}=\dfrac{2\times75\times8}{0.00012}=10^7$ dynes.

7. *A capillary tube of internal diameter 0·1 cm. and external diameter 0·5 cm. hangs vertically from the arm of a balance. The lower end of the tube dips in a liquid of surface tension 80 dynes/cm. If the liquid wets the tube, what is the change in apparent weight of the tube due to surface tension ?* (*cf. All. U. 1948*)

Ans. : The surface tension acts along the outer and inner circumferences of the tube, that is, along a length $2\pi\,(0.25+0.05)$ cm.$=0.6\pi$ cm.

Downward force due to S.T.

$=0.6\pi\times T=0.6\pi\times80$ dynes.

$\therefore$ Change in apparent weight

$$=\frac{0.6\pi\times80}{980}=0.1539 \text{ gm.-wt.}$$

8. *Two equal spherical soap bubbles of radius 'a' each coalesce to form one spherical bubble of radius b. If the atmospheric pressure be A, show that the surface tension of the bubble is* $\dfrac{A\,(2a^3-b^3)}{4(b^2-2a^2)}$. (*Pat. U. 1957*)

Ans. : When two bubbles coalesce to form a single one, the volume changes and so also the density of the enclosed air. But as the mass of air remains constant (assuming no air escapes), we have,

$2.\frac{4}{3}\pi a^3\,\rho_1=\frac{4}{3}\pi b^3\,\rho_2$, where ρ_1 and ρ_2 are the densities of air in the bubbles. As the temp. is constant, from Boyle's law, $pv=$const.$=k$ (say).

$\therefore$ If p_1 and p_2 be the internal pressures of bubbles of radii a and b respectively, then $\dfrac{p_1}{\rho_1}=\dfrac{p_2}{\rho_2}=k$; or, $\rho_1=\dfrac{p_1}{k}$ and $\rho_2=\dfrac{p_2}{k}$.

We know that the excess pressure inside a soap bubble of radius a is

$$A-p_1=\frac{4T}{a}\ ;\ \text{or, } p_1=A+\frac{4T}{a}.$$

Similarly for the bubble of radius b,

$$p_2=A+\frac{4T}{b}.$$

$$\therefore\ 2.\frac{4}{3}\pi a^3\,\frac{p_1}{k}=\frac{4}{3}\pi b^3\,\frac{p_2}{k}.$$

$$\text{or,}\quad 2a^3\left(A+\frac{4T}{a}\right)=b^3\left(A+\frac{4T}{b}\right);$$

$$\text{or,}\quad A\,(2a^3-b^3)=4T\,(b^2-2a^2);$$

$$\therefore\quad T=\frac{A\,(2a^3-b^3)}{4(b^2-2a^2)}.$$

Questions (Chapter IX)

1. Find an expression for height to which a liquid will rise above the general level in capillary tube. Can the liquid be depressed in any case? If so, why? (C. U. 1948; '62)

2. Define surface tension and show how it can be regarded as involving potential energy. From consideration of energy, show that (*i*) oil will spread on the surface of water, (*ii*) mercury will collect on a clean glass plate in drops of different shapes according to their size, and (*iii*) waetr will rise in a capillary tube. (Bom. U. 1935)

3. Account for the curvature of the surface of a liquid in the neighbourhood of a solid surface, *e.g.* the wall of a beaker.

Find an expression for the difference in pressure between the interior and exterior of a spherical soap film. Describe an experiment to determine the surface tension of such a film. (Pat. U. 1947)

4. Obtain an expression for the rise or fall of a liquid in capillary tube dipped in a vessel of liquid. (Bom. U. 1950; Pun. U. 1946)

5. Calculate the amount of energy needed to break a drop of water, 2 mm. in diameter, into 10^9 droplets of equal size, taking surface tension of water as 73 dynes/cm, ($\pi=3·14$) (Madr. U. 1947)

[*Ans.*: 9160 ergs]

6. A ring cut from a platinum tube of 8·5 cms. internal and 8·7 cm. external diameter is supported horizontally from a pan of a balance so that it comes in contact with water in a glass vessel. It is found that an extra weight of 8·97 gms. is required to pull away the ring from water. Calculate the surface tension of water. (C. U. 1943)

[*Ans.*: 72·17 dynes/cm.]

7. Define surface tension. Show that the excess pressure acting on the curved surface of a curved membrane is given by,

$$P=2S\left(\frac{1}{r_2}+\frac{1}{r_1}\right).$$

where r_1 and r_2 are the radii of curvature and S, the surface tension of the membrane. (Punj. U. 1950)

8. A soap film is formed on a rectangular frame of length 7 cms. dipping into a soap solution. This frame work hangs from the frame of a balance. An extra weight of 0·33 gm. must be placed in the opposite pan to balance the pull of the film. Calculate the surface tension of the soap solution.

[*Ans.*: 26·6 dynes/cm.] (Dac. U. 1942)

9. (*a*) Calculate the excess pressure in spherical soap bubble of diameter 1 inch blown with a soap solution of surface tension 25 dynes per cm.
[*Ans.*: 78·74 dynes/cm.^{2}] (C. U. 1957)

(*b*) Describe a method to determine the surface tension of a soap bubble. Deduce and explain the necessary relation. (Utk. U. 1962)

10. If a number of little droplets of water, of all the same radius r cms. coalesce to form a single drop of radius R cms show that rise of temperature of water will be given by $\frac{3S}{J}\left(\frac{1}{r}-\frac{1}{R}\right)$, where S is the surface tension of water and J, the mechanical equivalent of heat. (Saug.U. 1948)

11. Explain the terms, surface tension and angle of contact. Show that the pressure inside a spherical bubble of radius, r, exceeds that outside it by $\frac{4T}{r}$. If this excess pressure is balanced by that due to a column of oil (sp. gr.=0·8), 2 mm. high, when $r=1·0$ cm., find the surface tension of the film of the soap bubble. (Bom. U. 1939 ; C.U. '65)

[*Ans.* : 39·24 dynes/cm.]

12. Define surface energy. Give the theory and practice of the method of determining the surface tension of a liquid by weighing drops.

A drop of water, 0·4 cm. radius, is split up into 125 tiny drops. Find the increase in surface energy. The surface tension of water may be taken as 72 dynes/cm. ($\pi=3·14$) (Punj.U. 1946)

[*Ans* : 578·8 ergs] (*cf.* C.U. 1966 sp.)

13. (*a*) Describe an experiment for determining the surface tension of a liquid. Deduce the formula used. (Punj.U. 1948)

(*b*) Describe Quincke's method for measuring the surface tension of mercury. (*cf.* C.U.H. '65)

14. (*a*) A capillary tube of 0·5 mm. bore stands vertically in a wide vessel containing a liquid of surface tension 30 dynes/cm. The liquid wets the tube and has a specific gravity of 0·8. Calculate the rise of liquid in the tube.

[*Ans.* : 3·061 cms.] (Utk. U. 1950)

(*b*) Calculate the ascent of water in the tube having the inner radius 0·5 cm. (S.T. of water=70 dynes/cm.) (C.U. 1964)

[*Ans.* : 2·85 cms.]

15. Show that the maximum vapour pressure over a curved liquid surface differs from that over a plane surface. Deduce the value of this difference in terms of the densities of the liquid and the vapour, the radius of the curvature of the surface and the surface tension of the liquid. Discuss the bearing of the above fact on the formation of rain drops. (Madr.U. 1949)

16. Show that the vapour pressure over the surface of a liquid drop of radius, r, exceeds that over a flat surface by an amount equal of $2T\sigma/r(\rho-\sigma)$, where T is the surface tension of the liquid, ρ its density and σ the density of its vapour. Discuss the application of the result to the condensation of supersaturated vapour on dust particles and other nuclei. (Bom. U. 1937)

17. Find the work done in blowing a soap bubble of surface tension 30 dynes/cm. from 2 cms. diameter to 4 cms. diameter. (Pat.U. 1954)

[*Ans.* : 2262·8 ergs.]

18 A glass plate of length 10 cms., breadth 1·54 cms. and thickness 0·2 cm. weighs 8·2 gms. in air. If it is held vertically with its long side horizontal and its lower end half immersed in water, what will be its apparent weight ? S.T. of water=73 dynes/cm. and $g=980$ cms./sec.^{2} (Pat. U. 1957)

[*Ans.* : 8·17 gm.-wt.]

19. Discuss the various methods for measuring the surface tension of liquids, indicating the suitability of the method for different types of liquids. (C.U.H. 1967)

CHAPTER X

VISCOSITY

193. Flow of a Fluid and Equation of Continuity :—Whenever an ideal fluid flows at a steady rate, we may consider it to be consisted of a number of tubes of flow. The characteristic of a steady flow is that the amount of fluid crossing any section of a tube per second will remain constant throughout the flow.

If we consider two sections of area S_1 and S_2 at right angles to a tube of flow at two different positions, then the volumes of the fluid crossing them per second will be given by S_1v_1 and S_2v_2 respectively where v_1 and v_2 are the velocities of flow at those positions (Fig. 209). Again if the densities at S_1 and S_2 be respectively ρ_1 and ρ_2, the masses flowing through S_1 and S_2 per second are respectively $S_1v_1\rho_1$ and $S_2v_2\rho_2$. When the flow is steady, these two masses must be identical,

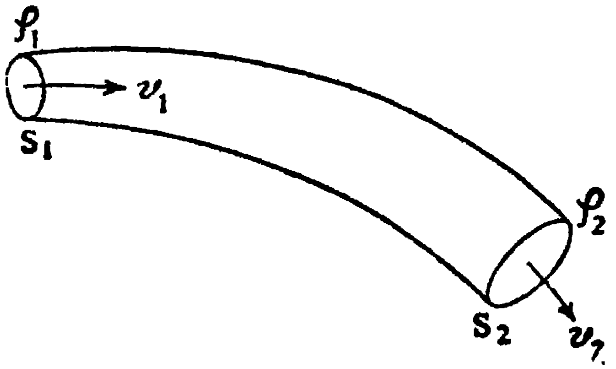


Fig. 209

$$\therefore \quad S_1v_1\rho_1 = S_2v_2\rho_2 \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

When the fluid is incompressible, the density is unaltered due to the flow (as in a liquid) and so $\rho_1 = \rho_2$.

Hence (1) reduces to $S_1v_1 = S_2v_2$; or, Sv = constant $\qquad \ldots \quad (2)$

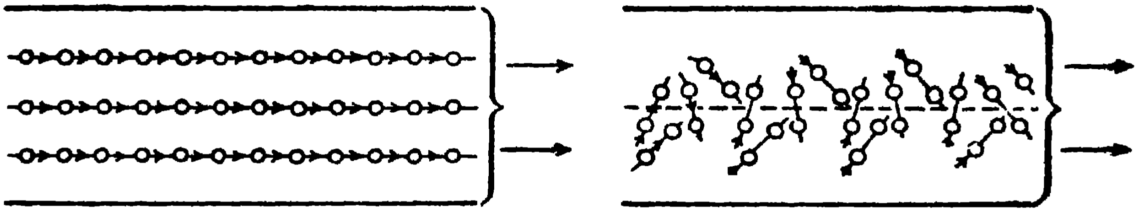
Eqn. (2) is generally known as the *equation of continuity of the liquid flow.*

194. Stream-line Motion and Turbulent Motion : Reynolds' Number :—If the path of every moving particle of a fluid coincides with the line of motion of the fluid as a whole [Fig. 210, left], the motion is said to be a ***stream-line motion.***

If the motion of the particles of a fluid are disorderly, *i.e.* in directions other than the line of motion of the fluid as a whole, the motion is said to be ***turbulent*** [Fig. 210, right].

In stream-line motion, under a given pressure gradient the flow of a liquid through a narrow tube is decided mainly by its viscosity.

whereas in turbulent motion it is solely governed by its density and very little by its viscosity. According to Prof. Osborne Reynolds the motion of a fluid changes from a stream-line motion to a turbulent one, if a certain velocity, whose value is fixed for the fluid for a given temperature, called its *critical velocity*, is exceeded. This critical velocity is given $V_c = k \frac{\eta}{\rho \times r}$, where r is the radius of the tube, ρ the density of the liquid and η the coefficient of its viscosity. This constant k is a useful parameter for ascertaining the nature of flow and is known as the **Reynolds' number.** Turbulent flow sets in when the Reynolds' number exceeds the numerical value of 1000 roughly in a narrow tube.



Stream-line motion *Turbulent motion*

Fig. 210

195. Motion of a body through a Real Fluid : Vortices :— A body, while moving through a real fluid, drags a part of the medium along with itself and drives the rest to the side. As a result some work is to be done at the expense of the kinetic energy of the body. In other words, a kind of resistance is offered by the medium. The resistance is small in the case of a body which is thin and pointed or sharp. It can further be decreased by making the body gradually tapering towards the rear side. Such a shape is called a *stream-lined shape.* In other types of the bodies generally the part of the fluid that is dragged along with it forms a *tail* at the rear and this tail is enclosed by the stream-line flow past the body on all the sides. So during its motion the body shall have to encounter the increased pressure in front (the *head pressure*) and a suction due to the diminished pressure behind (the *tail-suction*). In the rear region of this decreased pressure, the fluid (air or water) flows from sides thus giving rise to whirls and eddies (vortices) as shown in Fig. 211. The formation of these vortices are therefore

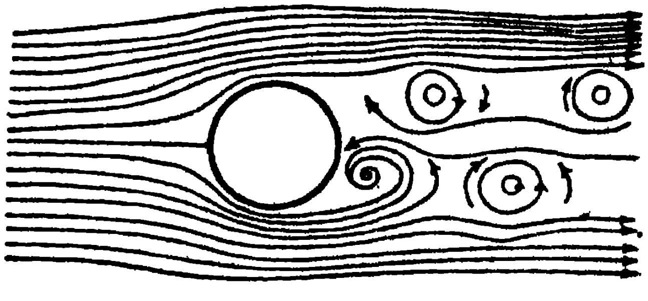


Fig. 211

responsible for the decrease of the kinetic energy of the moving body to a certain extent. As in a stream-lined body there is no scope of formation of such vortices at the tapering rear, the resistance to the forward motion is considerably diminished.

In real fluids vortices (eddies and whirls) are always formed when a body is moved through it. The cause is evidently the internal friction or viscosity which distinguishes a real fluid from an ideal one. The *peculiarity of a vortex motion* is that in a particular vortex the actual particles of the fluid always remain the same so that a vortex may be considered as a separate body. Though the shape and the position of a vortex may change the particles of fluids constituting the vortex must not alter. The energy of the vortices is often very great. The vortices of air or liquid are often found to possess high rigidity—due to which they are thought to be very dangerous for swimming and air flight.

196. Bernoulli's Theorem :—Bernoulli's theorem states that in the case of a *steady flow of an incompressible fluid* (free from viscous resistance) the sum of the potential energy, kinetic energy and the pressure energy per unit mass of the fluid is a constant along any section of flow. It is merely the principle of conservation of energy as applied to the flow of a liquid (incompressible fluid).

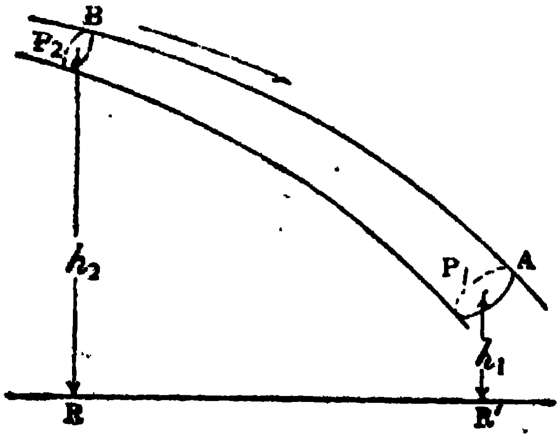


Fig. 212

To establish the theorem, let us consider a tube through which the liquid flows steadily. Let A and B be two sections of the tube at right angles to the direction of the stream-line flow at heights h_1 and h_2 from the reference plane RR' (Fig. 212). From the principle of continuity we know that the mass of the liquid flowing per second across A is equal to that across B. Let m be this mass of liquid flowing per second. If v_1 and v_2 be speeds of the liquid flow at A and B respectively, the sum of the kinetic and potential energies at the two sections are respectively $\frac{1}{2}mv_1^2 + mgh_1$ and $\frac{1}{2}mv_2^2 + mgh_2$.

The difference between these two amounts of energy at B and A is equal to the work done by the liquid in moving from B to A. If V be the volume of the liquid (mass m) and if P_2 and P_1 be the pressure at B and A, then the work done by the liquid in flowing from B to A is $(P_2-P_1)V$.

Hence from the principle of conservation of energy, the sum of $K.E.$ and $P.E.$ at B together with the work done must be equated to the sum of $K.E.$ and $P.E.$ at A.

$$\therefore\quad \tfrac{1}{2}mv_2^2+mgh_2+(P_2-P_1)V=\tfrac{1}{2}mv_1^2+mgh_1$$

Dividing throughout by V and transposing we get,

$$\tfrac{1}{2}\rho v_2^2+\rho gh_2+P_2=\tfrac{1}{2}\rho v_1^2+\rho gh_1+P_1 \qquad \ldots \quad \ldots \quad (1)$$

Since liquid flow per sec. across any section is constant, eqn. (1) will be valid for any section.

$$\therefore\quad \tfrac{1}{2}\rho v^2+\rho gh+P=\text{constant.} \qquad \ldots \qquad \ldots \quad (2)$$

This result is the famous Bernoulli's Theorem. Though it has been deduced in the case of an ideal incompressible fluid, it can be also applied to the case of stream-line flow in air with certain amount of approximation, because air may be supposed to behave roughly as an ideal fluid.

When a liquid moves horizontally, the ρgh term will be absent and eqn. (2) will reduce to $P+\frac{1}{2}\rho v^2=$ constant.

Here P is termed as the ***static pressure*** and $\frac{1}{2}\rho v^2$ as the ***dynamic pressure*** or ***velocity pressure*** of the fluid.

197. Efflux of a Liquid (Torricelli's law) :—The above equation (2) may be applied to find the velocity of efflux of a liquid contained in a vessel (Fig. 213) upto a level h above the level of the orifice O, through which it is escaping under the action of gravity. If P_a be the atmospheric pressure, v, the velocity of efflux, then just outside the orifice, $P=P_a$, $h=0$, and at the surface ab, $P=P_a$, $v=0$.

$$P_a+\tfrac{1}{2}\rho v^2=P_a+\rho gh$$

or, $v=\sqrt{2gh}$.

This is the velocity of efflux sometimes known as *Torricelli's law* or the *law of efflux* of a liquid.

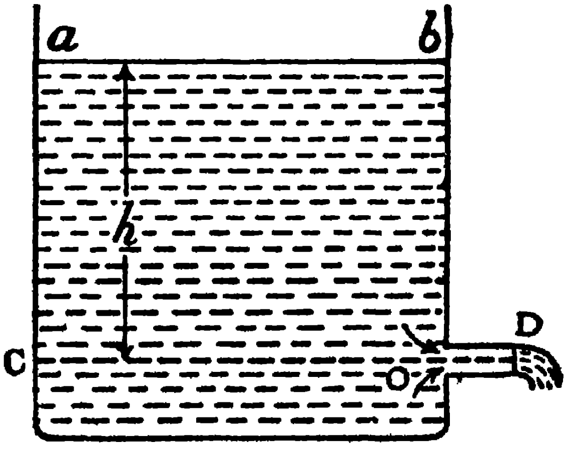


Fig. 213

198. Measurement of Efflux of Liquids :—The measurement of flow of a liquid (or fluid in general) is one of the most important measurements required for the construction of instruments.

based on hydrodynamical or aeronautical principles. There are various types of *flow-meters* constructed on different principles of which the venturi-meter and the Pitot tube both working on the differential pressure principle are described below.

(a) Venturi-meter.—

The construction of the Venturi-meter is based on the researches of Venturi, an Italian scientist, who showed that fluids under pressure gain speed and lose pressure while passing through a converging pipe and for fluids passing through a diverging pipe the reverse is the case. Clemens Herschel first used this principle to measure the liquid flow. The figure (Fig. 214) above shows the principle of construction of a Venturi-meter. It is a tube of cast metal. The size of an average Venturi tube varies from a few to 60 inches in diameter and 40 ft. in length having a flow capacity of 80,000 gallons per minute. Larger tubes are also available for special types of work. The up-stream side of the tube is first cylindrical and then tapers making an angle of nearly 35° with the axis and the down-stream portion is diverging making an angle of about 7° with the axis. The narrowest portion of the tube is known as the throat. When the liquid flows through such a tube it is found that the static pressure drops sharply as the throat of the tube is approached, and recovers the same on the drown-stream side. For the measurement of pressure difference between the throat and the up-stream or the down-stream side there are arrangements for attaching manometers to the side tubes at the corresponding positions of the Venturi tube.

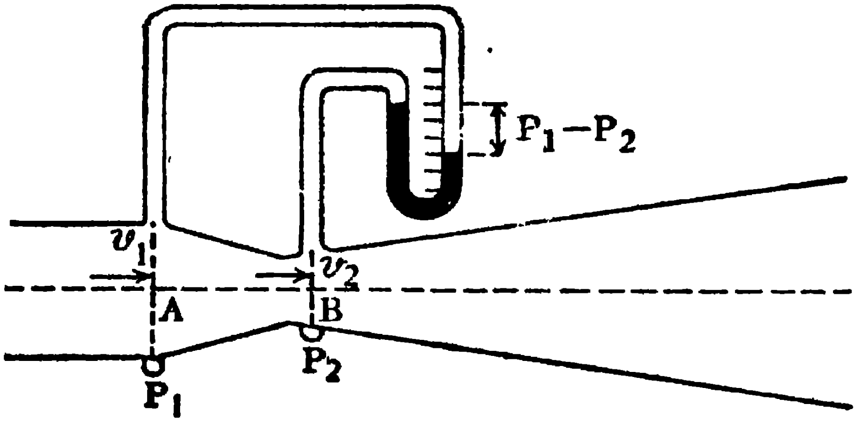


Fig. 214

The theory of the instrument follows from the equation of continuity of flow of an ideal incompressible fluid. When the liquid flows through the Venturi tube, the product of the velocity and cross-sectional area at *A* will be identical with that at *B*.

If a_1, a_2 and v_1, v_2 represent the areas of the sections and the velocities respectively at *A* and *B*, then $a_1v_1=a_2v_2$. $\therefore$ $v_1=\frac{a_2}{a_1}v_2$ and $v_2=\frac{a_1}{a_2}v_1$.

Since a_1 is greater than a_2, v_2 is greater than v_1. This is in agreement with Venturi's observations as stated above ; so where the speed of water is high the pressure is low.

From Bernoulli's equation as applied to horizontal flow, we get

$$P_1+\tfrac{1}{2}\rho v_1^2=P_2+\tfrac{1}{2}\rho v_2^2 \;;$$

$$\text{or,}\quad 2(P_1-P_2)=\rho(v_2^2-v_1^2)$$

$$=\rho\left(\frac{a_1^2}{a_2^2}v_1^2-v_1^2\right).$$

$$\therefore\quad v_1=\sqrt{2(P_1-P_2)\Big/\rho\left(\frac{a_1^2}{a_2^2}-1\right)}.$$

Thus knowing (P_1-P_2) from the manometer and the other quantities on the right-hand side by auxiliary measurements, v_1 can be determined. Similarly, v_2 can also be determined. When properly constructed and calibrated, the Venturi-meter is probably the most accurate unit used for measurement of liquid flow. But because of its unweildy size and high cost, it is sometimes desirable to adopt other methods.

(e) **Pitot tube.**—Pitot tube is also a differential pressure type of flow meter.

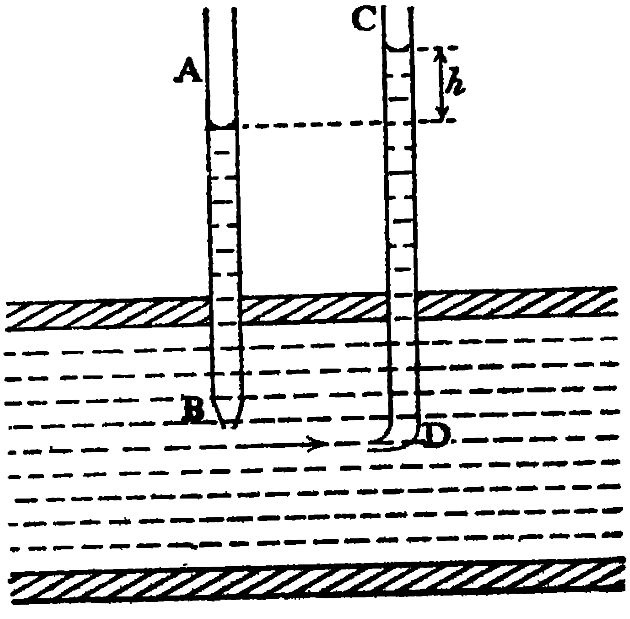


Fig. 215

The instrument consists of two vertical tubes AB and CD, the former having an aperture at the bottom and the latter an aperture turned sideways also near the bottom (Fig. 215). The plane of the aperture at B is set parallel to the direction of flow so that the liquid meniscus in AB directly gives the pressure at B. The tube CD is so set that the aperture at D faces the flow. So in the plane of the aperture D the flow (with $K.E.=\frac{1}{2}\rho v^2$) is stopped and as a result the pressure is gained by the amount $\frac{1}{2}\rho v^2$, v being the velocity of flow. This pressure difference $\frac{1}{2}\rho v^2$ corresponds to the hydrostatic pressure due to the difference of liquid level in AB and CD (i.e. h).

$$\therefore \quad h\rho g = \tfrac{1}{2}\rho v^2 \ ; \text{ or, } v = \sqrt{2hg}.$$

Thus v, the speed of the flow of the liquid is determined. This instrument is simple in construction but cannot be adopted for all types of experiments. If liquid contains dirt or solid particles, a continuous use of the tube CD may not be possible due to clogging and occasional cleaning of the tube CD would be essential.

199. Viscosity :—A rod held fixed at one end and twisted by the other can resume its original shape when the turning force (which is shearing force), is withdrawn. This is possible due to an inherent property of a solid known as its rigidity. All solids can stand shearing forces, though up to a maximum limiting value only. But this limiting value is very high for the solids. The ideal fluids differ in this respect from the solids. They cannot stand any shearing forces and so they have no definite shape of their own. This difference in behaviour arises from the fact that the interacting forces between molecules, known as cohesive forces, in a liquid are very much less than in a solid, while they are negligible in the case of gases.

If water in a pot, after stirring, is left for sometime, the motion of the water subsides. This is a very common observation. What stops the motion ? An enquiry into the question reveals that in an actual fluid, whether liquid or gas, when any relative motion between parts of the fluid is caused, internal forces are set up in the fluid which oppose the relative motion between the parts in the same way as the forces of friction operate when a block of wood is dragged along the ground. In short, a *less quickly moving layer of the fluid exerts a retarding force on the more quickly moving layer*, the motion of the latter being thereby reduced while that of former accelerated and in this way the relative motion between the two gradually decreases till finally it stops altogether with respect to the walls of the stationary pot in which the water is contained.

***The property, by virtue of which retarding forces are called into play within a fluid when any relative motion between its parts occurs, is an inherent property of all fluids, differing only in degree from one fluid to another, and is termed the* viscosity of fluids.**

A viscous liquid (or fluid) is such a liquid (or fluid) in which these retarding forces come into play appreciably.

200. Coefficient of Viscosity :—A measure of the viscosity of a fluid is given by what is termed the coefficient of viscocity. *The* *coefficient of viscosity is defined as that tangential force applied per unit area which will maintain a unit relative velocity between the layers of fluid at unit distance apart.*

Suppose two layers AB and CD in a fluid, distant dx apart (Fig. 216), both moving forward in a steady state execute *laminar motion* such that while still remaining parallel to each other, the layer AB moves faster than the layer CD, the former having a relative velocity dv with respect to the latter. A driving force, parallel to AB and directed from A to B, will be necessary to maintain this flow. On account of the viscosity of the fluid, the driving force acting on AB will be opposed by a force F, called the viscous force, exerted by the layer CD on AB. The layer AB will also exert a force, F on CD tending to accelerate the motion of the latter. According to Newton, the stress $\frac{F}{\alpha}$, where α = area of either layer, is proportional to the velocity gradient, $\frac{dv}{dx}$ (i. e. the relative velocity per unit distance), or $F/\alpha = \eta \frac{dv}{dx}$, where η is a constant called the **coefficient of viscosity** of the fluid whose value depends on the nature of the fluid and its temperature.

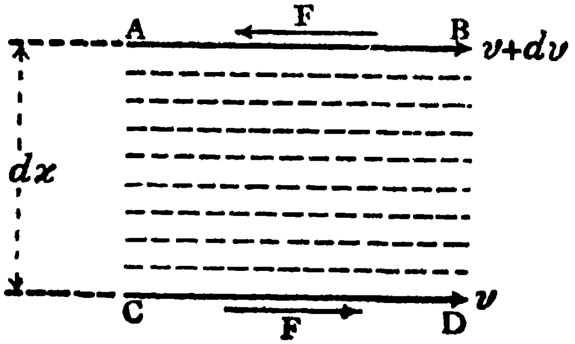


Fig. 216

That is, the viscosity coefficient, $\eta = \frac{F/\alpha}{dv/dx}$ = viscous stress intensity per unit relative velocity. Its dimensions will be force per unit area per velocity gradient, i.e. $\frac{MLT^{-2}}{L^2} / \frac{LT^{-1}}{L} = ML^{-1}T^{-1}$. In the *c.g.s.* system the unit of viscosity is the *poise*.

Note :—The liquids that obey the above law of viscous force are known as **Newtonian liquids**. For such liquids η is constant at a given temperature and pressure. There are a number of impure liquids, semi-liquids and semi-solids whose viscous property is found to be anomalous and is not in accordance with Newtonian law or other laws like Stokes' law or Poiseuille's law. These are known an **non-Newtonian liquids.** In the case of such liquids (*e g.* blood, sols, gels), the rate of shear is not proportional to the shearing stress. Here the viscosity that is experimentally determined is known as the **apparent viscosity.**

201. Viscosity is a Relative Term :—When water is poured into a funnel, it runs out quickly but glycerine or thick oil does so slowly and treacle much more slowly. Ordinarily, the liquids like water which flow readily are termed *mobile* while those of the treacle type which do not flow so readily are termed *viscous*. This does not mean that water has no viscosity. Its viscosity is only small. For treacle, it is very much greater. That is, these terms are used in our common language in the relative sense. It should be remembered that the gases, as a class, are much less viscous than the liquids, while a particular gas may be more viscous than another.

202. Viscosity and Friction :—Viscosity and friction have a good deal of similarity between them but in certain aspects they differ from each other. The following points of comparison may be discussed to show these similarities and dissimilarities.

(1) When relative motion occurs between the various parts of the same fluid internal stresses are set up within it which tend to resist such motion. The coefficient of viscosity is defined as the tangential force required to maintain a relative velocity of unity between two parallel planes in the fluid at unit distance apart. Friction is also a tangential force which are called into play between two surfaces in contact when one tends to slide over the other. It also acts in a direction tending to oppose the motion but is a purely surface effect. (2) The coefficient of kinetic friction is independent of relative velocity between two surfaces in contact but η depends on it. (3) Friction depends only on the nature of the surfaces in contact but η depends, in addition, on the relative velocity of the bodies upto a limiting value called the *critical velocity* as has been shown by Osborne Reynolds. (4) Pressure generally tends to increase the viscosity of liquids and has practically no effect on that of gases. The friction between two surfaces in contact does not alter due to any change in the normal reaction.

203. Viscosity may be regarded as Fugitive Elasticity :—A liquid may be regarded as capable of exerting and sustaining a certain amount of shearing stress (which is quite small for a short time after which the shear breaks down only to appear again. The idea is due to Maxwell. He regards viscosity as a limiting case of an elastic solid when the material of the solid breaks down under shear. It is from this standpoint that viscosity is often referred to as fugitive elasticity.

204. Demonstration of Viscosity :—

(1) **For a Liquid.**—Two identical weights are dropped at the same time, one into water and the other into glycerine (Fig. 217). In water the weight descends more quicky than in glycerine showing that the viscous drag in glycerine is greater than in water.

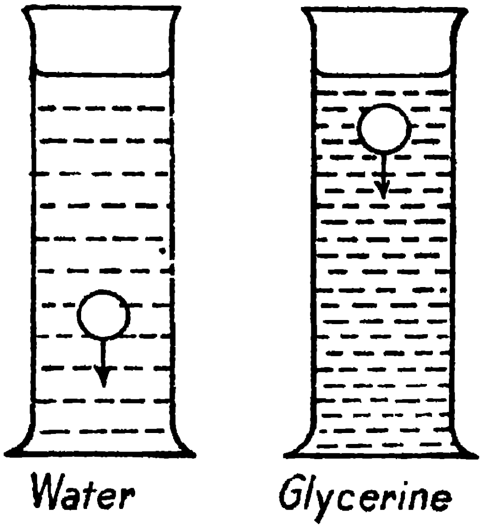


Fig. 217

(2) **For a Gas.**—A card-board disc A suspended from a rigid support by a thread attached to its centre is held in air just above but not touching a wooden disc B as shown in Fig. 218. When the disc B is rapidly rotated, the upper disc also is urged into rotation in the same direction as that of the lower disc. This is possible only because air has viscosity.

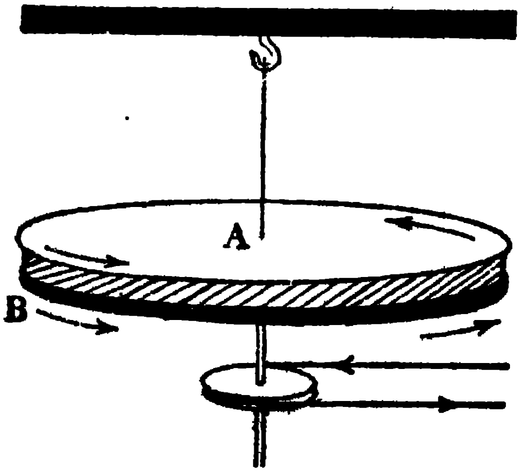


Fig. 218

205. Slow and Steady Flow of Water in a River :—Slow and steady flow here means stream-line motion. In such motion, it is found from flow measurements that the speed of motion is maximum (v_m) at the top surface of the water in the river and reduces gradually with the depth below, and finally to almost zero speed (v_0) at the bottom or bed of the river (Fig. 219). The idea is that the whole mass of moving water may be taken as consisting of a very large number of thin parallel layers in which each upper layer slides

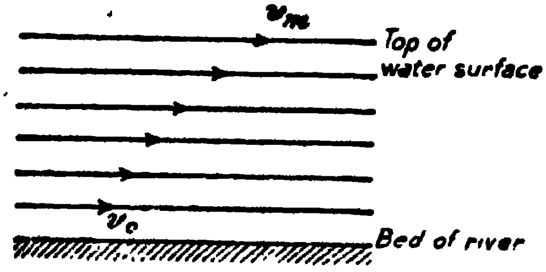


Fig. 219

over that below it. Because of the adhesive forces, the rigid bed of the river almost prevents the bottommost layer of the water from moving; this almost stationary layer owing to interlocking forces, called cohesive forces, tries to hold back the layer above it with a force which is less than in the previous layer. This layer again tries to hold back the layer above it; all the way up, the resistance of motion of a layer diminishes upwards. So the speed of motion of the liquid is maximum at the top and reduces downwards. The mechanism of flow as stated above shows how viscosity actually acts in determining the type of flow.

206. Flow of a Liquid through a Narrow Tube (Poiseuille's equation) :—In the case of a stream-line motion of a liquid through a narrow tube under a difference of pressure P between its ends, the pressure is constant over any cross-section since there is no flow transverse to the axis. Poiseuille made the further assumption that the liquid in contact with the walls of the tube is at rest and his assumption has been proved true by actual experiments.

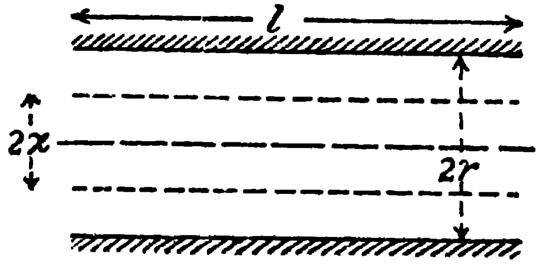


Fig. 220

Suppose the length of the tube is l (Fig. 220). Let the velocity of the liquid layer at a distance x from the axis of the tube be u in the steady state. Consequently, due to viscosity, a retarding force $F = \eta \frac{du}{dx}$ per sq. cm. is called into play, which acts along the surface of the coaxial liquid cylinder of radius r in opposition to the pressure gradient, where η is the coeff. of viscosity of the liquid. The force due to the pressure difference P which tends to accelerate the motion of this liquid cylinder is $P \times \pi x^2$. When a steady state is attained,

$P\pi x^2 = -\eta\frac{du}{dx}(2\pi xl)$, where $2\pi xl$ is the surface area of the liquid cylinder, the negative sign indicating that u decreases with the increase of x.

$$\text{Hence} \quad -x\,dx = \frac{2\eta l}{P}du.$$

At the wall of the tube, $x=r=$ radius of tube, and $u=0$ according to Poiseuille's assumption.

$$\therefore \quad \int_x^r -x\,dx = \frac{2\eta l}{P}\int_u^0 du\,;$$

$$\text{or,} \quad \frac{r^2-x^2}{2} = \frac{2\eta l}{P}u\,; \quad \text{or,} \quad u = \frac{P(r^2-x^2)}{4\eta l}.$$

This gives the velocity of the liquid at a distance x from the axis of the tube.

To calculate the total volume V of the liquid flowing out of the tube per second, let us consider an element of volume dv of the liquid flowing out through the tube per second between the radii x and $x+dx$.

$$\text{Then,} \quad dv = 2\pi u.x\,dx = \frac{P\pi}{2\eta l}(r^2-x^2)x\,dx.$$

$\therefore$ The total volume flowing out per second is given by

$$V = \int_0^r \frac{P\pi}{2\eta l}(r^2-x^2)x\,dx = \frac{P\pi}{2\eta l}\left\{\frac{r^4}{2}-\frac{r^4}{4}\right\} = \frac{\pi P r^4}{8\eta l}.$$

The above result is subject to correction due to the following reasons :—

(1) The pressure difference P is used partly in imparting kinetic energy to the liquid.

(2) It has been assumed that there is no acceleration along the axis of the tube—an assumption which does not apply near the admission end of the tube as the accelerations do not reduce to zero until an appreciable length of the tube has been traversed by the liquid.

However, in the case of a long tube of fine bore the result may be taken to be approximately true.

207. Determination of the Coefficient of Viscosity of a Liquid :—There are various methods for the determination of viscosity of liquid of which only a few important methods are described below.

(a) Poiseuille's method.—The apparatus used is shown in Fig. 221 where A is a moderately large glass vessel in which water is kept at a constant level by the inlet and outlet tubes M and N. The liquid is introduced into the vessel by the supply tap M which is connected to a raised reservoir and as soon as it comes up to the upper end of the outflow tube N it escapes through it and thus the level is kept constant. A long tube PC of small radius r and length l is horizontally fixed through a cork at a depth h below the free surface in A.

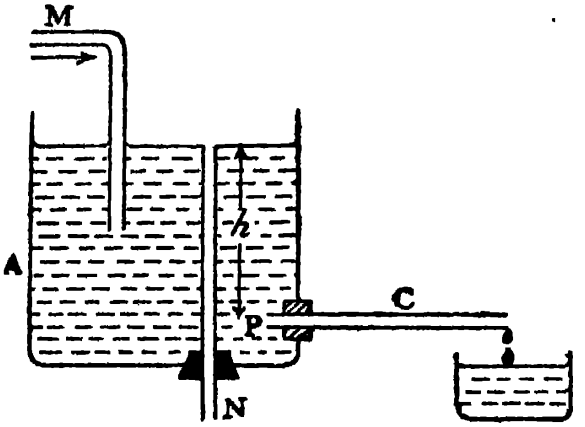


Fig. 221

In order to determine the viscosity of water (or any other liquid) by the above arrangement the inlet tube M is connected to a reservoir of water (or the liquid) and when the constancy of level has been attained a good quantity of water (or liquid) is collected in a beaker at the outlet end of the tube PC. The time, t, of collection is noted by a stop-watch. If the density of water (or liquid) be ρ and the mass collected be m gms., then the volume of water (or liquid) flowing per sec. is $V=\frac{m/\rho}{t}$. Knowing this V, η can be calculated with the help of Poiseuille's formula.

$V=\frac{\pi P r^4}{8\eta l}$, where P is the pressure difference between the ends of the tube, PC.

Here the pressure difference between the ends of the tube PC is the hydrostatic pressure at its end inside the vessel A. Hence $P=h\rho g$, h is directly measured either by a scale or more accurately by a cathetometer and ρ is found out at the temperature of water (or the liquid) from table.

The radius, r, of the flow tube requires an accurate estimation, since it occurs as r^4 in the working formula. It is usually measured by drawing into the tube a mercury thread and measuring its length by a travelling microscope. The length of the thread is to be taken a number of times at different places throughout the length l of the flow tube and the mean length of the thread x is taken. The mass of mercury introduced in the tube is now taken out in a previously weighed crucible and its weight found out by direct weighing. If σ be the density of mercury, the radius can be calculated from the relation, $m=\pi r^2 x.\sigma$. Thus knowing all the terms in Poiseuille's formula excepting η, the value of η can be found out by this method.

(b) **Rotating Cylinder Method** (Searle's Viscometer).—Searle's viscometer is essentially a rotating cylinder apparatus. The arrangements are shown in Fig. 222. Here a wide cylinder G of radius b, containing the experimental liquid L is rigidly fixed on the base of the instrument. A solid narrower cylinder C of radius, a, is fixed to an axle AH which is pivoted at its ends at A and H. A drum D and a graduated circular disc B are attached to the same spindle. The disc B, together with a fixed index R, is used to determine the speed of rotation of the cylinder C, when it revolves rather slowly under the joint action of the couple exerted by weights Mg and the viscous drag of the liquid. The weights Mg are suspended by strings passing over the frictionless pulleys PP, the other ends of the string being attached to a pin on D in such a way that when the weights go down, a rotating couple will cause the cylinder C to revolve. In the actual experiment when the torque applied by Mg and the viscous drag of the liquid act together the period of rotation T which should be made rather great, is to be carefully noted with the help of a stop-watch

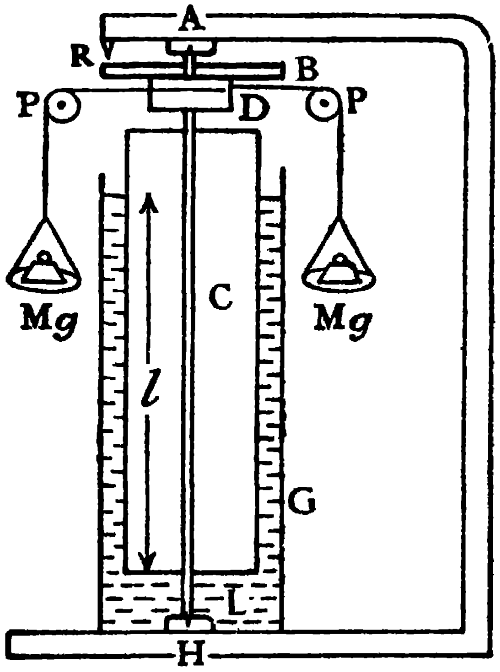


Fig. 222

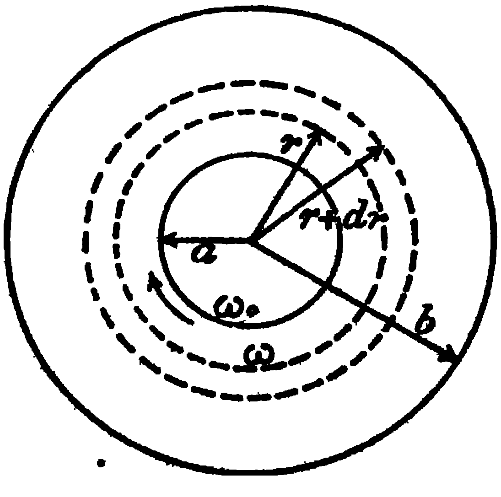


Fig. 223

Thus the steady angular velocity of rotation of the inner cylinder as found experimentally is $\omega_0=\frac{2\pi}{T}$.

In considering the theory of Searle's viscometer, since the inner cylinder rotates in the steady state with a constant angular velocity ω, let the angular velocity of rotation decrease by $d\omega$ across the ring between radii r and $r+dr$ (Fig. 223). The velocity gradient across this annulus is $-\frac{rd\omega}{dr}$ (as $v=\omega r$, $dv=rd\omega$). Hence if l be the length of the inner cylinder immersed in the viscous liquid the force due to viscous drag on the cylinder of radius r will be

$$F=-2\pi rl\eta\frac{rd\omega}{dr}.$$

Hence the torque about the axis of rotation

$$=Fr=-2\pi r^3 l\eta\frac{d\omega}{dr}.$$

This torque in the steady state of rotation is balanced by external torque applied by Mg i.e. Mgd where d=diameter of D.

$$\therefore\quad Mgd=-2\pi r^3 l\eta\frac{d\omega}{dr};$$

$$\text{or,}\quad Mgd\frac{dr}{r^3}=-2\pi\eta l d\omega.$$

From the condition of the experiment, when $r=a$, $\omega=\omega_0$ and when $r=b$, $\omega=0$. Hence integrating within these limits we get,

$$Mgd\int_a^b\frac{dr}{r^3}=-2\pi\eta l\int_{\omega_0}^{0}d\omega;$$

$$\text{or,}\quad Mgd\left(\frac{1}{a^2}-\frac{1}{b^2}\right)=4\pi\eta l\omega_0=4\pi\eta l\times\frac{2\pi}{T}.$$

$$\therefore\quad \eta=\frac{MgdT}{8\pi^2 la^2b^2}(b^2-a^2).$$

From this expression η can be determined. Here the velocity of rotation should be small so that the conditions are those of stream-line flow. This method is generally used for finding η of very viscous liquids. With modifications, this method has been utilised by Bearden for accurately determining the viscosity of air.

(e) **Stokes' Viscometer** (Falling sphere method)—This instrument is very simple in construction and is also very convenient to use. It has been shown by Stokes that when a sphere of radius a moves in a homogeneons incompressible fluid of infinite extent it finally acquires a uniform velocity, v (*terminal velocity*) and the retarding force due to the viscous drag under this condition is given by $F = 6\pi\eta av$, when η = coeff. of viscosity.

The working of the viscometer depends on this relation. It consists of a wide vertical glass cylinder containing the liquid as shown in Fig. 224. Small spherical globules of lead or of any other convenient substance are dropped in the liquid one by one. Let a and m be the radius and mass of one of such falling globules (shown magnified in the figure) and v be the terminal uniform velocity acquired by it after a short time. This velocity v is determined by timing the fall of the sphere over a known height h of the tube. Since the force mg acting downwards must be equal to the retarding force due to viscosity when a uniform velocity is attained, we have

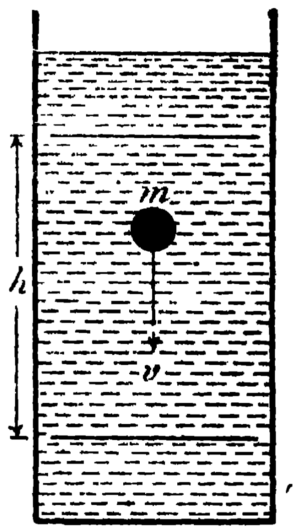


Fig. 224

$$m_1 g = 6\pi\eta av$$

where m_1 is the effective mass of the falling sphere in the liquid of density σ. If ρ be the density of the material of the sphere, then $m_1 = \frac{4}{3}\pi a^3(\rho - \sigma)$ due to buoyancy of the liquid, which is slightly smaller than m.

$$\text{Hence } \frac{4}{3}\pi a^3(\rho - \sigma)g = 6\pi\eta av\ ;$$

$$\text{or, } \eta = \frac{2}{9}\frac{a^2(\rho - \sigma)g}{v} \qquad \dots \qquad \dots \qquad \dots \quad (1)$$

Thus knowing a, v, ρ and σ, η can be found out from a knowledge of g at the locality. In actual determination a may be conveniently measured by a travelling microscope.

The above expression of η [Eq. (1)] is utilised in Millikan's oil drop experiment for determining the electronic charge.

For accurate value of η, **Ladenburg** showed that the above expression must be divided by the factor $\left(1 + 2{\cdot}4\frac{a}{r}\right)$ where a and r are respectively the radii of the sphere and the cylinder. This correction is needed because of the finite extent of the cylinder used in the experiment and is known as the *wall effect correction*.

Again a correction for the *end effect* of the cylinder is to be introduced and the expression for v is to be further multiplied by the correction term $\left(1 + 3{\cdot}3\frac{a}{h}\right)$. So for accuracy, η is to be evaluated from the expression, $\eta = \dfrac{2}{9}\,\dfrac{a^2(\rho-\sigma)g}{v(1+2{\cdot}4a/r)(1+3{\cdot}3a/h)}$.

[*Deduction of Stokes' formula from the Dimensional Equation* — When a sphere of diameter $2r$ moves uniformly through a viscous medium of coeff. of viscosity η and if the medium be of infinite extent, the retarding force F is experimentally found to depend on the uniform velocity v of the moving sphere, the coeff. of viscosity η of the medium and the quantity $2r$ of the moving body.

Hence $F = C_1 . v^x \times \eta^y \times (2r)^z$

$= C_0 v^x \times \eta^y \times r^z$, where C_1 & C_0 are constants.

But the force is dimensionally represented by MLT^{-2}, v by LT^{-1}, η by $ML^{-1}T^{-1}$ and r by L.

$$\text{Hence } MLT^{-2} = C_0 L^x T^{-x} \times M^y L^{-y} T^{-y} \times L^z$$
$$= C_0 L^{x-y+z} M^y T^{-x-y}$$

$$\therefore \quad y=1,\ x+y=2, \text{ or, } x=1;$$

$$x-y+z=1, \text{ or, } z=1,$$

$$F = C_0\, \eta r v.$$

For a sphere the constant, C_0, takes up a value 6π as has been theoretically established by Stokes.

Hence, $F = 6\pi\eta r v$.]

208. Comparison of Viscosities of Liquids

:—For the comparison of viscosities of two liquids a simple apparatus as shown in Fig. 225 is used. Here a U-tube fitted with two bulbs B and A, one at a higher level than the other, has its lower bent portion EF capillary. The stems are marked at m_1, m_2 and m_3. The bulb A must be placed at a level below m_2 and A and B should be of nearly equal capacity. The tube is now filled with a liquid of viscosity and density respectively equal to η_1 and ρ_1 up to the upper level m_1, the lower one being at m_3. The pinch cock C is now opened and the liquid flows on to the bulb A. The time t_1 required for the change of the liquid surface from m_1 to m_2 due to the above flow is noted. Next the experiment is repeated with a second liquid of viscosity and density, η_2 and ρ_2 and the time t_2 for the same change of liquid level is noted.

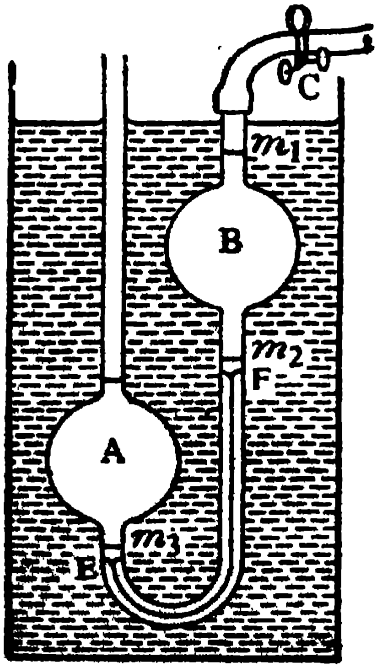


Fig. 225

Since the rate of flow of the liquid is proportional to the pressure difference which again varies directly as the density and inversely as the viscosity, the rate of flow of the liquid is $\frac{dV}{dt}=K\frac{\rho}{\eta}$, where V is the volume transferred in t seconds and K a constant.

$$\therefore \text{ Integrating } V=K\left(\frac{\rho}{\eta}\right)t.$$

$\therefore$ The values V_1 and V_2 for the two liquids are

$$V_1=K\frac{\rho_1}{\eta_1}t_1 \text{ and } V_2=K\frac{\rho_2}{\eta_2}t_2.$$

Since $V_1=V_2=$volume between the marks, m_1 and m_2,

$$\frac{\eta_1}{\eta_2}=\frac{\rho_1 t_1}{\rho_2 t_2}.$$

Thus viscosities of two liquids can be easily compared by this method. This comparative study can be carried out at any desired temperature because the temperature can be easily controlled in the arrangement.

209. Measurement of η at different temperatures :—The viscosity of a liquid at different temperatures may be determined by enclosing the capillary tube in a bath whose temperature can be controlled. The actual arrangement is shown in Fig. 226. The liquid in the vessel V is siphoned out by an inverted U-tube, bcd, of comparatively wide bore. Inside the liquid in V, is kept the capillary tube ab through which this siphoned liquid has to pass initially. So if the effective length of ab be taken as l and if the pressure difference P in Poiseuille's formula be replaced by the hydrostatic pressure due to the column of length, viz. h, then η of the liquid can be determined by the direct application of Poiseuille's formula. Evidently the liquid level should be kept constant during the experiment with the help of the pointer P and the constancy of temperature of the liquid is to be checked occasionally during the experiment.

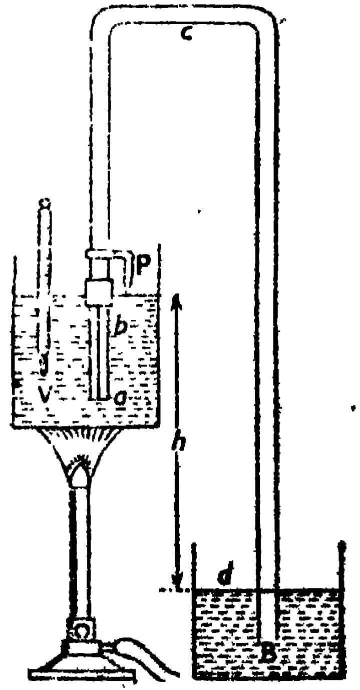


Fig. 226

210. Determination of Viscosity of Gases :—The viscosity of gases cannot be determined by the usual method followed in cases of liquids applying Poiseuille's formula. In calculating volume of liquid flowing per sec. through a narrow tube the compressibility was not taken into account as this factor is rather negligible in a liquid. But in gases the compressibility plays an important role.

If there is no accumulation in any part of the tube, the mass of gas flowing through every section of the tube per second must remain constant in the steady state. So Poiseuille's formula is to be modified accordingly.

If we consider an elementary section of the tube of length dx, the pressure difference between whose faces is dp, then, by Poiseuille's formula, the volume V of the fluid flowing per second through the section is given by,

$$V = -\frac{\pi r^4}{8\eta} \cdot \frac{dp}{dx} \qquad \ldots \qquad \ldots \qquad \ldots \quad (1)$$

Here $\frac{P}{l}$ in Poiseuille's formula has been merely replaced by $-\frac{dp}{dx}$, the negative sign indicating that the pressure (p) decreases as the distance (x) increases.

Since in the case of gas flow the mass flowing through each section remains constant.

$V\rho =$ a constant, where ρ is the density of the gas.

Again ρ being directly proportional to pressure, we can write, $\rho = Kp$ where K is a constant.

$\therefore$ $KVp =$ constant ; or, $Vp =$ another constant $= C$ (say).

Hence $Vp = C = -\frac{\pi r^4}{8\eta} . p . \frac{dp}{dx}$;

$$\text{or,} \quad Cdx = -\frac{\pi r^4}{8\eta} p.dp \qquad \ldots \qquad \ldots \qquad \ldots \quad (2)$$

Supposing the gas flow occurs in a narrow tube of length l, the pressures at the ends of which are p and P where $p > P$, we have by the integration of equation (2),

$$C \int_0^l dx = -\frac{\pi r^4}{8\eta} \int_p^P p dp \, ;$$

$$\text{or,} \quad Cl = \frac{\pi r^4}{16\eta} (p^2 - P^2)$$

$$\therefore \quad C = Vp = \frac{\pi r^4}{16\eta l} (p^2 - P^2) \qquad \ldots \qquad \ldots \quad (3)$$

Here V is the volume flowin through the narrow tube and p indicates the pressure on the inlet side o the tube. r is evidently the radius of the narrow tube under consideration and P is the pressure at the outlet side of the tube.

(a) U-tube and mercury pellet method :—

The simplest arrangement in the case of a gas is illustrated in Fig. 227(a). Here a long tube of glass is drawn into a capillary tube *cd* at one of its ends and the tube is bent in the form of a *U* with the capillary arm shorter than the arm *ab*. A pellet of mercury *I* drives air out of *ab* through *cd* when the *U*-tube is made vertical The time required for *I* to pass over the distance between two scratch marks M_1 and M_2 is noted. If the cross-sectional area of the arm *ab* be α and L its length, the total volume of the gas flowing through the capillary tube of length l is $L\alpha$. This volume flows in t seconds, which is the time of fall of the mercury pellet I from M_2 to M_1. Hence the volume V flowing per sec. is $\frac{L\alpha}{t}$. Again if Mg be the weight of the mercury pellet, the pressure at the inlet of the capillary will be,

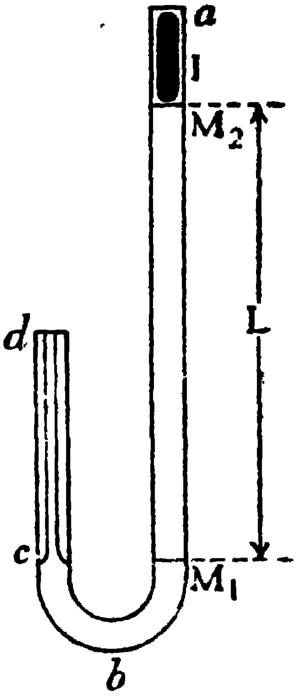


Fig. 227 (a)

$p=P+\frac{Mg-w}{\alpha}$, where P is the atmospheric pressure and w is a correction for the sticking of mercury to the walls of the tube.

Hence substituting in eqn. (3), we have,

$$\left(P+\frac{Mg-w}{\alpha}\right)\frac{L\alpha}{t}=\pi r^4\frac{\left\{\left(P+\frac{Mg-w}{\alpha}\right)^2-p^2\right\}}{16\eta l}.$$

From this relation knowing all the terms excepting η, the latter can be calculated. The value of the sticking correction can be found from readings taken with the different lengths of the mercury pellet.

*(b) Searle's method :—

Searle devised a simple method for determining the viscosity of air wherein air kept under a pressure in a large vessel is allowed to flow

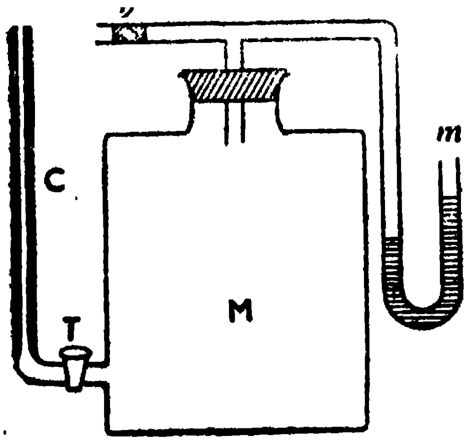


Fig. 227(b)

out through a long capillary side tube. In Fig.227(*b*), *M* is a large vessel of volume V_m fitted with a manometer *m* for the measurement of pressure. Air under pressure is introduced in *M* through a cycle valve *v* and the pressure is noted from the manometer, which should always be less than that corresponding to turbulent flow. Now the tap *T*, attached to a capillary side tube *C*, is opened and air is allowed to escape for a known interval of time and the final pressure in *M* is again noted from the manometer. From these observations and knowing the length and radius of the capillary tube, the viscosity of air can be determined as will be evident from the calculations given below.

Let p_1 and p_2 be the initial and final pressures in *M* before and after the flow. Let *V* be the volume of air flowing per second through the capillary tube at an instant *t*. During the next small interval *dt*, the pressure changes from its values *p* to $p+dp$. When the rate of flow is slow,

$$(p+dp)\,(V_m+V.dt)=pV_m.$$

Neglecting the small term containing $dp\times dt$, we have

$$pV\,dt=-\,dp.V_m\,;$$

$$\text{or,}\quad V_m\frac{dp}{dt}=-pV\ldots\ldots\ldots\ldots(A)$$

From relation (3) in Art. 210 above,

$$pV=\frac{\pi r^4}{16\eta l}(p^2-P^2),$$

where P= the external (atmospheric) pressure, η= coeff. of viscosity, r= radius of the capillary bore and l= length of the capillary tube. Hence substituting in (*A*),

$$-\frac{dp}{dt}=\frac{\pi r^4}{16\eta l V_m}(p^2-P^2)=K(p^2-P^2),\ \text{where}\quad K=\frac{\pi r^4}{16\eta l V_m}.$$

$$\therefore\quad \frac{dp}{p^2-P}=K\,dt$$

Integrating we have,

$$-\int^{p_2}\frac{dp}{p^2-P^2}=K\int^{t}dt,\quad (t=\text{time of flow})\,;$$

$$\text{or,} \int_{p_1}^{p_2} \frac{dp}{p+P} - \int_{p_1}^{p_2} \frac{dp}{p-P} = K \times 2Pt\,;$$

$$\text{or,} \log\left(\frac{p_2+P}{p_1+P} \cdot \frac{p_1-P}{p_2-P}\right) = 2KPt = \frac{\pi r^4 Pt}{8\eta l V_m} \quad \ldots \quad \ldots \text{(B)}$$

Hence the pressure in M falls exponentially.

From relation (B), η can be evaluated because all other terms excepting η in the relation are directly determinable.

In an actual experiment, it is very difficult to keep V_m constant because the manometric liquid is displaced due to the change of pressure as the flow continues. Hence to reduce the error due to this variation in volume, V_m should be sufficiently large.

*(c) Rankine's method :—

The above method of U-tube and mercury pellet, has been modified by Rankine in such a way that the experiment may be carried out with a small quantity of gas and the measurement may be made at any desired temperature by enclosing the apparatus in a suitable thermostatically controlled chamber. The arrangements are evident from figure 227(c). Here the capillary tube AD and the other tube BC are joined in series so that a falling mercury pellet M forces the gas below it through DA on the region AEB. The apparatus may be filled with the desired gas at a known pressure through the taps TT. Let p be this known pressure when the Rankine tube lies horizontally so as to ensure equal pressure everywhere in the tube. If V be the effective internal volume of the apparatus and ρ_0 the density of the gas *at unit pressure*, the mass of the gas in the tube is $\rho_0 pV$.

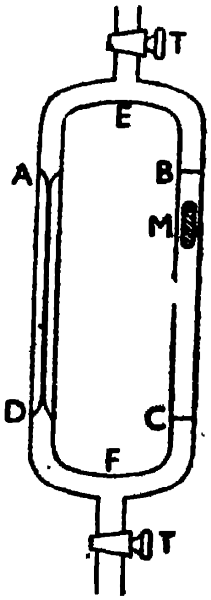


Fig. 227(c)

The tube is now made vertical and time of fall of the mercury thread is noted from the instant when its upper meniscus is at B to the instant when its lower meniscus just touches C. The marks B and C are so scratched that the volume AEB above B is equal to the volume CFD below C. Let each of these volumes be V_1. Let M be the mass of the mercury pellet, α its cross-sectional area and p_1 the pressure in AEB when the pellet is at B. Then the pressure in the space upto D below the mercury thread is $p_1 + \frac{Mg}{\alpha}$. The sticking correction of the pellet must be made as in method (a) above. The mass of gas above

B is $\rho_0 p_1 V_1$ and that in space below the thread is $\rho_0(V-V_1)\left(p_1+\frac{Mg}{\alpha}\right)$ neglecting the volume of the capillary.

$$\therefore \quad \rho_0 pV=\rho_0 p_1 V_1+\rho_0(V-V_1)\left(p_1+\frac{Mg}{\alpha}\right).$$

$$\text{Hence, } p_1=p-\frac{Mg}{\alpha}+\frac{Mg}{\alpha}\cdot\frac{V_1}{V}.$$

So the pressure in the region below the thread is

$$P_1=p_1+\frac{Mg}{\alpha}=p+\frac{Mg}{\alpha}\frac{V_1}{V}.$$

If the time of fall of the thread from B to C be t seconds, after this interval, the pressure p_2 above the thread may be similarly found,

$$\rho_0 pV=\rho_0 p_2(V-V_1)+\rho_0\left(p_2+\frac{Mg}{\alpha}\right)V_1.$$

$$\text{Hence, } p_2=p-\frac{Mg}{\alpha}\cdot\frac{V_1}{V}.$$

So the pressure below the thread (in CFD),

$$P_2=p_2+\frac{Mg}{\alpha}=p+\frac{Mg}{\alpha}\frac{V-V_1}{V}.$$

$\therefore$ The mass of gas below the thread (while at B)

$$=\rho_0(V-V_1)\left(p+\frac{Mg}{\alpha}\frac{V_1}{V}\right)$$

and that below the thread (while at C)

$$=\rho_0 V_1\left(p+\frac{Mg}{\alpha}\cdot\frac{V-V_1}{V}\right).$$

Hence the mass of the gas passing through the capillary in t secs. is $\rho_0(V-V_1)\left(p+\frac{Mg}{\alpha}\cdot\frac{V_1}{V}\right)-\rho_0 V_1\left(p+\frac{Mg}{\alpha}\cdot\frac{V-V_1}{V}\right)=\rho_0 p(V-2V_1)$.

where $V-2V_1$ is the volume of the space BC minus the volume of the mercury thread.

Hence the average rate of flow in gms. per sec. is

$$\frac{\rho_0 p}{t}(V-2V_1) \qquad \ldots \qquad \ldots \quad \text{(I)}$$

Now from eqn. (3) of Art. 210, the mass of gas entering the capillary per second at the start is given by

$$\frac{\pi r^4 \rho_0}{16\eta l}[P_1{}^2 - p_1{}^2] = \frac{\pi r^4 \rho_0}{16\eta l}\left[\left(p + \frac{Mg}{l}\cdot\frac{V_1}{V}\right)^2 - \left(p - \frac{Mg}{\alpha} + \frac{Mg}{\alpha}\frac{V_1}{V}\right)^2\right]$$

$$= \frac{\pi r^4 \rho_0}{16\eta l}\cdot\frac{Mg}{\alpha}\left[2p - \frac{Mg}{\alpha} + \frac{2Mg}{\alpha}\frac{V_1}{V}\right] \qquad \text{(II)}$$

Similarly, the mass of gas entering the capillary at the end of the interval per second is

$$\frac{\pi r^4 \rho_0}{16\eta l}\left[\left(p + \frac{Mg}{\alpha}\frac{V - V_1}{V}\right)^2 - \left(p - \frac{Mg}{\alpha}\frac{V_1}{V}\right)^2\right]$$

$$= \frac{\pi r^4 \rho_0}{16\eta l}\cdot\frac{Mg}{\alpha}\left[2p + \frac{Mg}{\alpha} - \frac{2Mg}{\alpha}\frac{V_1}{V}\right] \qquad \text{(III)}$$

∴ The average rate of flow from (II) and (III) is

$$\frac{\pi r^4 \rho_0}{16\eta l}\frac{Mg}{\alpha}\cdot 2p \text{ gm. per sec.}$$

Equating this will the rate calculated in (I), we have,

$$\frac{\rho_0 p}{t}(V - 2V_1) = \frac{Mg}{\alpha}\cdot\frac{\pi r^4 \rho_0}{16\eta l}\cdot 2p\ ;$$

$$\text{or,} \quad \eta = \frac{\pi r^4 .t.Mg}{8l(V - 2V_1)\alpha}.$$

With the help of this relation, the viscosity of gas at any desired pressure and temperature can be determined by Rankine's apparatus.

***211. Determination of viscosity of air by rotating cylinder method (*Bearden's method*)** :—Bearden in 1939 determined the viscosity of air by the rotating cylinder method and his method is supposed to be one of the standard methods for determining the viscosity of a gas.

In the method the inner cylinder is rotated and the outer deflecting cylinder is provided with a guard-ring arrangement to avoid the distortion of lines of flow at the ends. The deflecting cylinder is suspended from a torsion head H by means of a wire and a spider-claw arrangement. If K be the torsional couple per unit twist of suspension wire and if θ be the twist produced in it in the steady state of rotation of the inner cylinder, the restoring torque developed in the wire $= K\theta$. This restoring torque is developed in the wire as a result of the deflecting torque acting on the outer cylinder due to the viscous drag. As shown in Art. 207(*b*), this deflecting torque acting on a cylinder of radius r is $-2\pi\eta l r^3 \frac{d\omega}{dr}$, where η = coefficient of viscosity, $-\frac{rd\omega}{dr}$ = the velocity gradient and l = effective length of the deflected outer cylinder which includes half the clearance between it and the guard-ring at both the ends.

Hence equating the deflecting and the restoring torques we get,

$$-2\pi\eta l r^3 \frac{d\omega}{dr} = K\theta.$$

or, $$-2\pi\eta l \int_{\omega_0}^{0} d\omega = K\theta \int_{a}^{b} \frac{dr}{r^3}.$$ where ω_0 = constant angular velocity of the inner cylinder ; a and b represent the outer radius of the inner and the inner radius of the outer cylinder respectively.

Hence, $$2\pi\eta l\omega_0 = \frac{K\theta}{2}\left(\frac{1}{a^2} - \frac{1}{b^2}\right) = \frac{K\theta}{2}\,\frac{b^2 - a^2}{a^2 b^2};$$

or, $$\eta = \frac{K\theta}{4\pi l\omega_0}\left(\frac{b^2 - a^2}{b^2 a^2}\right) = \frac{KT(b^2 - a^2)}{8\pi^2 l a^2 b^2}\theta \quad \ldots \quad (1)$$

where $\omega_0 = \frac{T}{2\pi}$, T being the time-period of steady revolution of the inner cylinder. Hence knowing all the quantities on the right hand side of relation (1), η can be determined.

A simplified form of Bearden's apparatus is shown in Fig. 228. Here J represents the inner cylinder which is rotated at a uniform speed from outside by means of a suitable device. In the original experiment a magnetic drive was used. On the base B, a vertical stand I is fitted for supporting the torsion head assembly on the shelf S attached to it. To a second shelf F the upper guard-ring cylinder NN is fixed so as to be concentric with a rotating inner cylinder. The pivot about which the inner cylinder rotates is also embedded in F. The outer deflecting cylinder L is held in position by a spider-claw arrangement from C. L is guarded by the cylinders NN and NN at the top and bottom, the inner radii of L and the guard-ring cylinders being made exactly identical and the clearance between them very small. The system C, from which the spider-claws support the deflecting cylinder, is suspended by a wire from the suspension head H. A mirror m is attached to this suspension wire for the necessary lamp and scale arrangement.

The whole arrangement can be put inside a bell jar D, which can be evacuated to any desired degree and the moist air can be completely pumped out. Now inserting the air or any other gas at a known pressure its coefficient of viscosity η can be determined. For measurements at low pressure, the actual value of the low pressure, can be determined by means of an ionisation gauze G attached to the bell jar. The torsional constant K, of the suspension wire can be measured by an auxiliary experiment wherein the time period of oscillation is measured with a body of known moment of inertia. In the actual experiment of Bearden a small correction for the influence of slip was also introduced. All the measurable quantities involved in relation (1) were measured by standard methods at a temperature of 20°C and the mean value of η for air as obtained from a number of similar experiments was computed as,

$$\eta = (1819{\cdot}20 \pm 0{\cdot}06) \times 10^{-7} \text{ c.g.s. unit}$$ at 20°C.

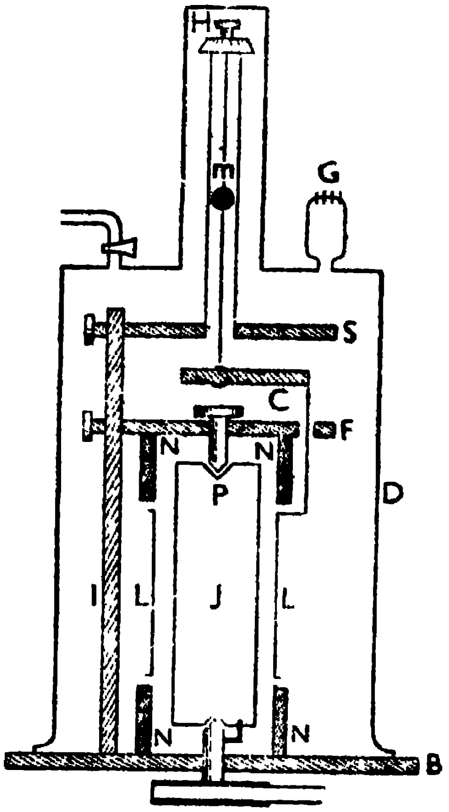


Fig. 228

212. Effects of Pressure and Temperature on Viscosity :—

(1) Liquids :—

In ordinary liquids the effect of pressure on viscosity is small. Except in the case of water the effect of pressure is more pronounced in all liquids at higher pressures. At high pressures the deviation is mainly due to deformation of molecules under the superincumbent pressure.

The viscosity of liquids generally decreases with rise of temperature. As yet no satisfactory theoretical relation has been deduced to exactly represent the variation of η with temperature. There are, however, a number of empirical relations of which two important formulae are given below :

$$(1) \quad \eta_t = \eta_0/(1+\alpha t+\beta t^2)$$

and $$(2) \quad \eta_t = A/(1+\beta t)^n$$

where A, α, β and n are constants.

Andrade gave a relation from the standpoint of a new theory of viscosity proposed by him. His relation is,

$$\eta v^{\frac{1}{3}} = A e^{b/vT},$$

where A and b are constants an the specific volume of the liquid at absolute temperature T.

(2) Gases.—

According to the kinetic theory of gases, the viscosity should be independent of pressure. This conclusion is found to be true over a wide range of pressures. But at low pressures due to the increase of the mean free path of the molecules, the viscosity gradually decreases with the fall of pressure.

The viscosity of gases increases with temperature and nature of variation is given by Sutherland's formula which can be written in the simple form as

$$\eta_t = \eta_0 \frac{A}{T+B} . T^{\frac{3}{2}}$$

where A and B are constants and T is the absolute temperature.

Table of Coefficients of Viscosity

Substance	η in $C. G. S.$ unit	Temperature
Water	0·01006	20°C.
,,	0·01793	0°C.
Mercury	0·0156	20°C
Ether	0·00234	20°C.
Glycerine	8·5	20°C.
Air	$1{\cdot}72 \times 10^{-4}$	0°C.
Hydrogen	$0{\cdot}86 \times 10^{-4}$	0°C.
Oxygen	$1{\cdot}92 \times 10^{-4}$	0°C.
Nitrogen	$1{\cdot}67 \times 10^{-4}$	0°C.

213. Viscosity of Fluids and its Practical Applications:— When contiguous layers of fluid move relative to each other internal forces are called into play which tend to destory this relative motion. The more quickly moving layer is acted on by a retarding force while motion of the less quickly moving layer is accelerated. The property of the fluid by virtue of which it resists this relative motion between its parts is called *viscosity*. It is due to the viscosity that mist drops fall slowly through air, the waves on the sea subside when the wind stops. This property is an inherent characteristic of a fluid. The magnitude of these internal forces differs from liquid to liquid. It is due to this difference in viscosity that different forces are experienced in working a churning machine in milk and in **water**. All fluids are more or less viscous.

The coefficient of viscosity is defined as the tangential force per unit area necessary to maintain a relative velocity of unity between two parallel planes of the fluid at unit distance apart. In an elastic solid the shearing stress on any plane, which is a tangential stress, is proportional to the displacement gradient normal to the direction of shear. According to Newton, viscous drag produced internally in a fluid on any plane is also proportional to the velocity gradient normal to the direction of the motion. It is then possible to regard a liquid as capable of exerting and sustaining a certain amount of shearing stress for a short time, after which it breaks down but the shear appears again. This lead Maxwell to regard viscosity as a limiting case of an elastic solid when the material breaks down under shear. So viscosity is sometimes called fugitive shearing elasticity.

Prof. Osborne Reynolds has found that for every fluid, there is a critical velocity below which the path of a moving particle in a liquid coincides with the motion of the liquid as a whole, *i.e.* the motion is stream-lined and above which the path does not correspond to the motion of the fluid, *i.e.* the motion becomes turbulent. This means that below this critical velocity of rate of flow depends mostly on viscosity and above it, the effect of viscosity is very small and the rate of flow is governed mostly by the density of the liquid. This explains the comparatively rapid flow of the very viscous lava during volcanic eruptions wherein the critical velocity is exceeded. Moreover, it has been found that the critical velocity of flow is directly proportional to the viscosity, inversely to the density of the liquid and also inversely to the radius of the tube (or passage). It follows that in a large tube treacle would flow as readily as water, at velocities exceeding the critical velocity for treacle. Thus it is said that if the Mississipi were a river of treacle it would flow at the same rate as that at which water actually flows.

The viscosity of a liquid decreases with the increase of temperature as has been found by Slotte, Thorpe, Roger, Andrade and others. But in the case of gases it increases with temperature as shown by Sutherland and others. Whether viscosity of gases directly varies as the square root of absolute temperature as predicted by Maxwell on the basis of the kinetic theory of gases, has of course yet to be proved decisively by experiments.

Andrade, Bridgman and others have found that the viscosity of liquids increases with pressure. The effect is more pronounced is some liquids like mineral oils. But the behaviour of water is anomalous below 30°C at which viscosity decreases showing a minimum at 100 atmospheres. Maxwell, from kinetic theory, showed that *viscosity of gases is independent of pressure.* This is found to be true except at very low pressures.

Just as viscosity of a solution has been found in many cases to be greater than that of the pure solvent, there are again, solutions of certain salts of potassium and caesium in water and alcohol, between certain limits of temperature, which have got less viscosity than the solvent.

The range of sound depends on the viscosity of the medium through which it passes. In any homologous series the viscosity increases regularly with molecular weight.

Between viscosity and kinetic friction, there is a very prominent distinctive feature, in spite of their otherwise similar nature. This lies in the fact that the kinetic friction is independent of the relative velocity of the bodies in contact while the viscosity depends on it up to a critical velocity.

The practical importance of viscosity cannot be exaggerated. It is evident from the facts that (*i*) the viscous resistance offered by the sea water to a ship in motion is an essential factor in the matter of designing of ships ; (*ii*) the viscosity of the fountain pen ink is an essential criterion of its quality ; (*iii*) the viscosity of the human blood forms an important point of test for the physicians in many cases. But perhaps the widest application of its knowledge is required in the preparation of lubricants whose use cannot be exaggerated in the present day world ruled as it is by machinery.

Examples

1. Water flows through a horizontal pipe of non-uniform cross-section. If the pressure at the position where the velocity is 25 cms. per sec., be 0·75 cm. of mercury, calculate the pressure at the point where the velocity is 75 cms. per sec.

Ans. : Here, $p_1 = 0{\cdot}75 \times 980 \times 13{\cdot}6$ dynes/cm.^{2} and $v_1 = 25$ cms./sec ;

$v_2 = 75$ cms./sec., $\rho = 1$ gm./c.c. and p_2 is unknown.

We know, for a horizontal flow,

$$p_1 + \tfrac{1}{2}\rho v_1^2 = p_2 + \tfrac{1}{2}\rho v_2^2, \text{ (Bernoulli's theorem).}$$

$$\therefore\ p_2 = p_1 + \tfrac{1}{2}\rho (v_1^2 - v_2^2)$$

$$= 0{\cdot}75 \times 980 \times 13{\cdot}6 - \tfrac{1}{2}(75^2 - 25^2)$$

$$= 980 \times 13{\cdot}6 \left[0{\cdot}75 - \tfrac{1}{2} \times \frac{100 \times 50}{980 \times 13{\cdot}6} \right]$$

$$= 980 \times 13{\cdot}6\ [0{\cdot}75 - 0{\cdot}1876]$$

$$= 980 \times 13{\cdot}6 \times 0{\cdot}5624 \text{ dynes/cm.}^2$$

$$= 0{\cdot}5624 \text{ cm. of mercury.}$$

2. A gas bubble of diameter 2 cms. rises steadily through a solution of density 1·75 gms./c.c. and of viscosity coefficient 0·00109 poise. Calculate the upward velocity of the bubble.

Ans. : Here $r=d/2=1$ cm., $g=981$ cms./sec.^{2}, $\eta=0{\cdot}00109$ poise, and $\rho=1{\cdot}75$ gms./c c. Neglecting air density σ, we have from Art. 207,

$$v=\frac{2}{9}\frac{r^2.\rho\, g}{\eta}=\frac{2\times1\times981\times1{\cdot}75}{9\times0{\cdot}00109}=0{\cdot}35\text{ cm./sec.}$$

3. In a certain experiment on the flow of a liquid through a capillary tube, the following data were obtained :

Volume of liquid coming out per minute = 15 c.c.

Head of liquid = 30 cms.; Length of tube = 25 cms.; Radius of tube = 1·0 mm. Calculate the coefficient of viscosity of the liquid, its density being 2·3 gms./c.c. *(C. U. 1954)*

Ans. : We know, $\eta=\dfrac{\pi P r^4}{8Vl}$.

Here $V=15$ c c. per min.$=15/60=0{\cdot}25$ per sec.;

$P=h\rho g=30\times2{\cdot}3\times980$ dynes/cm.^{2} ;

$r=0{\cdot}1$ cm. and $l=25$ cms.

$$\therefore\quad \eta=\frac{\pi\times30\times2{\cdot}3\times980\times(0{\cdot}1)^4}{8\times{\cdot}25\times25}$$

$=0{\cdot}425$ c g.s. unit (poise).

4. Assuming Stokes' formula for terminal velocity, calculate the limiting velocity of a raindrop weighing $\frac{11}{10}$ mgm. falling through air.

($g=981$ cms./sec.^{2}, $\eta_{air}=1{\cdot}8\times10^{-4}$, neglect density of air)

Ans. : From Stokes' formula, we know $mg=6\pi\eta av$.

Again, $m=\frac{4}{3}\pi a^3\rho=\frac{4}{3}\pi a^3$, ($\rho=1$ for water).

$\therefore$ Putting a in terms of m in Stokes' formula, $v=\dfrac{2}{9}\dfrac{\left(\frac{3m}{4\pi}\right)^{\frac{2}{3}}g}{\eta}$

$$\therefore\quad v=\frac{2(1{\cdot}1\times10^{-3}\times3/4\pi)^{\frac{2}{3}}\times981}{9\times1{\cdot}8\times10^{-4}}=4968\text{ cms./sec}$$

5. Air from a bomb of 3·5 litres capacity at pressure 10 atoms, leaks through a capillary tube 100 cms long and 0 25 mm. internal bore out into free atmosphere. If the pressure falls to 5·5 atmos, in 15 mins., calculate the viscosity coefficient of air, given 1 atmos. $=1{\cdot}013\times10^6$ dynes per cm.^{2} *(C. U, Hons. 1950)*

Ans. : From Art. 210 (*b*), $\eta=\dfrac{P\pi r^4 t}{8lV_m\log\left(\dfrac{p_2+P}{p_1+P}\cdot\dfrac{p_1-P}{p_2-P}\right)}$

Here $P=1{\cdot}013\times10^6$ dynes/cm.$^2=1$ atmos. :

$r=0{\cdot}0125$ cm ; $t=60\times15$ secs. ; $l=100$ cms. ;

$V_m=3500$ c.c. ; $p_1=10$ atmos. and $p_2=5{\cdot}5$ atmos. ;

Hence substituting.

$$\eta=\frac{1{\cdot}013\times10^6\times3{\cdot}14\times(0{\cdot}0125)^4\times900}{8\times100\times3500\times2{\cdot}303\log_{10}\left(\frac{6{\cdot}5}{11}\cdot\frac{9}{4{\cdot}5}\right)}$$

$$=\frac{1{\cdot}013\times3{\cdot}14\times(0{\cdot}0125)^4\times9\times10^8}{10^4\times8\times35\times2{\cdot}303\log_{10}1{\cdot}182}=1{\cdot}491\times10^{-4}\text{ c.g.s units.}$$

Questions (Chapter X)

1. (*a*) Enunciate and prove Bernoulli's theorem and mention some of its important applications. (C. U. 1968)

(*b*) Explain the principle of Venturi meter for the measurement of the volume of a liquid flowing through a tube with its help.

Why do two ships moving side by side and parallel to each other tend to come closer together ? (C. U. 1966)

2. A venturi meter is connected to two points in a main where its radii are 20 cms. and 15 cms. respectively, and the levels of the water column in the two limbs differ by 10 cms. How much water flows through the pipe per hour ? ($g = 980$ cms./sec.^{2}) [*Ans.* : $48{\cdot}11 \times 10^4$ litres]

3. (*a*) Define the coefficient of viscosity of a liquid and find its dimensions.

Describe how different parts of a viscous liquid move when flowing through a fine tube. What change takes place if the motion is increased ?

Describe a method of finding the coefficient of viscosity of water, deriving the formula you use. (Gau. U. 1949)

(*b*) Distinguish between stream-line flow and turbulent flow of a liquid. What is Reynolds' number ? State its utility. (C. U. 1964)

4. What do you understand by the viscosity of a liquid ? Describe carefully how you would measure the coefficient of viscosity of liquid. Establish the formula you use. (Pat. U. 1947)

5. Define coefficient of viscosity for a liquid. Describe, how the coefficient of viscosity is determined for water. (Ban. U. 1949)

6. Derive the formula for the flow of a liquid through a capillary tube. Why does the formula fail in the case of a wide bore ? (*cf.* Utk. U. '70S ; Agra U. 1941)
Give the dimensions of viscosity. (C.U. 1962)

7. Derive Poiseuille's equation for the viscosity of a liquid flowing through a narrow tube. State clearly the assumptions made and discuss any correction you have to employ. (Bom. U. 1948)

8. A flat plate of area 10 cm.^{2} is separated from a large plate by a layer of glycerine 0·1 cm. thick. If the force required to keep the plate moving with a velocity of 1 cm. per sec. be 2000 dynes, calculate the coefficient of viscosity of glycerine.
[*Ans.* : 20 c.g.s. units.]

9. What is meant by the term coefficient of viscosity ? Obtain as expression for the rate of flow of a liquid through a capillary tube of circular cross-section.

Note the precautions to be adopted in the experimental determination of this coefficient, using this expression. (Madras U.1949)

10. Define coefficient of viscosity. Give examples of some viscous substances. How would you determine the viscosity of a liquid ?
Why do you find cloud particles floating in the sky ? (Agra U. 1934)

11. (*a*) Explain what is meant by the viscosity of a liquid. How will you study experimentally the variation of viscosity with temperature ?
What are the dimensions of viscosity in terms of length, mass and time ?
(All. U. 1946)

(*b*) Describe how the different parts of a viscous liquid move when flowing through a fine tube. What change takes place if the velocity is increased to a high value (Utk. U. 1966)

12. What kind or kinds of energy result from the work done by a fluid against viscosity ? How can the viscosity of a liquid be determined ?
(Bom. U. 1947)

13. Give Poiseuille's method of measuring the viscosity of liquids. Derive the formula used. Indicate a method which could be employed to measure the viscosity of liquid at different temperatures. (Bom. U. 1947)

14. Define coefficient of viscosity.

Derive Stokes' formula for the velocity of a small sphere falling through a viscous liquid. Explain how this is utilised to determine the viscosity of a liquid like castor oil. Mention one more application of Stokes' formula. (Madr. U. 1951)

15. A capillary tube 1 mm. in diameter and 20 cms. in length, is fitted horizontally to a vessel kept full of alcohol, of density ·5 gm./cm^3. The depth of the centre of the capillary tube below the surface of alcohol is 40 cms. If the viscosity of alcohol is 0·012 c.g.s. units, find the amount that will flow out in 5 minutes. Prove the formula you use. (Bom. U. 1933)

[*Ans.* : 57·8 gms.]

16. Define coefficient of viscosity.

Describe fully what you would do to compare the viscosities of two liquids.

(Madr. U. 1949)

17. Define the coefficient of viscosity of a liquid.

What is the effect of temperature upon it ? How do you determine the viscosity of water at different temperature ? Derive the formula you use.

(Punj. U. 1947)

18. Water is escaping from a cistern by way of a horizontal capillary, 10 cms. long and 0·4 mm. in diameter at a distance of 50 cms. below the free surface of water in the cistern. Calculate the rate at which water is escaping. (Bom. U. 1926)

[*Ans.* : $\frac{8\cdot082 \times 10^{-4}}{\eta}$ c.c. per sec. where η is the coef. of viscosity of water.]

19. A glass bottle is fitted with a siphon made of capillary glass tubing. Compare the times taken to empty the bottle when it is filled with (*a*) water, (*b*) petroleum of sp. gr. 0·8. The viscosities of water and petroleum are 0·01 and 0·02 c.g.s units respectively. (Bihar U. 1957)

[*Ans.* : 5 : 2]

20. Find the terminal velocity of an oil drop of density 0·95 gm./c.c. and radius 10^{-4} cm. falling through air of density 0·0013 gm./c.c., if the viscosity of air is 181×10^{-6} c.g.s. units. [$g = 980$ cm./sec.^{2}] (Pat. U. 1956)

[*Ans.* : $1\cdot333 \times 10^{-2}$ cm /sec]

21. Deduce Poiseuille's formula for the flow of a liquid through a capillary tube. How is this formula modified in the case of a gas ?

Describe how you would determine the viscosity of a liquid accurately.

(C. U. H. 1967)

22. Describe an accurate method of measuring the viscosity of gases at any desired temperature and pressure.

CHAPTER XI

DIFFUSION AND OSMOSIS

214. Diffusion :—If a jar, filled with a light gas like hydrogen, is inverted over another containing a relatively heavier gas like carbon dioxide, it is found that after some time a uniform mixture is produced, in apparent contradiction to the pull of gravity. Similarly, if a strong potash permanganate solution, which is a coloured liquid, is kept at the bottom of a cylinder and water is slowly and carefully added from above without appreciably disturbing the solution, the solution gradually works up and the coloured liquid spreads throughout the whole mass. Such processes of self-mixing, sometimes even in opposition to gravity, of one fluid with another is termed *diffusion.*

The explanation of the phenomenon of diffusion as based on the kinetic theory is that the molecules of fluids are in perpetual motion in all possible manners even irrespective of the rules of gravity and as such they make this intermixing possible. This perpetual but chaotic molecular motion was first observed by Dr. Brown and is known as Brownian motion. Molecular motion being more vigorous in gases than in liquids, the gaseous diffusion is more rapid than the diffusion of liquids. Graham made an extensive study of both the gaseous and the liquid diffusion and found that the rate of diffusion depends on the nature of the two fluids concerned, their concentrations and temperature.

215. Fick's Law :—The above facts regarding the diffusion of fluids, particularly for liquids, have been mathematically formulated by Fick on the line of arguments which are similar to those forwarded in the case of conduction of heat in a solid. Fick's law states that *the rate of diffusion in any direction is proportional to the gradient of concentration of the solute in that direction.*

If Q be the quantity of solute diffusing per sec per unit area from a plane of higher concentration c_1 to that of another of a lower concentration c_2 separated by a distance x, then $Q = K\frac{c_1 - c_2}{x} = K\frac{dc}{dx}$, where K is a constant called the *coefficient of diffusion*, whose value actually depends on the nature of the salt, the solvent and the temperature. A law of the same form as Fick's law for liquids is applicable to gaseous diffusion also. Here the mass Q of any one of the two gases diffusing per sec. per unit area at any height is given by, $Q=K\frac{d\rho}{dx}$, where $\rho=$ density of the gas at that height, its distance x being measured from some fixed horizontal plane.

216. Time of diffusion and length of the Column :—From Fick's law it can be shown that time required to pass from one distribution of concentration to a different one in any column of a solution is proportional to the square of the length of the column. To establish this let us suppose two columns of length l_1 and l_2 having identical distributions of a solute along them at a given instant. Evidently the time intervals t_1 and t_2 required to attain identical concentrations must be directly proportional to the corresponding lengths of the columns l_1 and l_2 and also inversely proportional to the absolute velocities of the solute at each layer.

$$\therefore \quad \frac{t_1}{t_2}=\frac{l_1}{l_2}\cdot\frac{v_2}{v_1}.$$

Again since the mass of solute diffusing per sec. per unit area is Q, we can write $Q=1\times v\times c=vc$, where $v=$ velocity at the layer and $c=$concentration.

$$\therefore \quad vc=k\frac{dc}{dx}; \quad \text{or,} \quad v=\frac{k}{c}\cdot\frac{dc}{dx}$$

where $\frac{k}{c}$ is constant for the layer.

If the distribution of concentration be the same in both the columns, $\frac{dc}{dx}$ is inversely proportional to the length of the column.

$$\therefore \quad v\propto\frac{1}{l}; \quad \text{or,} \quad \frac{v_2}{v_1}=\frac{l_1}{l_2}.$$

$$\therefore \quad \frac{t_1}{t_2}=\frac{l_1}{l_2}\times\frac{v_2}{v_1}=\frac{l_1^2}{l_2^2}.$$

Thus the time of diffusion is directly proportional to the square of the length of the column.

217. Graham's Law of Gaseous Diffusion :—Graham enunciated a law for diffusion of gases, which has been subsequently explained from the Kinetic theory. This law is known as Graham's law of diffusion. The law states that *the rates of diffusion of two gases are inversely proportional* to the square roots of their densities.

$$i.e. \frac{\text{rate of diffusion of a gas } A}{\text{rate of diffusion of a gas } B} = \frac{r_a}{r_b} = \sqrt{\frac{\rho_b}{\rho_a}},$$

where ρ indicates the density.

The gaseous mixtures were separated by Graham's law with the help of the principle of different rates of diffusion. This process of separation of gases is known as "*atmolysis*".

218. Effusion :—When a gas passes from a vessel into an evacuated one through a small hole in a thin plate such that the width of the hole is greater in comparison with its length, the process is called *effusion*. Graham showed that the rate of effusion of a gas varies directly as the square root of the difference of pressure on the two sides of the hole and also inversely as the square root of its density.

$$\text{So the velocity of effusion} \propto \sqrt{\frac{\text{Difference of Pressure}}{\text{Density}}}.$$

219. Determination of coefficient of Diffusion :—If the concentration at various depths of a vertical column can be successfully determined at known intervals of time, the coefficient of diffusion can be easily calculated from Fick's Law. Lord Kelvin determined the distribution of concentration by placing floats of various densities within the column. From the positions of the floats the densities at different positions are known and hence knowing the concentration-density relation of the solution, the distribution of concentration was found. Then by directly applying Fick's law the coefficient of diffusion was determined.

Later on Clack measured the concentration gradient by using the movement of the interference fringes when two slightly displaced beams of light pass through the solution.

220. Crystalloids and Colloids :—Thomas Graham drew a distinction between classes of solution—Crystalloids and Colloids.

Crystalloids consists of mineral acids, salt solutions, sugar solution, etc. which have a comparatively greater rate of diffusion, while the *Colloids* consist of generally amorphous gelatinous substance such as albumen, gum, starch, etc. which diffuse rather slowly than the crystalloids. The distinction between two varieties of solution is very much marked in some physical properties. Crystalloids exert osmotic pressure, lower the vapour pressure, depress the freezing point and raise the boiling points of the solvent while colloids have no such effect. Colloids in suspension are all electrically charged similarly while crystalloids in solution dissociate into oppositely charged ions. Though apparently crystalline substances form the crystalloids, the line of demarcation is very vague ; for example, sodium stearate acts as a colloid in aqueous solution and as a crystalloid in alcoholic solution. Graham found that if a solution containing both crystalloids and colloids are kept in a vessel whose bottom is made of a semipermeable membrane such as a fish bladder and partly immersed in pure water, the crystalloids gradually pass through while the colloids are left behind in the vessel. Such separation of colloids from crystalloids by the help of a semipremeable partition has been called by Graham as **Dialysis**.

Modern X-ray researches, however, have shown that all matter is essentially crystalline in form. Therefore the distiuction between crystalloids and colloids is only artificial. The difference between the two types of solution is really a difference in the degree of dispersion or subdivision of the dissolved substance. If the particles in solution are within the limits of visibility which ranges today up to a diameter of 10^{-7} cm (with the help of Zsigmondy's ultramicroscope arrangement), the solution is classified as a colloidal one and beyond that it is termed a true solution. In a proper dispersive medium the so-called colloids may also turn into true solutions as has been illustrated in the case of sodium stearate above.

OSMOSIS

221. Osmosis and Osmotic Pressure :—If a crystalloidal solution is separated from the solvent by means of a partition which is permeable to the molecules of the solvent only but not to those of the solute, transmission of solvent molecules from the solvent side takes place. Such one way or selective transmission of solvent molecules is known as *Osmosis*. The copper-ferrocyanide earthenware pot[*] acts as a good semipermeable partition for aqueous solutions. If such pot is filled with an aqueous solution of sugar and its mouth is closed by a rubber stopper R through which passes a glass tube connected to a closed mercury manometer M and the pot is wholly immersed in a bath of water W (Fig. 229), it is found that gradually the pressure within it increases as shown by the manometer and attains a final steady value.

The *excess pressure as shown by the manometer is called the osmotic pressure of the solute in the solvent at the temperature of the experiment*, or, in the alternative, the *pressure which is required on the*

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[*] If a porous pot of unglazed earthenware be filled with cupric sulphate solution and then immersed in a dilute solution of potassium ferrocyanide, the two solutions meet and interact within the pores of the pot forming an insoluble precipitate of cupric ferrocyanide. The precipitate acts as a very satisfactory semipermeable membrane for aqueous solutions and stands moderately high pressures.

solution side to stop osmosis is a measure of the osmotic pressure of the solute in the solvent at the temperature of the experiment.

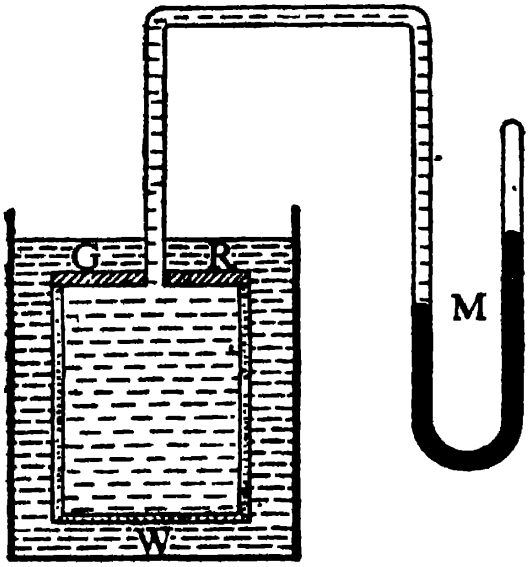


Fig. 229

The explanation of the phenomenon of osmosis lies in the fact that the concentration of solvent molecules per unit volume on the solvent side is greater than that on the solution side and hence the pressure due to the bombardment of the membrane by the solvent molecules is greater on the solvent side. As a result of this inequality, the solvent molecules are transmitted through the semipermeable partition to the solution side, thus giving rise to the phenomenon of osmosis.

Osmotic pressure must, however, be distinguished from the pressure of the solute molecules on the membrane, otherwise called diffusion pressure, and also from the thermal pressure of the solution caused on account of the disorderly motion of the molecules in it due to their kinetic energy. In fact, the osmotic pressure is not an actual pressure exerted by a component of a solution but rather is a difference in pressure which must be established between the solution and the solvent in order to make the escaping tendency of the solvent equal to that of the solution. It depends on the concentration of the solution and its temperature.

The osmotic pressure as measured by the above method by Pfeffer is not accurate specially for concentrated solutions. Later on Morse and his co-workers made improvements in respect of (1) quality of the semipermeable membrane, (2) connection between cell and the manometer and (3) measurement of pressures. The method devised by Earl of Berkeley and Hartley, described below, is superior to other methods in that the strength of the solution remains constant throughout in this method since any influx of the solvent into the solution is here offset by an externally applied counter pressure.

222. Berkeley and Hartley's method for measuring Osmotic Pressure :—A semipermeable membrane *M* of cupric ferrocyanide is formed by the electrolytic deposition near the outer surface of a porous porcelain tube (Fig. 230).

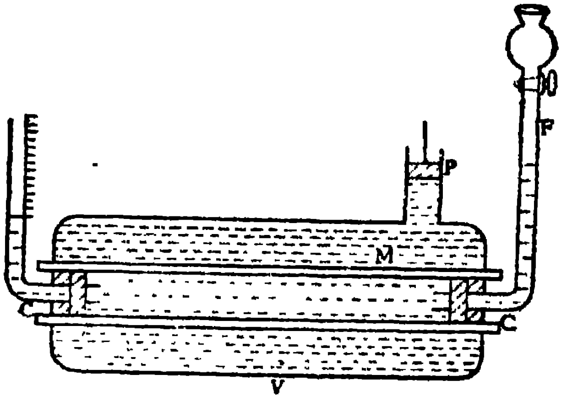


Fig. 230

The tube is held coaxially within a large cylindrical vessel V of gun-metal placed horizontally. The two are jointed by an ingenious system of rings and clamps. The open ends of the porcelain tube are closed with rubber stoppers C fitted with capillary tubes bent at right angles, of which, one F is fitted with a funnel and stopcock. The solvent (water) is introduced into the porcelain tube till it is filled up and stands up to a certain level in the capillary tubes. The outer tube is then filled up with the experimental solution and then by means of an auxiliary pump P connected to it a gradually increasing pressure is applied to the solution. This applied pressure can be measured by a suitable pressure gauge (not shown in the figure) connected to the auxiliary pump arrangement. If no such pressure were applied the solvent would have gradually passed through the membrane into the outer cylinder and the liquid level in the capillary tubes would fall. By suitably adjusting the applied pressure the liquid level is to be kept constant in the capillary tubes. This is done by applying gradually increased pressure till it just counterbalances the osmotic pressure. At this stage the reading of the pressure gauge gives the required osmotic pressure.

The values of osmotic pressure obtained by this method for concentrated solutions of cane sugar have been found to be greater than the theoretically expected values. According to Callendar, it is due to the hydration of the solute, as each molecule of cane sugar is associated with five molecules of water of crystallisation.

223. Laws of Osmosis : (1)—Pfeffer's experimental results show that for weak electrolytic solutions, the osmotic pressure at any particular temperature is proportional to the number of gramme-molecules per c.c. of the solution, *i.e.* $p \propto c$ where p is the osmotic pressure and c is the gramme-molecular concentration.

(2) An analysis of Pfeffer's data for osmotic pressure at different temperatures made Van't Hoff to conclude that the osmotic pressure for weak solutions is proportional to the absolute temperature, i.e. $p \propto T$.

Thus for dilute solutions,

$$p = B_1 cT \qquad \ldots \qquad \ldots \qquad \ldots \quad (1)$$

where R_1 is a constant.

Again if v c.c. of the solution contains 1 gm. mol. of the solute, $c=\frac{1}{v}$.

$$\text{From (1) we get, } p=\frac{R_1T}{v}\text{ ; or, } pv=R_1T \qquad \ldots \quad (2)$$

Both these laws of concentration and temperature are commonly known as Van't Hoff's laws of osmotic pressure.

Van't Hoff calculated the value of R_1 in eqn. (2) using Pfeffer's data and found it to be equal to 0·0813 litre-atmosphere per gm.-molecule of the solute—a striking resemblance with a gas. Hence he pointed out that the pressure exerted by any substance in a *dilute solution* is the same as it would have exerted if present as a gas in the same volume as that occupied by the solution provided that the solution is so very dilute that the actual volume of the solute may be neglected in comparison to that of the solvent. From these considerations it may be said that a solute in a dilute solution behaves as a gas. This assumption is further corroborated by the fact that solutions of non-electrolytes exerting equal osmotic pressures contain equal number of gm.-moles per c.c.—an observation similar to Avogadro's in gases. Such solutions having identical osmotic pressures are called ***isotonic solutions***.

The above equation of Van't Hoff applies to non-electrolytic solutions or at best weakly electrolytic solutions. For solutions of electrolytes the laws of osmosis are also the same ; but for any concentration, the value of the osmotic pressure is greater than that of a non-electrolyte. On the basis of Arrhenius' theory of electrolytic dissociation the larger osmotic pressure is due to increase in the number of ions in the solution due to dissociation. So the osmotic pressure of a solution is the sum of the osmotic pressures of different kinds of ions in the solution plus the pressure of the undissociated molecules. So if i be the degree of dissociation, Van't Hoff's eqn. (eqn. 2) can be modified in case of an electrolyte as $pv=iR_1T$.

Example

1 gm. of cane sugar (mol. wt.=342) is dissolved in 100 c c. of water. Calculate the value of osmotic pressure at 0°C of this solution. (*C.U. 1959*)

(Assume $R=0{\cdot}082$ litre-atmos.)

Ans. : Here 1 gm. of sugar is contained in 100 c c.

∴ 342 gms. of sugar is contained in 100×342 c.c. $=34{\cdot}2$ litres.

From $PV=RT$, we have $P=\frac{RT}{V}$.

$$P=\frac{0{\cdot}082\times273}{34{\cdot}2}=0{\cdot}6546 \text{ atmosphere.}$$

Questions (Chapter XI)

1. What do you understand by osmosis, dialysis and diffusion ? State the laws of osmosis and describe an arrangement for measuring osmotic pressure.

(C. U. 1944)

2. Define osmotic pressure and state its laws. How will you demonstrate this pressure ? (Madr. U. 1949)

3. At 10°*C* the osmotic pressure of an aqueous solution of urea is 500 mm. of mercury. The solution is diluted and its temperature raised to 25°*C*., when the osmotic pressure becomes 105·3 mm. Determine the extent of dilution.

[*Ans.* : 5 times diluted].

4. Calculate the strength of a cane sugar solution whose osmotic pressure at 27°*C* is one atmosphere. (Mol. wt. of cane sugar=342 gms. and $R=8·4\times10^7$ ergs. per °*C*).

[*Ans.* : 13·76 gms./litre]

5. State the laws of osmotic pressure.

Give an account of the relation between osmotic pressure and other properties of a solution. (Madr. U. 1947)

6. Enunciate Fick's law of diffusion.

Starting with the same concentration of a given solute at the bottom of columns of the same liquid, show that the times required to set up a given concentration at the top are proportional to the squares of the heights of the column. (Madr. U. 1947)

7. State Fick's law of diffusion in liquids.

Indicate a method of determining experimentally the coefficient of diffusion of a salt in a solution. (Madr. U. 1949)

8. Define and explain the following :—

Effusion, osmotic pressure, isotonic solution, dialysis, Van't Hoff's laws of osmotic pressure.

Discuss the shortcomings of Pfeffer's method of determining the osmotic pressure. Describe any other method which you consider superior to the former one.

9. A solution containing 3·12 gms. of a substance per litre gives an osmotic pressure of 1·27 atmospheres at 25°*C*. Calculate the molecular weight of the substance. (C. U. 1942)

[*Ans.* : 60]

CHAPTER XII

PUMPS AND GAUGES

224. **Introduction** :—In the recent developments of science and technology, vacuum technique or production of vacuum plays a very important role. Vacuum is produced to the desired extent by various types of *vacuum pumps*. The vacuum is that has been attained by these pumps ranges from the atmospheric pressure down to a pressure of 10^{-9} mm. of mercury. The instruments by means of which the pressure of these evacuated spaces are measured are generally known as *vacuum gauges*. The important types of pumps and gauges are described briefly in the subsequent articles.

225. **Pumps** :—For the production of moderately low pressures ordinary single barrel or double barrel plunger air pumps are used and for still higher order of evacuation jet pumps and Töepler pumps may be used according to the need. But for the production of high vacuum, we first need a fore-pump which is to be connected to the final pump producing the desired high vacuum. The function of this fore-pump is to evacuate the vessel to the desired extent previous to final evacuation. The best type of the fore-pump used in the laboratories as well as in industries is the rotary pump. A few important pumps are described below.

(a) **Filter Pump (or Water Jet Pump.)**—It is an exhaust type of air-pump ordinarily made of glass and is used when the degree of vacuum required is not lower than about 7 mm. Its special feature is that it needs no attention.

The pump is shown in Fig. 231. The side-tube B is connected with a rubber tubing to the vessel intended for evacuation. The upper end of the vertical tube A which tapers below and ends in the nozzle N is connected to the water mains, the pressure of which should remain constant. As a strong jet of water forces out of the nozzle with a very high speed, some air from around the nozzle is also entangled and carried down the tube. The draught produced thereby draws out the air from within the vessel at the same rate.

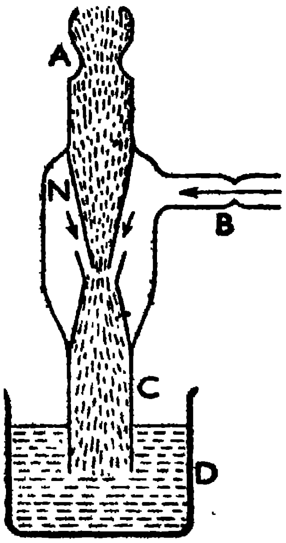


Fig. 231—Filter Pump.

(b) Sprengel Pump.—The attached figure (Fig. 232) shows the working principle of a simple type of Sprengel pump, which is a mercurial air pump. It consists of a tube *ACB* connected to a mercury reservoir *R* through a pinch-cock *P*. The vessel *V* to be evacuated is connected to the tube *ACB* at *C*. When mercury passes *C*, it separates into drops which can be properly adjusted in size and speed by controlling the pinch-cock and the height of the reservoir. The falling mercury drops entrail bubbles of air in between them. Thus the air in *V* will be dragged down with the descending mercury droplets in *CB*. By adjusting the speed of the falling drops, the rate of suction can also be regulated. In some type of this pump, the mercury in the reservoir is automatically replenished by means of a capillary tube joining the mercury trough *T* with the reservoir the upper end of the capillary tube being above the mercury level in *R*. Here the suction of mercury from *T* to *R* is effected by connecting *R* to a water-jet pump. In some other form a jet arrangement is also made at *C* as in a water-jet pump. This device increases the efficiency of the pump to a certain extent.

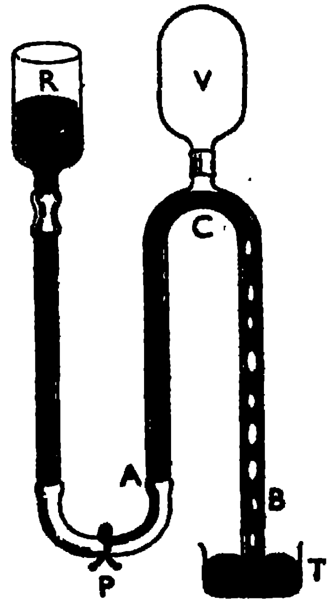


Fig. 232

(c) Töepler Pump.—It is essentially a mercury air pump operated by hand. It was designed in 1862 by Töepler and had been since then used for the production of high vacuo. But due to its slow speed it has recently been completely replaced by more efficient pumps. In high vacuum technique it is now-a-days almost obsolete. It is, however, often used for collecting gases at a particular pressure, *i.e.* for filling gases in discharge tubes etc. The pump, though slow in operation, can produce a vacuum of the order of 10^{-4} mm. of mercury.

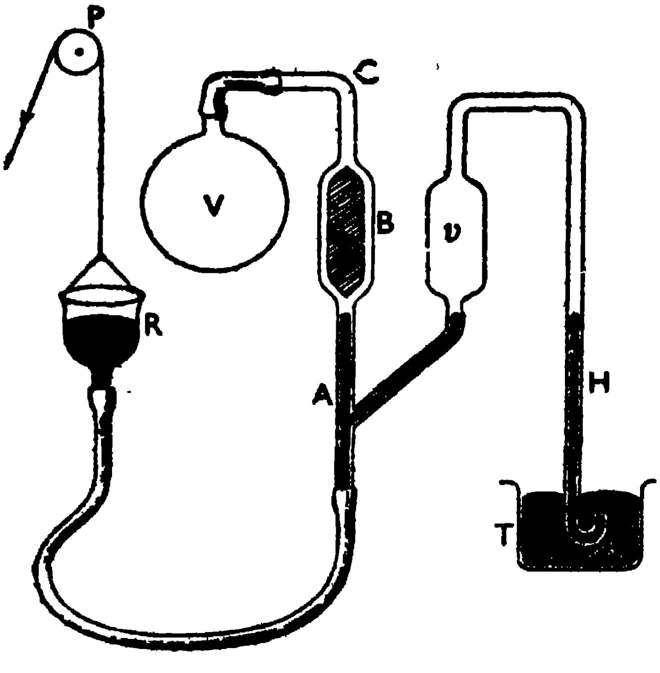


Fig. 233

A modified design of the pump is shown in Fig. 233. Here the vessel *V* to be evacuated is connected to a mercury reservoir *R* through a connecting tube *ABC* and a rubber tubing. The height of *R* can be adjusted by means of a string attached to it which moves over a pulley *P*. A side chamber *v* is connected at the bottom to *ABC* in the way shown in the figure. At the top, *v* is attached to a barometric tube *H* dipped in a mercury trough *T*. A bit above the side tube connecting *ABC* with *v*, there is a comparatively wider portion *B* in *ABC* wherein a glass bead is inserted. This bead serves as a valve and prevents the flow of mercury in *V* when *R* is raised. By this device the length of the connecting tube has been sufficiently shortened. In the original apparatus (without this valve arrangement) this length was sufficiently large so that mercury could not enter *V* when the latter is partially evacuated.

To operate the pump, *R* is sufficiently raised when mercury in *AB* cuts off the air communication between *V* and *v*. On further raising *R*, the rise of mercury in *BC* is arrested by the valve whereas the rising column of mercury in *v* forces out air from it through the mercury trough. *R* is then lowered again till air communication is established between *V* and *v* whereby some amount of air from *V* passes on to *v*. *R* is then again raised and the air in *v*, is forced out. In this way slowly repeating the operation a sufficiently high degree of vacuum is ultimately obtained in *V*.

(b) **Rotary Pumps.**—A pump of the rotary class is valuable for use where lack of space prevents the adoption of an ordinary plunger pump. Its discharge is continuous and can be worked over a wide range of speeds. Moreover, a rotary exhaust pump, is superior to a piston pump, for it is simpler, faster and can produce higher vacua. Its principal disadvantage is due to leakage past the rotating surfaces, which results in a loss of efficiency.

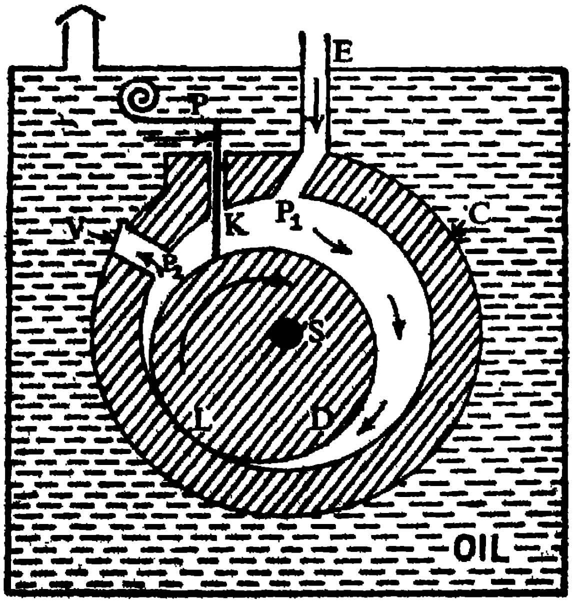


Fig. 234—The Rotary Pump

The principle of a **Hyvac rotary pump** is illustrated in Fig. 234. It is practically a technically improved form of the original type of rotary pump invented by Gaede. Such a pump can be used to reduce the pressure in a vessel to about 0·0001 mm. of mercury.

A cylindrical drum D acts in it as rotor and it is mounted eccentrically to a shaft S which passes along the axis of a cylinder C. The shaft is rotated by an electric motor. The drum and the cylinder are machined accurately such that the surface of the drum just slides on the inner surface of the cylinder as the shaft rotates. In the figure the line of contact is shown by L at some instant of time. P_1 is the entrance port through which air from the vessel to be evacuated enters into the cylinder, and P_2 the exit port through which the air leaves the cylinder. The exit port P_2 is provided with a simple valve V opening only outwards. A scraping vane K is constantly pressed on to the drum between the entrance and exit ports, by the action of a spring P, whatever be the position of the eccentric drum in course of its rotation. The whole arrangement is immersed in some oil of low vapour pressure contained in a box as shown in the figure Pipe E connected to the entrance port projects out and is to be connected to the vessel intended for evacuation. The oil lubricates the shaft and prevents air leakage along the shaft into the high vacuum in the cylinder.

The drum D, as shown, is rotated in the clockwise direction at the rate of a few hundred revolutions per minute. At the instant shown in the figure. the volume, on the entrance side of the line of contact L, *i.e.* at the tail end of the rotor, is increasing and so the pressure diminishing thereby causing the air in the vessel to flow into it. On the other hand, the volume, on the exit side of L, *i.e.* at the head end of the rotor, is decreasing. This means that the air in front of the rotor is driven out through the exit valve V. The scraper vane K prevents any air from flowing from the head end of the rotor to its tail end.

As the rotor D continues to rotate, a time comes when the line of contact L passes the exit port P_2. So the exit port then becomes exposed to the vessel to be evacuated and the atmospheric pressure closes the valve V. Soon after, the line of contact passes the entrance port P_1, when the volume of air in front begins to be swept out again as in the previous cycle.

(e) **Gaede and Langmuir Molecular Pump.**—It is a type of pump requiring a fore-vacuum for its successful working. It consists of a rotor R revolving round an axis through its centre, inside and closely adjacent to the walls of the stator S. R and S are eccentrically fitted (Fig. 235). An outlet and an inlet port, O and I respectively are provided in the stator. The annular space between I and O is compara-

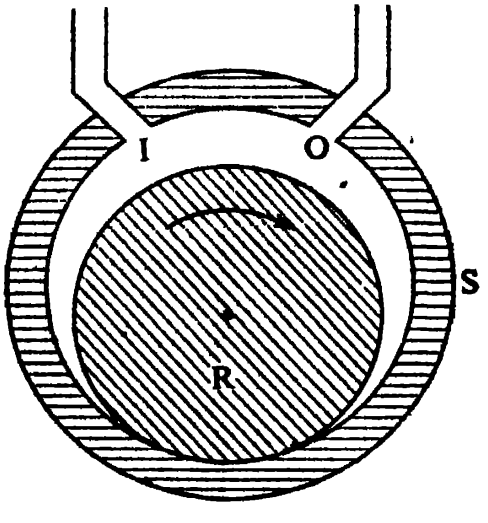


Fig. 235

tively greater than at other regions of the stator.

When the rotor revolves with a high speed in the direction shown in the figure, it drags with it the gas from the inlet port towards the outlet one due to the viscosity of the gas. Thus a pressure difference is created. This pressure difference cannot be solely due to the viscosity of the gas for the viscosity is independent of pressure at ordinary temperatures. In order to have a higher degree of evacuation the rotor should have a speed exceeding 5000 rev./min. and the clearance shown much magnified in the figure between the stator and the rotor should be of the order of ·03 mm. or even less. With a fore-vacuum of about 2 mms. of Hg and with a speed of 10,000 rev./min. for the rotor, the pressure attained by Gaede-Langmuir pump will be of the order 10^{-6} mm. of mercury.

(f) **Diffusion and Condensation Pumps.**—These pumps also require a fore-vacuum for their working which should be of the order of 1 mm. of Hg. Generally rotary hyvac pumps serve as the fore-pumps in this case.

(i) **Diffusion Pump.**—To understand the principle of the diffusion pump, consider the arrangement shown in Fig. 236. A stream of mercury vapour rushes up the tube *AB*. The passage of the vapour jet up the tube *AB* gives rise to a partial vacuum in the system to the right.

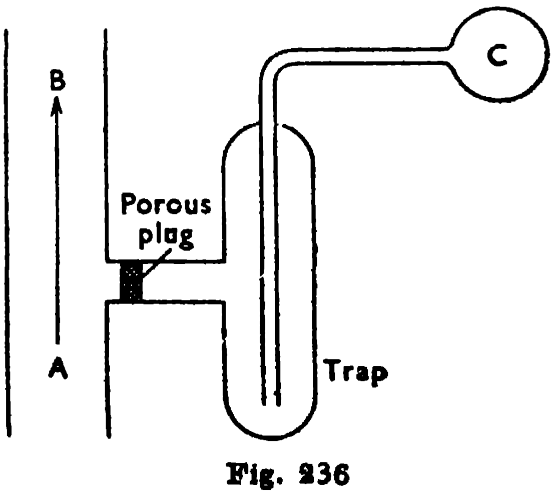


Fig. 236

The degree of evacuation produced in the system to be evacuated (i.e. *C*) depends on the relative rates of diffusion of mercury vapour and the gas through the porous plug (see Fig. 236).

Mercury, being heavier diffuses at a slower rate than the gas to be evacuated. The efficiency is sufficiently increased by the introduction of the vapour trap cooled by water or liquid air. The fore-pump maintains a pressure of a few mm. in the tube *AB*. The air molecules passing out of the plug are carried along the jet of the mercury vapour towards the fore-pump. The traces of mercury vapour that may pass through the plug are condensed in the trap. The jet of mercury that ascends up may be condensed and restored to the boiler. Gaede first applied this principle and the apparatus so constructed was called a diffusion pump. Subsequently porous plug was replaced by adjustable slits and it was found that when slit width was of the dimension of the mean free path of the gas molecules, the efficiency becomes maximum. The pressure attained by this type of diffusion pump is near about 10^{-6} mm. of mercury. Theoretically, of course, there should be no limit to the vacua attainable by this pump.

(i) **Diffusion-condensation Pump.**—In practice, there are two disadvantages in the above arrangement—

(*i*) The exhaust speed is slow.

(*ii*) Careful regulation of temperature of mercury vapour is not possible.

These difficulties are removed by cooling the mercury vapour thoroughly at the jet so that condensation sets in, the back diffusion being thus almost completely eliminated. The arrangement requires no critical conditions for its successful operation and the size of the orifice may vary over a wide range. These improvements were introduced by Langmuir and the pump acting on this principle is called a diffusion-condensation pump because both these processes are of vital importance in the operation of the instrument. Sometimes this type of pump is referred to as 'diffusion' pump or a 'condensation' pump. There are various designs of this pump. Langmuir's modified diffusion-condensation pump as shown in Fig. 237 is described here.

Mercury in the bulb *A* is boiled by heating it with a gas burner or by an electric heater. The vapour passes through the orifice *E* in a vessel *P* which is covered by a water jacket *J*. The condensed mercury returns to *A* through *R*. The bulb *A* and the tube *B* are lagged with asbestos to prevent the mercury vapour from condensing before it issues through *E*. The vessel to be evacuated is connected to *F* directly or through a liquid air-trap *T* which is partly immersed in liquid air contained in a Dewar flask (not shown) wherein the mercury atoms diffusing through *F* may be condensed. At the bottom *P* is connected to the fore-pump through the side tube *N*, which reduces the pressure to nearly 1 mm. of mercury before the diffusion-condensation pump begins to operate.

If there is a large concentration of mercury vapour in *P* immediately above the level of *E*, it would tend to cut down the speed of pumping because the gas molecules will then have to diffuse through mercury before coming in contact with the stream of the jet. So the level of water in J should always be sufficiently above the level of E.

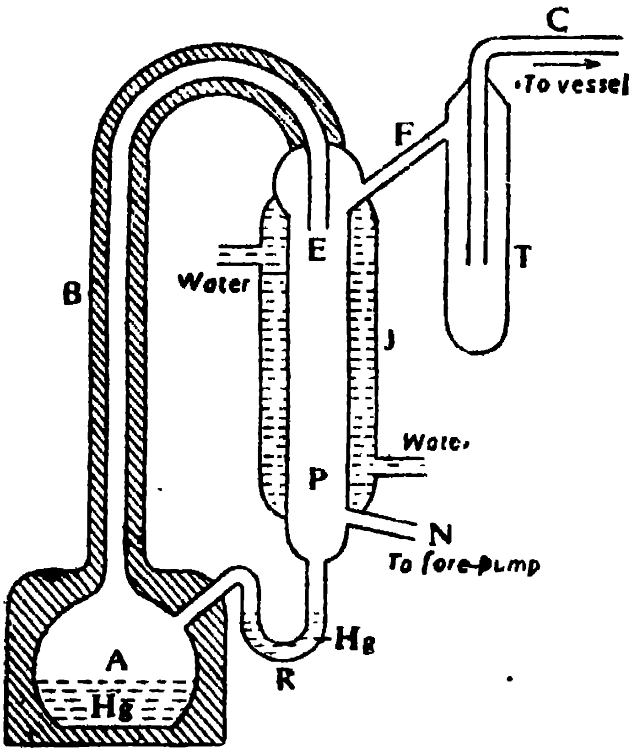


Fig. 237

These pumps can produce vacua of the order of 10^{-6} to 10^{-8} mm. of mercury depending on the design. Without the liquid-air trap T, the mercury atoms diffuse into the vessel to be evacuated and as such the pressure attained will not be so low.

The dimension of the connecting tubes are very important in determining the speed of exhaustion. To utilise the maximum speed of evacuation the connecting tubes should be short and wide.

Sometimes instead of mercury, organic liquids (oils) with high boiling points and low vapour pressures are used in these pumps.

226. Processes of further Reduction of Pressure.—To push the vacuum below 10^{-6}mm. of mercury several methods may be adopted.

(1) Sorption Process.—

The high vacuum system is connected to a tube surrounded by liquid air and containing freshly heated cocoanut charcoal. Traces of N_2, CO_2, NH_3 and H_2 may thus be readily removed and the pressure is further reduced.

(2) Flashing.—

When in a vessel connected to the high vacuum system a metal like Mg, Ca or Na is vaporised, oxygen in the system is removed due to formation of an oxide whose vapour pressure is negligible.

(3) Baking.—

This consists of heating the system to a temperature of a few hundred degrees while the pump is still operating. The occluded gases are removed by this baking process.

(4) Electrical Process.—

This process is adopted for obtaining the high order vacuum required in incandescent lamps, and thermionic valves. Here the residual gas molecules (or atoms) are ionised by either thermions or by a glow discharge and the ions thus formed are made to adhere to the walls of the vessel by suitably cooling the walls.

227. Manometers and Gauges :—For the measurement of low pressures various types of instruments have been devised. They may be broadly classified under the following heads :—

(1) Mercury Manometers.—

Under this heading we shall discuss the ordinary differential types of manometers and McLeod gauge. The former type may again be subdivided into two varieties : Open tube type and Closed tube type.

(a) Open Tube Manometer.—When the pressure to be measured is nearly atmospheric and differs from it only by a small amount, an open tube manometer is a convenient instrument.

The instrument consists of a glass U-tube *BUA* open at both ends containing a liquid of known density (Fig 238). The liquid used may be water, oil or mercury. If the liquid used is mercury the instrument may be used to measure pressures, say, from $\frac{1}{2}$ to $1\frac{1}{2}$ atmospheres. One end of the U-tube is connected to the vessel *V* in which the pressure is to be measured. The other end is open to the atmosphere. If the pressure of the gas in the closed vessel is the same as the atmospheric pressure, there will be no difference of levels, and in other cases, there will be a difference of levels *h* between the surfaces *A* and *B* of the liquid in the two limbs of the U-tube. If ρ be the density of liquid, the pressure of the enclosed gas in *V*=atmospheric pressure ± $h.\rho.g$. Thus, **the height 'h' of the liquid column in the open tube measured from top of the liquid column in the other limb gives a measure of the difference between the pressure of the enclosed gas and the pressure of the atmosphere.**

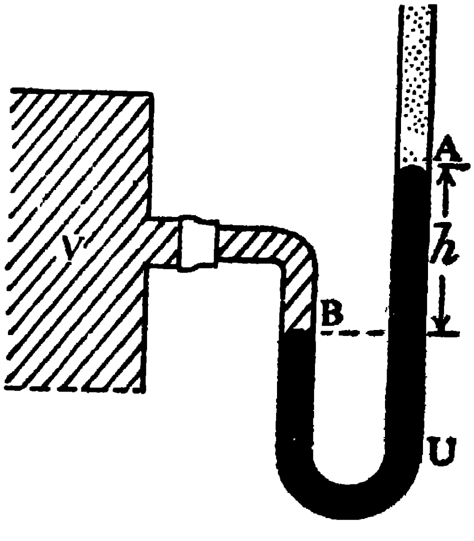


Fig. 238—An Open Tube Manometer

If the level in the open tube *A* is higher than the level in the connecting limb *B* as in Fig. 238, the plus sign is to be used and the pressure of the gas in *V* is greater than the atomspheric pressure ; and if the level in *A* is lower than that in *B*, the minus sign is to be used and the pressure of the gas is less than the atmospheric pressure.

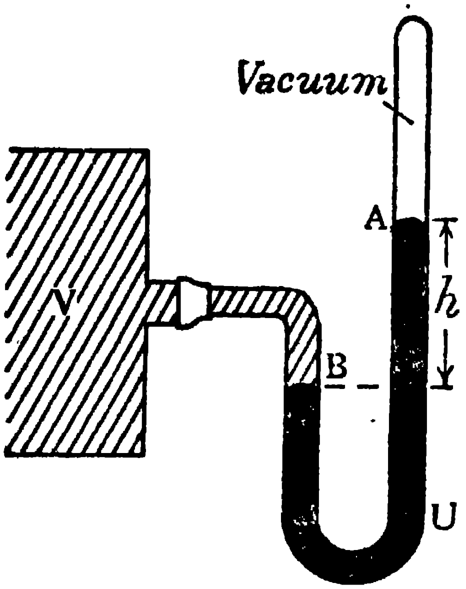


Fig. 239—A Manometer for low Pressures.

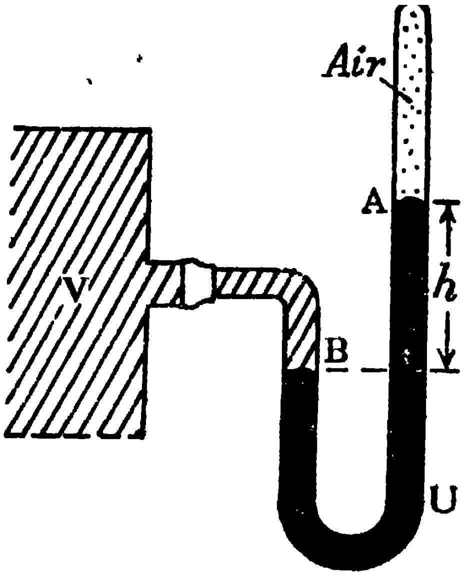


Fig 240—An Air-manometer.

(b) Closed Tube Manometer.—(*i*) When the pressure to be measured is only a small fraction of the atmospheric pressure, one limb of the U-tube is sealed (Fig. 239) after the space above the manometric liquid in that limb is evacuated. The atmospheric pressure being thus eliminated from that limb, the gas pressure *p* of the enclosed gas *V* will be indicated by the liquid column *h*, and will be given by, $p = h \times \rho \times g$.

(*ii*) For pressures, say above 1½ atmospheres and up to moderate pressures, a convenient instrument is an air-manometer (Fig. 240). This is a U-tube with the bend filled with some convenient liquid. The sealed-up limb *A* contains some dry air above the liquid surface. This air gets compressed as the end *B* is connected to the vessel *V* containing the gas under pressure. The sealed-up limb is graduated so that the pressure (which is inversely proportional to the volume of the air in (A) can be directly read off from it.

(c) **McLeod Gauge.**—For measuring the pressures of the order of the vacuum in electric lamps, X-ray tubes etc. (nearly 10^{-5} mm. of Hg) usually a McLeod gauge is used (Fig. 241). A tube $ABCD$ closed at the top, having a portion AB of small uniform bore and graduated in c.c. with a wider bulb BC and a branch at C leading to the tube EF of the same bore as AB (to compensate for the capillary effect) is connected to the low pressure supply. The lower end D is joined by a flexible rubber tubing to the mercury cistern G which can be raised or lowered vertically. In measuring the low pressure of a supply, the cistern G is lowered below C so that the space above C in the tubes EF and AC is filled with the gas at the same pressure as that of the supply. The cistern G is then gradually raised until mercury seals the gas in ABC. On further raising the cistern, let h be the difference of the mercury levels in the tubes AB and EF, the gas entrapped in ABC being compressed into AB. If p be the pressure of the supply, the pressure of the gas entrapped in AB is $(p+h)$. If V=volume of ABC and v=volume of the gas entrapped in AB, then from Boyle's law, $pV=(p+h)\,v$, assuming the process to be isothermal ;

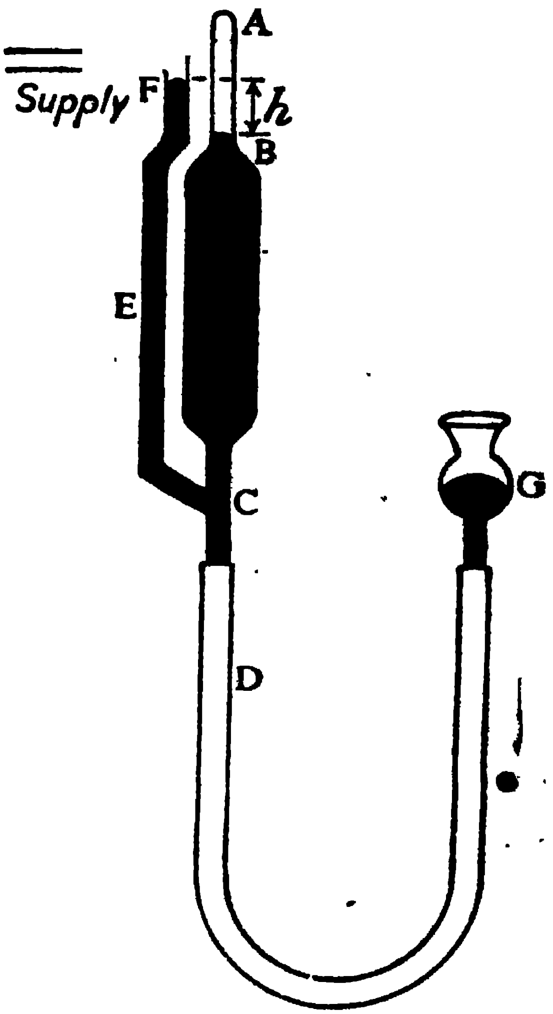


Fig. 241

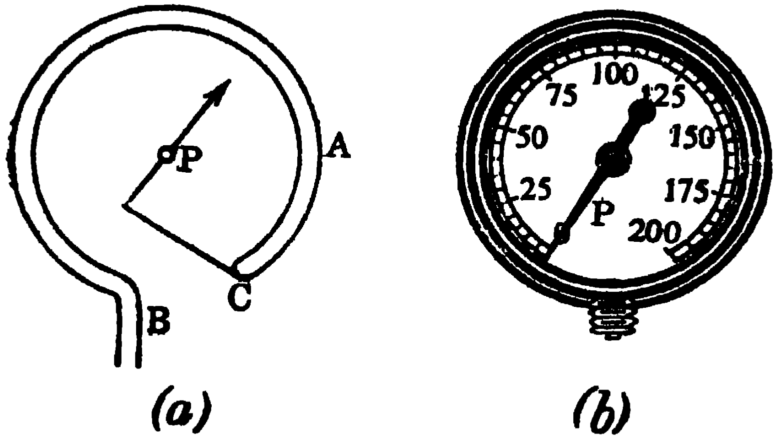
$$\text{or, } p(V-v)=hv \text{ ; or, } p=\frac{hv}{V-v}=\frac{hv}{V} \text{ nearly.}$$

($\because$ v is very small compared to V).

Thus knowing h, v and V, the pressure p can be determined.

(2) Mechanical Manometers.—

Under this class of manometers we shall describe only the Bourdon Gauge which is commonly used in all medium pressure measurements.



(a) (b)

Fig. 242– The Bourdon Gauge

The Bourdon Gauge.—When the pressures to be measured differ greatly from the atmospheric pressure, both above and below such as those in steam boilers, compressed air-reservoirs, condensers etc. or in evaporators, a Bourdon Gauge which is a metallic pressure gauge, is generally used. Fig. 242(*a*) represents how the instrument works, while Fig. 242(*b*) illustrates the appearance of the instrument.

The instrument consists simply of a hollow tube *BAC* open at *B* to be connected to the supply and closed at other end *C*. It is a spring tube usually made of a special quality of bronze or sometimes of brass and is of elliptical cross-section. The pressure of the supply changes the cross-section of the tube to more and more circular from until the stress-developed in the material of the tube is balanced by the pressure within, and due to this change in cross-section, the free end *C* of the tube is displaced with respect to *B*. As a result, the pointer *P* connected to it moves over a circular scale. The instrument is previously graduated by referring to a standard gauge. Since the manometer tube is surrounded by the atmospheric air the movement of *C* measures the excess or deficit of the internal pressure referred to the atmospheric pressure. To get the absolute pressure of a high pressure supply, the atmospheric pressure at the place of observation at the time of experiment is to be added to the pressure indicated by the gauge.

(3) Viscosity Manometers.—

Under this heading we can include *two types* of gauges :—

(a) Dushman's molecular gauge.—

Suitable for measurement of pressure within the range 10^{-3} mm. to 10^{-7} mm. of mercury.

(b) Quartz fibre gauge.—

Here the pressure is ascertained from the damped oscillation of the vibrating quartz fibre in the vacuum. As the residual gas will always have some damping effect on the vibrating fibre due to its viscosity, the period will be damped and as such the degree of damping will depend on the pressure of the gas. The range of this gauge is from 10^{-}[illegible] mm. to 10^{-5} mm. of mercury.

(4) Radiometer gauges.—

The most important type belonging to this class is ***Knudsen's Absolute gauge***, having a range of 10^{-4} mm. to 10^{-7} mm. of Hg. Here the pressure is ascertained by noting the deflection of a suspended cold plate in the vacuum due to the continuous impingement of the molecules rebounding from a neighbouring hot plate.

(5) Conductivity gauges.—

The most important of this class is the ***Pirani-Hale gauge.*** Here a wire is heated by an electric current. The resistance of the wire, which is a function of the temperature and hence also of the pressure of the gas, is measured. The wire used is generally of platinum or tungsten about 0·03 to 0·06 mm. in diameter and 10 to 20 cms. in length. The pressure range in a suitably chosen gauge is almost the same as in a McLeod gauge. The resistance wire is included in a Wheatstone bridge, the three other resistances of which are of negligible temperature coefficients (preferably of manganin). The connections of the Wheatstone's bridge should be as shown in Fig. 243, where V is a voltmeter, R an adjustable resistance and F is the tungsten or platinum filament enclosed in a glass bulb connected to the vacuum chamber. By adjusting the value of R, the deflection of the galvanometer is made zero, the current through the filament F being such that its temperature is nearly 100°C. The voltage V, across the bridge is so adjusted that the balance of the bridge is restored each time when the pressure in P is varied. If V be the voltage applied at a pressure p and V_0 be that at $p=0$ at the balance condition of the bridge then it is found that $\frac{V^2 - V_0^2}{V_0^2} = Kf(p) = Cp$ when pressure is below 0·1 mm., K and C being different constants. The constant C for air may be determined by calibrating the instrument against a McLeod gauge.

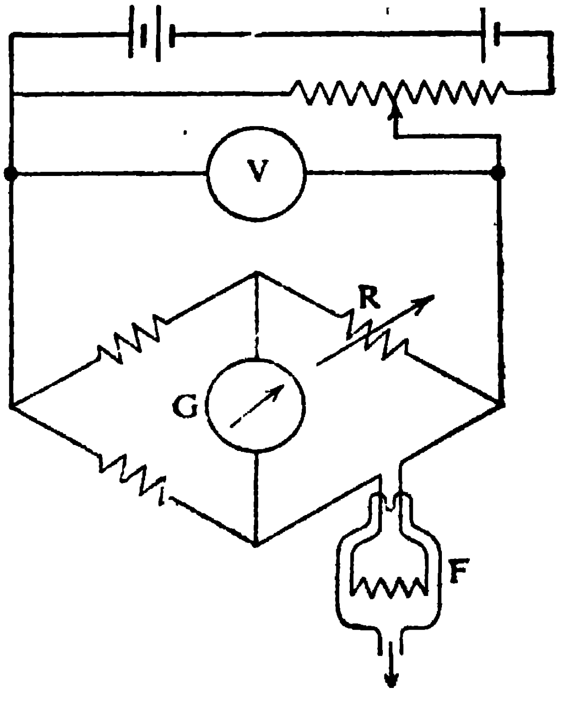


Fig. 243

(6) Ionisation Gauge.—

This type of gauge is similar in construction to a triode valve and can record pressures of the order of 10^{-2}mm. of Hg. Here a tungsten filament is surrounded by a special grid of tungsten and this system is mounted coaxially inside a cylinder of metal serving as the plate. The whole system is then placed inside a bulb whose degree of evacuation is to be ascertained. As usual the filament circuit is completed. The plate is given a negative potential and the grid maintained at a comparatively high positive potential. No electrons can evidently reach the plate but the positive ions rush towards it and the galvanometer attached to the plate records the plate current which is proportional to the number of ions reaching the plate. This number evidently depends on the pressure within the bulb because the greater the pressure the greater will be the secondary ionisation produced inside the bulb and as such greater will be the number of positive ions reaching the plate. This is the principle on which the ionisation gauges work.

228. A Complete arrangement for the production of high vacuum :—A simple but complete arrangement for the production of high vacuum is shown in Fig. 244.

In the figure,

(1) *V* is the vessel to the evacuated,

(2) *G* is a pressure gauge connected very close to *V*,

(3) *T* is a liquid-air trap immersed in liquid air contained in the Dewar flask D,

(4) *M* is a Langmuir type of mercury vapour pump,

(5) *m* is a manometer to record the fore-vacuum,

(6) *D. C.* is a drying chamber containing P_2O_5,

(7) *A* is the fore-pump.

(8) S_1 and S_2 are two stop-cocks of which S_2 connected to *V* must be of a large bore.

In making the above vacuum lines complete, drying agents, stopcock greases, waxes and thick rubber tubings may be used only in the fore-vacuum region. In the high-vacuum side substances with very low vapour pressures may be used and if possible there should be nothing but glass and metal. The glass connecting tubes on the high-vac. side should be short and of a large bore so as to secure the fullest advantage from a rapid pump. The connections should be perfectly leak-tight and the vessel to be evacuated should preferably be heated because the rate of outflow of gas is greater at higher temperatures. Traps, cutoffs and angles which are essential in the lines effectively decrease the rate of flow. So their number should be minimised.

The speed of a pump indicates the rapidity with which the gas is exhausted from a vessel. It is measured by the number of cubic centimetres of gas removed per second at the mean pressure. Mathematically it is given by the expression,

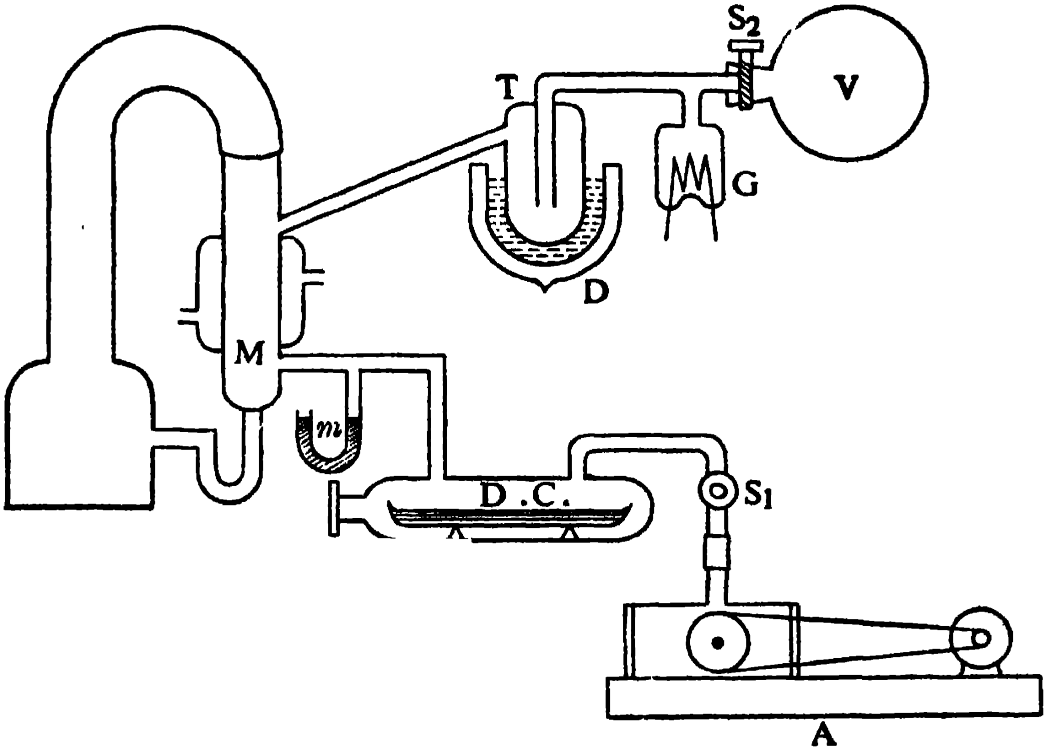


Fig. 244

$$S = 2{\cdot}303 \frac{V}{t} \log_{10} \frac{p_1}{p_2} \text{ c.c. per sec.,}$$

where V = volume in cm.^{3} exhausted from the pressure p_1 to the pressure p_2 in t secs.

Questions (Chapter XII)

1. Give an account, with a neat sketch, of some form of a gas pump, suitable for attainment of low pressures. (Punj. U. 1945)

2. Describe with a neat diagram, the construction and working of (*a*) any modern high vacuum pump and (*b*) a gauge ,which can measure the low pressure this pump produces. (Madr. U. 1949)

3. Describe the working of a rotary oil pump for producing low pressures. How are these pressures measured ? (O. U. 1966. Sp. ; All. U. 1959)

4. Describe the various types of gauges that are suitable for measurement of low pressures.

5. Describe a mercury pump of diffusion-condensation type. What are its advantages over the ordinary diffusion pumps ?

What is a McLeod gauge ? Explain its working. (*cf.* C. U. 1962, '68)

6. Write notes on :—

(*a*) Bourdon gauge ; (*b*) McLeod gauge : (*c*) Ionisation gauge ; (*d*) Gaede molecular pump.

7. Write an essay on the production and measurement of high vacuum. (C. U. H. 1967)

CHAPTER XIII

AERONAUTICS

THE ATMOSPHERE

229. **Aerodynamics and Aeronautics** :—*Aerodynamics* is a general name for that part of Physics which deals with the properties of *any* gaseous medium in motion. *Aeronautics* is a specialised branch of it which, in particular, deals with the behaviour of atmospheric air when an *aircraft* moves through it.

230. **Facts about the Atmosphere** :—Before proceeding further to study of principles on which the flight of an aircraft depends, the following facts about the atmosphere should be well remembered :—

(A) Extent of the Atmosphere and the Variations of Pressure and Temperature with Altitude.—An elaborate study of the composition of the atmosphere reveals that nitrogen, oxygen, argon, and small traces of some other gases are the constituents of air and the percentage of composition slightly varies from one place to another. As the atmosphere extends upwards, the density of the air diminishes. Opinions, however, vary as to how high the atmosphere reaches. Some estimate the height to be as great as 200 miles even. It has been described earlier how the temperature of the atmosphere falls as the height increases. Roughly speaking, in the lower belt of the atmosphere which is known as the *troposphere*, the temperature steadily falls at the rate of about 1°*F*. for every 300 ft. increase in height, and in the upper belt which is known as the *stratosphere*, the temperature is more or less steady at about—60°*F*. and does not alter with the increase of height.

The average pressure of the atmosphere at sea-level is about 14·7 lbs.-wt. per sq. inch, which changes from place to place and from day to day with changes of weather and temperature. The pressure decreases with increase of altitude. It has been estimated that about one-half of the total weight of the atmosphere is concentrated in the first 18,000 ft. It should be remembered, however that the pressure exerted by *air in motion* may be greater or less than the pressure exerted by *air* when *stationary*, according to the nature of its motion, and from these pressures the forces of *lift* and *drag* (discussed later) on an aircraft are obtained.

(B) Air Resistance.—Due to the fact that air has weight and that it is always subject to convection currents, air offers resistance to any body which moves through it, and this resistance, for a body of a given shape, and a given relative motion, depends upon to the properties of (*i*) viscosity, (*ii*) elasticity, and (*iii*) inertia of air.

(i) **Viscosity.**—It is an inherent property of all fluids. Due to its existence when any relative motion occurs between parts of a fluid, internal forces of frictional character are set up within the fluid which tend to retard the relative motion. This phenomenon clearly shows that the molecules of a fluid are mutually interlocked, the strength of the bonds of interlocking vary, however, from one fluid to another depending on the viscosity. So when a body moves through air (which is a kind of fluid) and the layers of air in contact with it are moved, they also cause layers next to them to move to some extent. The types of movements that are caused in the neighbouring layers depend on the shape of the moving body and the magnitude of its motion relative to the air. When this relative motion is large *eddies* or *vortices* are formed in the air around the body. It will be seen later that these eddies cause many phenomena connected with *flight*.

(ii) **Elasticity** :—The tendency of the particles to re-occupy former space from which they are disturbed is due to that property of air which is known as its *volume elasticity*. With increase of altitude when the pressure falls, the tendency of air to expand and thus to reduce in density arises out of this property.

(iii) **Inertia.**—It is a property common to all matter (arising out of mass or density) due to which air tends to be at rest or in steady motion, and resists any attempt to change such rest or motion.

(C) **Density.**—The density of the air depends on the atmospheric pressure. It is greatest at the sea-level and decreases with altitude. At sea-level the density of air is about 0·08 lb. per cu. ft. and at 20,000 ft. it is only 0·042 lb. per cu ft., which is about one-half of the first value. It is the density of air which makes all flight possible, as an aircraft is supported in the air by forces entirely dependent on the density ; the less is the density the less the weight lifted and more difficult does the flight become, and in vacuum any flight is impossible.

An idea about how the density of air decreases with increase of altitude will be obtained from the following table :—

Altitude	Density (lb./cu. ft.)	Altitude	Density (lb./cu. ft.)
Sea-level	0·0803	15,000 ft.	0·0503
1,000 ft.	0·0778	20,000 ,,	0·0426
5,000 ,,	0·0689	30,000 ,,	0·0298
10,000 ,,	0·0590	40,000 ,,	0·0197

(D) **Humidity.**—At the lower levels of the atmosphere water vapour is always present. The amount of it varies with the season and diminishes with the increase of altitude. Under identical conditions of temperature and pressure, the density of water vapour is only ***three-fifths*** of that of air and so the pressure of water vapour diminishes the pressure and density of the atmospheric air.

AIR RESISTANCE

231. Streamlines :—Whenever a body is moved through air (or any other fluid), or the fluid flows past a body, there is always produced a definite resistance to its motion. This resistance is usually termed drag in aeronautical work. The effect of this resistance in the viscous fluid is to set up displacements in the shape of eddies in the fluid.

In such cases two modes of flow take place : (*a*) ***turbulent flow*** and (*b*) ***streamline flow***. These types of flow have been described earlier in the text and it has been pointed out that the streamline flow degenerates into the turbulent flow when a certain relative velocity known as ***critical velocity***, is exceeded. So when a body moves with an excessive velocity through a viscous medium turbulent motion causing eddies and vortices results and the resistance to motion of the body increases, the magnitude of which depends also largely on the shape of the body. If a body is so shaped as to produce the least possible eddy motion so that the resistance to motion is also much reduced thereby, then it is said to have a ***streamline shape***, and the lines round the body interposed in the fluid showing the directions and shapes of the disturbances are called **streamlines**. These streamlines enable us to understand the nature of the flow of the fluid past the body.

As it is difficult to investigate the disturbances on an aircraft, while in actual flight, most of the aeronautical experiments for studying the phenomena of flight are carried out by scientists in the laboratory by using some form of **Wind Tunnel,**[*] in which air is made to flow past a model of aeroplane which remains at rest relative to the tunnel. The effect is the same as if the body were made to move through ***still*** air, because it is the ***relative motion*** of air to the aircraft or of the aircraft to air, which really matters in the investigation.

Air Speed and Ground Speed.—True ***air speed*** of an aircraft is the speed relative to air, that is, the speed with which it would travel in the absence of wind ; while ***ground speed*** means its speed relative to the earth, or the actual speed over the ground. For instance, if the normal speed (air speed) of an aircraft flying from A towards B be 100 m.p.h. while wind blowing at 40 m.p.h. from B towards A, the aircraft will reach B with an actual speed (ground speed) of 60 m.p.h.

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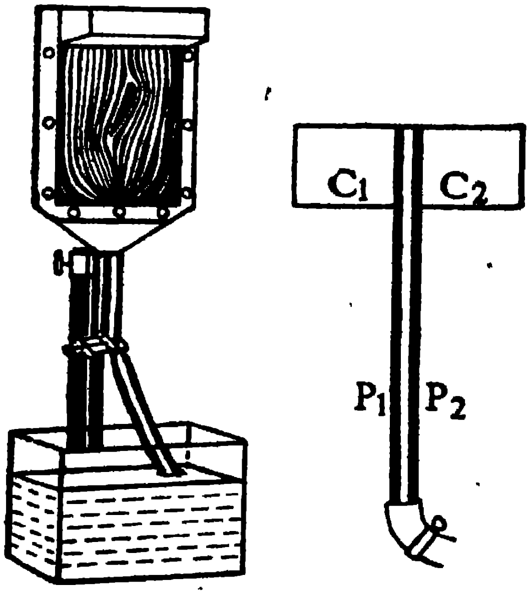
[*] A *wind tunnel* is nothing but a suitable chamber in which, say, a model of an aeroplane is kept and an artificial high speed air-current is produced across it by the action of air-screw. The temperature of this blast is also simultaneously kept very low by means of a refrigerating plant.

It is possible to study and photograph streamlines and eddy motions by introducing smoke into the air-flow in wind tunnels, or coloured jets into the Water Tank Experiment described below.

Water Tank Experiment.—The apparatus for demonstrating streamline flow of liquids consists of a rectangular reservoir at the top divided into two compartments C_1 and C_2 [Fig. 245 (*b*)] by two glass plates P_1 and P_2 separated by a distance of about 1 mm. These plates have equidistant perforations inside the reservoir, the perforations in P_1 being alternate to those in P_2. One of the compartments C_1 is filled up with clear water and the other C_2 with coloured water, say, water coloured with potassium permanganate. Now, the liquid flowing down between the plates, from both the compartments collects at the bottom and finally flows out through a rubber tube provided with a pinch-cock. On opening the pinch-cock clear water from C_1 and coloured water from C_2 will flow down between the plates through alternate perforations. The violet coloured tracks will show the parallel stream lines along which the water flows, and they finally curve inwards towards the end. Due to the colouring material the streamlines are made visible to an observer. The actual apparatus is shown in Fig. 245(*a*), where a thin body made of gutta-percha has been introduced in the stream between the plates to show the distortion of streamlines due to its shape. Similarly, small bodies of different shapes can be introduced to show how the streamlines are distorted in each case.



(*a*) (*b*)

Fig. 245—Water Tank Experiment

Effect of shape.—One great object of the designer of æroplanes is to reduce the eddy resistance to an absolute minimum, and much experimental work has been carried out with this point in view. Results show that the shape of a body has striking effect on the amount of drag produced, and that enormous advantage is gained by adopting a **streamline shape**,—the example of which in nature is the outline of a fish. When air flows past a perfectly streamlined body, no eddies are created in its neighbourhood.

Fig. 246 shows some of the streamlines flowing past a few bodies of different shapes. It will be noticed in Fig. 246(*a*) that, in the case of a flat plate the airflow breaks up after passing the edge of the plate into a series of eddies and vortices, the size and nature of which will also be influenced by both the velocity of the airflow and the linear dimension of the plate. It will also depend on the inclination of the plate to the direction of airflow. Fig. 246(*b*) shows that owing to its position both sides are affected by the air-current. Streamlines at the bottom are deflected downwards and eddies are formed at the lower edge, whilst on the top there are similar eddies and also regions of lower pressure due to the distortion of the straight line motion of the air current. When, however, the obstacle has got a suitably curved shape as in Fig. 246(*c*), the air or fluid passes over and behind the body in unbroken smooth lines, termed *streamlines*, and the obstacle giving rise to a definite streamline pattern is usually called a ***streamline body***.

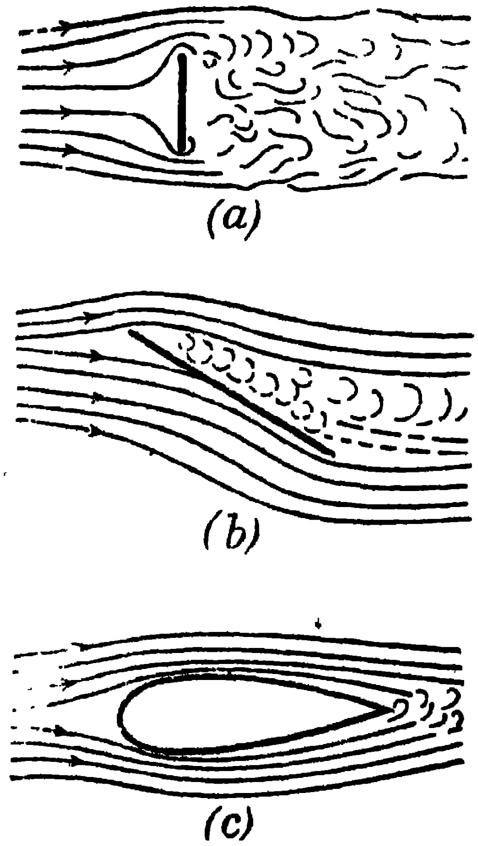


Fig. 246 – Effect of shape.

On comparing the flow past a rough obstacle with that past a streamline body, we notice that in the former case large portions of the fluid spin around as if they were detached portions of the fluid. These isolated portions of the fluid are called ***eddies***. A ball thrown in air and moving with spin will require more energy than when it is moving without spin. An eddy differs from a fluid moving in a streamlined manner in the same way as a ball moving with or without spin in air. For an aeroplane having a rough shape, the energy of the spinning fluid of the eddies must ultimately be derived from the engine, and so, such bodies will tend to slow down the motion and produce inefficient flight. ***Streamline shapes are***, therefore, ***necessary for the efficiency of the aircraft.***

232. Resistance to Motion :—The above experiments show that the resistance offered to a body in *relative motion* with a fluid is made up of two parts : (*a*) *Eddy Resistance* ; (*b*) *Skin Friction.*

(**a**) **Eddy Resistance.**— This is the portion of the resistance due to eddies formed when a viscous fluid flows past a solid body and we no longer get a smooth streamline flow. The extreme example of this type of resistance is the case of a flat plate held at right angles to the fluid or wind [Fig. 246(*a*)]. The resistance in this case is very large and the pressure in front of the plate is greater, and that behind the plate is less, than that of the atmosphere which causes a kind of '*sucking effect*' on the plate. The space immediately behind the plate is not traversed by streams of air and is called a *dead space.*

(**b**) **Skin Friction.**—This is the resistance due to friction between the surface of a body and the layer of air next to it. Also, due to the viscosity of the air, layers near the surface will tend to retard these farther away.

233. Resistance Formula :—It can be proved and verified by experiment that when a body passes through air, the resistance R on it depends within certain limitations, on the following factors—

(*i*) The shape of the body ; (*ii*) the surface ; (*iii*) frontal area of the body exposed (in sq. ft.) ; (*iv*) the square of the velocity V of air-flow (in ft. per sec.) ; (*v*) the density ρ of the air (in lb. per cu. ft.) ; (*vi*) acceleration due to gravity g (in ft. per sec^2.).

Thus, we have resistance formula, $\mathbf{R}=\dfrac{\mathbf{K\rho AV^2}}{\mathbf{g}}$ lb.-wt., where K is a constant depending on the shape of the body, the value of which for a flat *plate* is 0·6 and that for a streamline is 0·03.

234. Bernouilli's Theorem :—Before proceeding further we should consider here a theorem, known as *Bernouilli's theorem which states* that *in the flow of an ideal* (*i.e.* not viscous) ***fluid the sum of the potential energy, kinetic energy, and the pressure energy is a constant.*** This theorem can be roughly verified by an experiment with the Venturi-tube illustrated in Fig. 247 (for details see chapter on 'Viscosity').

Venturi-Tube Experiment.—A metal tube *AB* diverging at the two ends and having a narrow neck in the middle is connected to a horizontal glass tube *TT*, containing a coloured liquid by means of a number of narrow manometers (*h*). When, the reservoir *R* is half-filled with the coloured liquid, the liquid stands at the same level in all the tubes. This happens at normal pressure over all the tubes. When, however, an air-stream from a wind tunnel passes through the Venturi-tube, the speed of the air in *AB* will change from place to place due to which the liquid level in several of the inner manometers will rise. The greatest increase in height will be observed in the narrowest part of the tube, where the speed of the air is also greatest. But as the liquid level rises in the manometers due to reduction of pressure, we have this somewhat unexpected fact that ***the pressure of the air falls when its speed increases*** (vide chapter on 'Viscosity').

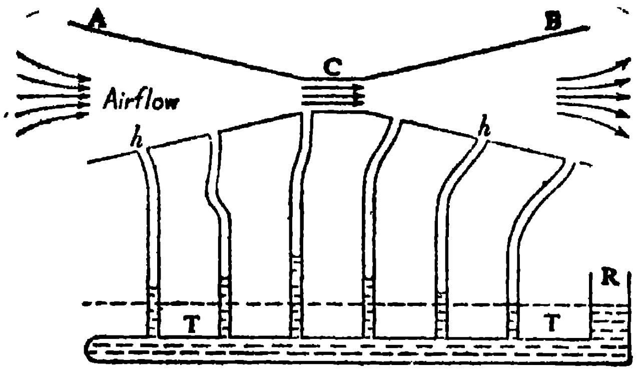


Fig. 247—Venturi-Tube.

As the change of potential enegry is negligible, the increase of speed (and hence of kinetic energy) is obtained by losing some of its pressure energy. Hence it illustrates the Bernouilli's theorem stated above. This Venturi-effect, as it is called, is employed in many scientific devices in order to produce a reduced pressure.

AEROFOILS (OR WINGS): FLAT AND CAMBERED SURFACES : LIFT AND DRAG

235. Principles of Flight :—Let us proceed now to consider the question of why it is that an aeroplane is capable of flying through air. In order that an heavier-than-air machine can fly, there must be some means of forcing the air downwards so as to provide the equal and opposite reaction which will lift the weight of the machine, and in the conventional aeroplane this is provided for by wings, which are inclined at a small angle to the direction of motion. The necessary force driving the machine forward is obtained by the thrust of an airscrew. The wings (or aerofoils) are always slightly curved, but let us consider the case of a flat plate first, as in the original attempts of flight flat surfaces were used.

236. Flat Plate inclined to Air Current :—For simplicity we suppose that a flat plate *AB* is at rest and that the air-current flows past the plate AB which is inclined at an angle α to the direction of the airflow (Fig. 248). In Fig. 246(b), it has been found that in this position both sides of the plate are affected by the air-current, due to which pressure of air on the top surface is decreased while that underneath the plate is increased. Each of these pressure changes produces forces R_1 and R_2 acting upwards on the plate giving rise to a resultant force R, which is practically normal to the surface when the angle α is small. The force R_1 arising from the decrease of pressure pulls the plate up, and the force R_2 arising from the increase of pressure pushes the plate up (*vide* Venturi-Tube expt.). The force R, called the **total reaction**, can be resolved into two components at right angles—one horizontal, D, and the other vertical L, acting upwards. The component L, called the *lift*, balances the weight of the plate, and the component D, called the *drag*, resists the motion through the air. Obviously, the L component which supplies the lifting force to the plane is of profound importance. For equilibrium the L component must equalise the weight W of the plate. If W is greater than L, the plate will fall, and if less, it will rise.

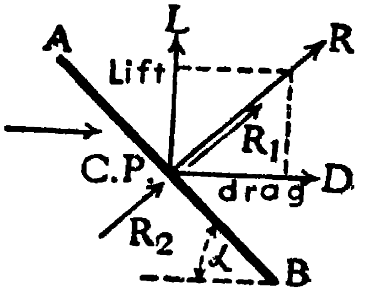


Fig. 248

Actually in practice the flat surface is inefficient as a lifting device because of the total resistance offered, and therefore the total engine power which has to be employed, is very high in comparison with the lift obtained from it.

237. Aeroplane and kite :—The flight of an aeroplane is much like that of a kite floating in air. In the case of the aeroplane the rush of air past the wings is due to the motion of the aeroplane itself through the air rather than to a wind, which supplies the rush of air in the case of a kite. The tension of the kite string here corresponds to the forward thrust of the propeller. The L component balances the weight of the machine while for equilibrium, the D component must be counter-balanced by an equal force which is obtained by the action of the screw-propeller. On increasing the propeller speed, the forward thrust and R increase. Consequently, the L component becomes greater than the weight and so the aeroplane rises. It should be noted that the air-pressure depends only on the rate and direction with which the air and the body meet, and the result is the same whether the body moves to meet the air, or the body remains still and the air flows against it. Obviously, the greater the velocity with which the aeroplane and air meet the greater will be the air-pressure.

238. Cambered Surface :—The advantage of using a suitably curved (or *cambered* as it is termed) surface, instead of a flat one, was soon discovered. By this device a much greater lift, especially when compared with the drag, could be produced. In this, the eddy disturbances due to the distortion of the streamlines can be diminished and so the efficiency of the system can be increased. Thus the modern aerofoil has both the top and bottom surfaces cambered. The top camber is greater than that of the bottom surfaces as due to this the lift component has an appreciably higher value over a wider range of the angle of incidence. The additional advantage of the curved surface is that it automatically provides a certain amount of thickness which is necessary for the structural strength. The thickness is expressed as a percentage of the chord and for general use the best top surface camber is about 11 per cent. of the chord, while for high speed it should be only 7 or 8 per cent.

AIRFLOW AND PRESSURE OVER AEROFOIL

239. **Some Definitions** :—A transverse section of a wing (or aerofoil, as it is called) of an aircraft is shown in Fig. 249, where along the front of the aerofoil at *A* is the **leading edge** and the rear at *B* is the **trailing edge**.

The line *AB* joining the centres of curvature of the leading and trailing edges is called the **Chord**.

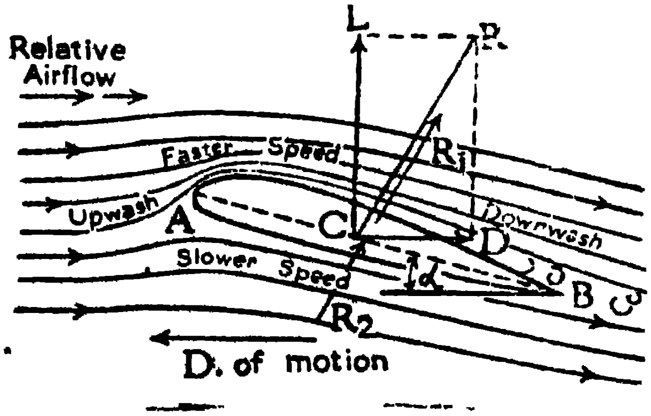


Fig 249—Airflow over Aerofoil.

Camber is the curvature of the aerofoil of both the top and bottom surfaces. The greatest height of the top or the bottom surface when divided by the chord length, is called the *camber* of the respective surface. Camber decides thickness of the aerofoil.

Angle of Attack is the angle between the chord line and relative airflow which is the direction of the airflow with reference to the aerofoil.

[**N.B.**—The angle of attack is often referred to as the angle of **incidence**, but it is better not to use this term in order to avoid confusion with the *Riger's angle of incidence* which is the angle between the chord line and some fixed horizontal data lines in the aeroplane. For a given aeroplane this angle is fixed, whereas the angle of attack may alter during flight.]

The total length of the aerofoil perpendicular to the section is called the **span**; and the ratio of the span to the chord is called the **aspect ratio.**

240. Airflow past an Aerofoil:—Experiments show the following results when a typical aerofoil moves through air at a small angle of attack (*vide* Fig. 249)—(*a*) A slight upward deflection, called *upwash*, occurs in front. (*b*) A considerable downward deflection called *downwash*, occurs behind the wing. The downwash is important as it affects the direction of the air striking the tail plane or other parts of the aeroplane in the rear of the main plane. (*c*) A smooth streamline airflow takes place over the top and the bottom surfaces. (*d*) The streamlines are closer above the top surface than over the bottom. (*e*) Above the top surface the speed of airflow is increased and below the bottom surface it is decreased. (*f*) The pressure of the air above the wing is reduced below the normal atmospheric pressure due to the increased speed of the airflow. (*g*) The pressure below the wing is increased due to the decreased speed.

Though the facts stated in (*f*) and (*g*) appear to be puzzling at first, it can be explained by the Venturi-tube experiment. Here the upper surface is somewhat similar in shape to the lower half of the Venturi-tube and the closer streamlines above the highest part of the camber resemble those passing through the neck of the Venturi-tube.

As stated in the case of a flat plate the decrease of pressure above the wing surface produces a force R_1 which is an important part of the total force pulling the wing up and the increase of pressure below the wing gives rise to the force R_2 pushing the wing up. These two upward forces gives us the resultant force R acting approximately at right angles to the chord line. But the decrease of pressure above the wing surface is more important, for to this is due the greater part of the lift force. Roughly about two-thirds of the total load on the wing may be attributed to this decrease of pressure while about one-third may account for the increase of pressure on the lower surface.

[**Note**—It should be noted here that the common idea is that the airflow moving away from the upper surface of the wing causes a partial vacuum and thus provides a lift force, but this is wrong. In fact, the greater will be the increase of speed as the air is drawn closer on to the upper surface of the wing, and by the consequent reduction of pressure the upward force produced will be greater.]

241. The Centre of Pressure:—The point in the chord line through which the total force R may be considered to act is known as the **centre of pressure.** It has no fixed position but varies according to the speed and the angle of attack.

(a) Distribution of Pressure over an Aerofoil.—The distribution of pressure over the surface of an aerofoil has been experimentally determined, and its study is of great importance. The method consists in distributing a number of glass tubes, which are placed parallel to the direction of motion, over the upper and lower surfaces of the aerofoil. These are connected to a manometer, and different pressures are ascertained. Fig. 250 shows the pressure distribution over an aerofoil at an angle of 5°, from which the following observations are made :—(*a*) The pressure is not evenly distributed – both the decreased pressure on the top surface and the increased pressure on the lower surface being most marked over the front portion of the aerofoil ; (*b*) the greatest pressure-decrease (and the largest forces) occur on the top surface, and it is near the leading edge and over the highest part of the camber ; (*c*) the decrease in pressure over the top surface is greater than increase on the lower surface.

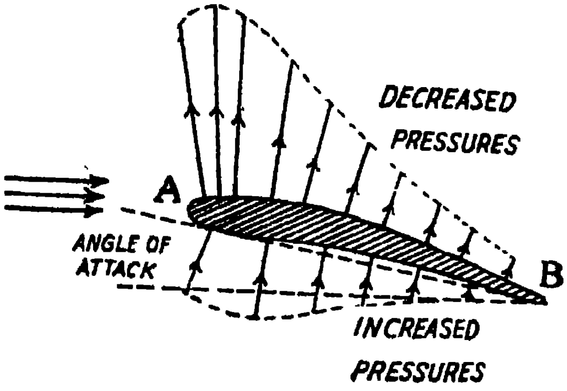


Fig. 250—Pressure Distribution over an Aerofoil

From this it is seen that the shape of the top surface is of great importance. It is the top surface, which by means of its decreased pressures, provides the greater part of the lift, and, at some angles of attack, this decrease of pressure on the top surface gives us as much as four-fifths of the lift.

(b) Movement of Centre of Pressure.—Experiments show that the distribution of pressure over the aerofoil changes considerably with the change of the angle of attack and consequently the centre of pressure C.P. moves. The position of C.P. is usually defined as a certain proportion of chord behind the leading edge. The movement of C.P. is an inconvenient property of the aerofoil, for unless its centre of gravity C.G. and C.P. coincide, there will be a turning effect about C.G. To understand this let us suppose that for a certain angle of attack the C.G. and C.P. coincide. Now when the angle of attack increases there will be a forward movement of C.P. and so there will be a turning moment about C.G. equal to Rx, where R is the total wind thrust and x the distance between C.P. and C.G. This moment will rotate the aerofoil and will still further increase the angle of attack and thus the equilibrium will be disturbed.

In any case, large movements of C.P. will make the aeroplane difficult to control and so in a good aeroplane the movement of C.P.

should be limited which is obtained by a suitable bi-convex cross-section or by increasing the aspect-ratio, for example, by tapering the wing.

242. Lift and Drag :—In practice the direction of motion of an aeroplane is not always horizontal. So it is usual to split up the *total reaction R* into two components L and D *relative to the airflow*, the component L which is always perpendicular to the direction of the airflow (or motion) is called **lift**, and the component D parallel to the direction of the airflow is called **drag**, which is always opposite to the direction of motion. Lift is used to balance the weight of the aeroplane and keep it in the air in level flight. Other parts of the aeroplane as tailplane, elevator, etc., may provide further lift forces when desired. *Drag is the enemy of flight* and every effort must be made to reduce it to a minimum. Only in normal level flight the lift is vertical and the drag horizontal, but if, in turning, the wings of an aeroplane assume a nearly vertical position, then the lift L is nearly horizontal. Lift is always perpendicular to the direction of motion and drag is always opposite to it.

243. Lift and Drag Formulae :—In Fig. 249, R is the resultant force on a transverse section of the wing of an aircraft whose angle of attack is α and whose velocity is V. We have already seen in Art. 233 that the total reaction (or resistance) $R=\frac{K\rho AV^2}{g}$ lb.-wt.

We have in Fig. 249, the lift component $L=R\cos\alpha$, and the drag $D=R\sin\alpha$, whence

$$L=K\cos\alpha\,\frac{\rho AV^2}{g}\text{ lb.-wt}\ldots(1),\quad\text{and } D=K\sin\alpha\,\frac{\rho AV^2}{g}\text{ lb.-wt}\ldots(2)$$

where ρ represents the air density (in lb. per cu. ft.), A the surface or plane area of the wing projected on the plane of the chord (in sq. ft.) V the velocity of air speed (in ft. per sec.) and g the acceleration of gravity ($=32{\cdot}2$ ft. per sec.^{2}).

Since K is a constant for some given conditions in a machine, we may write the symbol K_L for $2K\cos\alpha$ and K_D for $2K\sin\alpha$, which are spoken of as **lift and drag coefficients** respectively.

[**Note.**—(*a*) These symbols should not be confused with L and D which give the actual lift and drag in pounds-weight whereas K_L and K_D are constants only. (*b*) The above relations are strictly true when α is small, for we are not justified in assuming that R is at right angles to AB for large angles of attack.]

Then, we have, $L=K_L\frac{\rho AV^2}{2g}$; and $D=K_D\frac{\rho AV^2}{2g}$, and hence, dividing one by the other, we get the important relation, $\frac{L}{D}=\frac{K_L}{K_D}$; and L/D is known as the **lift-drag ratio.**

Note that when L is exactly equal to the weight W of the aerofoil we get, for a normal horizontal flight, $W=\frac{K_L \rho A V^2}{2g}$.

244. Factors affecting the Lift-Drag Ratio :—The factors affecting the *Lift-Drag* ratio are—

(*i*) *The angle of attack.*—We get maximum *Lift-Drag* ratio at an angle of attack of about 4° (Fig. 251).

(*ii*) *The air speed.*—Both lift and drag are directly proportional to the square of air speed. Hence increase in air speed will increase the lift and drag, other factors remaining the same.

(*iii*) *Increase in wing surface or plane area* (*i.e.* the area projected on the plane of the chord).—This will increase the lift and drag when the plane is flying at the same speed and the same angle of attack in air of the same density. (In practice, however, the angle of attack rarely remains constant even for a very short time.)

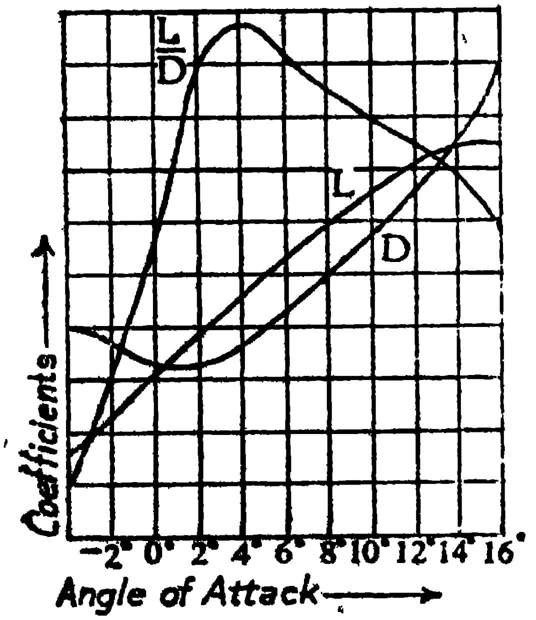


Fig. 251—Lift and Drag Curves.

(*iv*) *Increase in density of the air.*—V and $\angle$ remaining the same, the increase in density will increase and the decrease in the density will decrease the lift and drag.

245. Lift and Drag Curves :—In order to get some idea of what happens when the angles of attack of a typical aeroplane wing is gradually altered, we shall consider the lift and drag curves shown in Fig. 251. Considering the curve drawn with the lift co-efficient and the angle of attack, it will be seen that there is a definite lift at 0°, and that the lift increases steadily between 0° and 12°, where the graph is practically a straight line. The maximum value reaches at about 15° after which the lift begins to decrease rapidly. This rapid falling off is called *stalling* and the angle of attack at which the lift reaches a maximum value is known as the **stalling angle.**

Now for the *Drag Curve* we find that its value is always positive. It is least at about 0°. The increase of drag up to the stalling angle is not very rapid, but after it the increase becomes more and more rapid.

246. Lift-Drag Ratio Curve :—We have considered lift and drag separately, but it should be realised that the ratio *L/D* under varying conditions is of great importance. We know that an æroplane travelling through the air must employ power to create a propeller thrust in order to overcome the drag of the ærofoils and so it is desirable to require as little power as possible for a given lift or, in other words, for the sake of efficiency we want as much lift, but as little drag, as possible for our ærofoil. In fact, we want the highest possible values for the *L/D* ratio for any given working range. From Fig. 251 we find that the lift is highest at about 15° and the least drag we get at about 0° ; so at neither of these angles we really get the **best lift-drag ratio.** This shows the importance of the curve showing *L/D* ratio of the ærofoil against the angle of attack. Here we find that the greatest value of *L/D* occurs at about 3° or 4° at which angles the lift is about 20 times the drag. Thus, it is seen that an ideal ærofoil must be moving at an angle of attack of about 3° or 4° when it will give its best all-round result. This angle at which the best result is obtained is sometimes called the **optimum angle.**

[**Note**.—In Fig. 251, the values of lift and drag co-efficients are taken instead of the total lift and drag as the former will be practically independent of the air-density, the scale of the ærofoil, and the velocity employed, whereas the total lift and drag will depend on the actual conditions at the time of the experiment.]

247. Stalling :—At values greater than that corresponding to the maximum lift, the lift falls off rapidly and this rapid falling off is called **stalling,** when the æroplane is said to be stalled. Stalling is accompanied by a loss of lift as well as much increase in drag. The airflow no longer shows a smooth streamline flow and it finally changes into a turbulent flow It is extremely dangerous if stalling happens at the time when the æroplane is very near the ground.

One of the devices in reducing the risk of stalling is the **Handley Page slot,** which is shaped rather like a wing and fitted on the leading edge of the main wing. On moving forward the slot at a time when the angle of attack of the ærofoil is increased a smaller angle of attack is presented to the on-coming air causing an increased air-flow over the wing surface, and the lift is restored.

248. Aerofoil Characteristics :—The lift and drag co-efficients of an ærofoil depends on the shape of the ærofoil and they will change with changes in the angle of attack. The result of experiments on ærofoils can be easily demonstrated by drawing graphs to show how

L co-efficient, *D* co-efficient, and *L/D* ratio alter with changes in the angle of attack. These three may be called the *characteristics of the aerofoil.*

249. The Ideal Aerofoil :—The characteristics of the ideal ærofoil are given by the curves in Fig. 251. Thus the ideal ærofoil should have,

(1) *A high maximum lift co-efficient* in order to lower the landing speed for the safety of the plane. The higher the lift co-efficient of the æroplane, the lower will be its landing speed and greater will be the safety of the plane.

(2) *A low minimum drag co-efficient*—not only at a certain angle of attack, but it should remain low over a larger range of angles. Thus the æroplane will have a low resistance and will be able to attain high speed.

(3) *A high Lift-Drag ratio* for the sake of efficiency, good weight carrying capacity for a small expenditure of engine power and so less expense.

(4) *A small movement of centre of pressure to* improve stability.

(a) Compromises.—In actual practice, however, we find that no ærofoil will meet all the requirements. Therefore some sort of compromise is made just as in the case of a good physical balance. We cannot get an æroplane which will serve all our different purposes and the shape of the æroplane is the first, and perhaps the greatest compromise to be made. So different degrees of cambering is made according to the different purposes the æroplane is desired to serve. For instance, for high speed the *top surface* camber should be about 7 or 8 per cent of the chord while for general use it should be about 10 or 11 per cent of the chord.

Both lift and drag are increased by increasing the camber of the upper surface. The alterations in the camber of the *bottom surface* of the ærofoil have a much smaller effect. Modern ærofoils have their lower surface flat or slightly convex.

250. Normal Horizontal Flight :—Without taking into account the forces on the tail unit, an æroplane, when flying straight and level—which we refer as normal horizontal flight—may be said to be under the influence of the four main forces :

(1) The lift *L* of the main planes acting vertically upwards through the centre of pressure.

(2) The weight *W* of the æroplane acting vertically downwards through the centre of gravity.

(3) The thrust *T* of the propeller airscrew pulling horizontally forward along the propeller shaft.

(4) The drag *D* acting horizontally backward. This is the tota drag on the aircraft consisting of the drag of ærofoils and also of the remaining parts of the æroplane.

251. Conditions of Equilibrium :—In the ideal case when the aeroplane is flying level at a steady speed in a fixed direction, the main condition of equilibrium of those four forces, which must obey the simple laws of mechanics, is that all the forces would act through the same point. Then we have,

(*i*) $L=W$. This condition will keep the aeroplane at a constant height. If $L>W$ (this is secured by increasing air-speed by increasing engine power), the aeroplane will ascend, and if $L<W$, the aeroplane will descend.

(*ii*) $T=D$. This condition will keep the aeroplane moving at a constant speed. If $T>D$ the aeroplane will move with an acceleration and if $T<D$ there will be a retardation. In practice, however, these forces are never constant owing to varying conditions, *e.g.* the weight of the aircraft does not remain constant in value, since L is not constant as the angle of attack may change due to wing-thrust, the position of C.G. is not constant. Due to these difficulties the ideal arrangement of the forces is not possible.

Now when the size and position of forces change, the turning effect of the aircraft is controlled by the pilot by (*i*) control column movement (discussed later on) and (*ii*) mainly by adjustable tail plane.

AEROPLANES AND THEIR CONTROLS : MANOEUVRES

252. Component Parts of the Aeroplane :—We have already mentioned some parts of an aeroplane and especially have dealt with one of its main parts, *i.e.* the wings or aerofoils. An aeroplane mainly consists of the following parts (Fig. 252) :—

(*a*) *Fuselage* ; (*b*) *Wings or aerofoils* ; (*c*) *Propeller or Airscrew* ; (*d*) *Tail plane* ; (*e*) *Aileron* ; (*f*) *Elevator* ; (*g*) *Rudder and Fin.*

The Fuselage.—The main body of the machine is referred to as the fuselage, which must be large enough to contain engine, tanks, pilot, bombs, goods, passengers, etc. that the machine has to carry.

Tail Plane.—It is a small plane fitted at a considerable distance behind the main plane in order to provide the upward or downward forces necessary to counteract the unruly action of the four main forces mentioned in Art. 250.

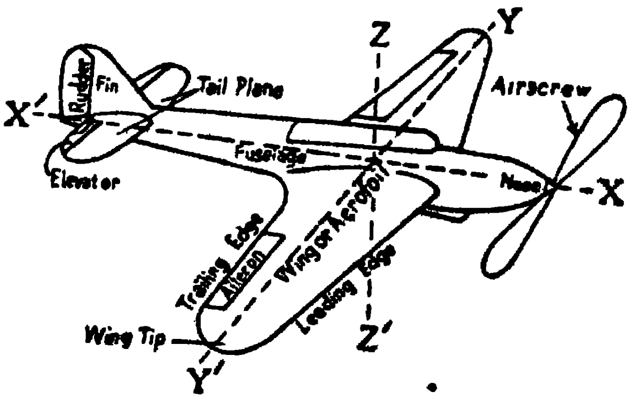


Fig. 252—Aeroplane.

253. The Propeller or Airscrew :—The theory of airscrew is too advanced to be considered here, but a general idea of the work of an airscrew will be given here.

A propeller, also called an airscrew, is much like an ordinary electric fan in appearance, but while a fan sucks air from behind and throws it forward, an airscrew sucks air from the front and throws it backward. The result is that due to reaction the fan tends to move backwards, while the airscrew is thrust forward, and thus pulls the aeroplane along with it. The thrust of a propeller is the force with which it drives the air backwards or urges the aeroplane forwards. The propeller is the means by which the power of the engine, which rotates it, is transformed into a forward thrust, and thus gives the aeroplane a translational velocity. Thus, the aeroplane forces its way through the air by means of propellers rotating in a vertical plane, and we may say in effect that an airscrew screws itself through the air pushing or pulling the aeroplane to which it is attached. The propellers are situated either in front of the body of the machine, when it will cause tension in the airscrew shaft and will thus pull the aeroplane forward (in which case the aeroplane is called a **tractor**) or in the rear of the body when it will push the plane forward (in which case it is called a **pusher**). Airscrews vary in the number of blades from two to four, but the two-bladed variety is easiest to manufacture and slightly more efficient. The shape of each part of an airscrew blade, taken in a direction at right angles to its length is found to be similar to that of an aerofoil.

The diagram (*A*, *B*, *C*, in Fig. 253) shows several cross-sections taken at various distance from the centre. The airscrew also derives, from the airflow, forces similar to those giving lift and drag in the case of wings but owing to variations in camber, chord and speed, the lift and drag components increase and decrease from section to section. The airscrew may be considered to be exactly like an aeroplane wings, but instead of moving in straight line and supporting the aeroplane, the airscrew moves in a spiral path and produces the thrust which overcomes the drag of the aeroplane. Due to their different functions the plane form of an airscrew blade differs from that of a wing ; and the airscrew blade is twisted so that the angle to the shaft of the propeller is greater at the base than at the tip, while the angle of the wing is almost the same throughout. Thus the forward thrust of the

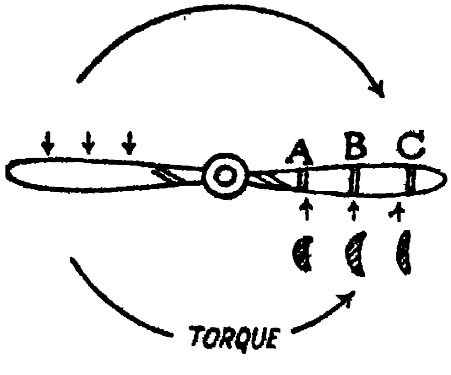


Fig. 253—Airscrew-Torque

airscrew corresponds to the upward lift of the aerofoil, and drag in this case is represented by the resistance of the air to the rotatory motion of the airscrew.

The total airscrew thrust is the sum of the thrusts on each blade section and it is the force which pulls the aeroplane through the air. The total drag on all the blade sections constitutes a couple—known as **airscrew torque**—which resists the rotatory motion of the airscrew and opposes the **engine torque** (or *the turning moment*) applied to the airscrew shaft by the engine. The airscrew torque has to be overcome by the engine torque. This is analogous to the thrust and drag in the case of an aeroplane.

(a) **Pitch.**—The airscrew is a screw which screws its way through the air in the same way an ordinary screw does through wood but some important differences are to be noted. In the case of an ordinary screw the distance moved forward in one revolution is a fixed quantity and is called the *pitch* of the screw, the value of which depends on its geometric dimensions, and is usually called the *geometric pitch*. But, in the case of the airscrew, the distance moved forward in one revolution (called the *advance per revolution*) is not a fixed quantity as it depends entirely on the forward speed of the aeroplane. Another important difference between the airscrew and the ordinary screw is that the airscrew has no actual grip on the air comparable to an ordinary screw in wood, and there is a certain amount of **slip** so that the distance moved forward is less than the geometric pitch This distance is not also constant as it varies with the speed of the aeroplane. Thus the *slip* of a screw is the difference between the distances it should travel theoretically and its actual progress.

(b) **Pitch Angle.**—We should all know that the twisted appearance of the airscrew blades is not without any meaning—rather it is the product of highly skilful design. The sections of the blade near the tip are moving with a much greater velocity than those near the root, and so most of the thrust is produced by the portions near the tip. For this reason the *pitch* (or *blade*) *angle* is not the same throughout the airscrew blade in order that every part of the airscrew may move the same distance forward during one revolution of it. Other things being equal, a large propeller moving comparatively slowly gives more thrust than a small one driven at high speed. The *pitch* (or *blade*) *angle* is the angle which the chord of any given blade section makes with horizontal plane when the airscrew is laid flat on this plane, its axis being vertical.

The *Experimental Mean Pitch* is the distance the airscrew moves forward in one revolution when the thrust is zero. When the thrust and efficiency of the airscrew is maximum, the pitch is called the ***Effective Pitch***.

(c) **Efficiency.**—The efficiency of an airscrew is the ratio of the useful work done by it to the work put into it by the engine. In actual flight for the same rotational speed of the airscrew, a forward motion—which means some useful work done—may be attained at which each blade section meets the airflow at the angle of attack of about 3°, which is the most efficient angle of attack for an aerofoil having its maximum lift-drag ratio. So here the ratio of the airscrew thrust to the torque is a maximum ; and so at this speed the screw has the maximum efficiency.

254. **Fixed Pitch and Variable Pitch Airscrews** :—It has been seen that only at a particular speed of the aircraft a fixed pitch airscrew has got its maximum efficiency at a given rotational speed, but, in practice, the actual speed of an aircraft varies over a more or less wide range An airscrew whose pitch can be varied by the pilot when in flight, is called a **variable pitch** airscrew the mechanism of which is rather complicated, though this is very effective for all conditions of flight. But whether a variable pitch airscrew is advisable or not depends on the speed range of the aeroplane. For a high speed range, a variable pitch airscrew is essential, and when the maximum speed is relatively low, a fixed pitch airscrew will work quite well. With the first type of airscrew an aeroplane might be brought home safely when in danger, which would have been impossible with the fixed pitch type. So for a modern machine of high speed range a V. P. airscrew is essential.

STABILITY AND BALANCE

255. **Stability and Balance** :—If an aeroplane, when disturbed, tends to return to its original position, it is said to be **stable** and the **stability** of the machine means its capacity to return to some particular condition of flight after it is slightly disturbed from that condition.

[**Note.—Stability** should not be confused with **balance**. Suppose an aeroplane flies with one wing more dipping than the other and it may, when disturbed from this state, return to its former position. Such an aeroplane is not unstable but only out of its proper balance.]

256. **Stability** :—An aeroplane may rotate about three axes all mutually at right angles to each other and all passing through the centre of gravity of the aircraft. These axes are as follows.—*The Longitudinal* (*or rolling*) *axis* XX' running from nose to tail and *the Lateral* (*or pitching*) *axis* YY' lying in the same horizontal plane and *the Normal* (*or yawing*) *axis* ZZ' (Fig. 252).

(1) The rotatory motion of the aeroplane about the lateral axis is called **pitching** caused mainly by a wind gust resulting in the nose

rising or depressing. During pitching the longitudinal axis moves in a vertical plane.

The capacity to correct pitching is defined as *longitudinal stability.*

(2) Any rotatory motion of the aeroplane about longitudinal axis is called **rolling**, resulting in one wing rising and the other dropping. The lateral axis moves in a vertical plane during rolling. The ability of the aeroplane to correct rolling is called *Lateral stability.*

(3) The rotatory motion about the normal axis is called **yawing.** It results in the nose and tail being deflected to one side. The capacity to correct yawing is called *Directional Stability.*

(a) **Longitudinal Stability.**—This is achieved by the tail plane by setting it at an angle less than that of the main plane. Suppose that due to wind gust the nose of the machine is thrown up. The tail plane is then turned so that it presents an angle of attack less than that of the main plane, and thus a force is obtained on the tail plane in such a direction as is necessary to counteract the movement of C.P. of the main plane, which is detrimental to stability, and thus to bring the machine to equilibrium position. Another condition for longitudinal stability is that the position of the centre of gravity of the aeroplane must not be too far back.

(b) **Lateral Stability.**—During normal flight the lift on the wings is vertical, and equal and opposite to the weight, but when a roll takes place one wing drops and the other goes up. In this position the lift is inclined and is no longer in the same straight line as the weight. As a result of these two non-parallel forces, the machine cannot be in equilibrium and moves bodily sideways, called *side slip*, in the direction of the lower wing. To overcome this lateral instability a small positive *dihedral angle* is introduced between the two wings by setting the wings to be inclined upwards by a small angle to the lateral axis. Now the vertical component of the lift on the lower wing is increased, the angle of attack being greater, and that on the other side is decreased and thus a couple is introduced which brings the aeroplane to the normal position. Lateral stability depends also on the position of the centre of gravity of the aeroplane.

[The **dihedral angle** is the angle between each plane and the horizontal for the normal position. It is positive when the plane is sloping upwards and negative when sloping downwards.]

(c) **Directional Stability.**—This is secured by fitting a small aerofoil vertically at the centre of the tail plane. This acts in a way similar to that of the tail plane and produces a force which opposes any tendency to spin round the normal axis. This small aerofoil is known as the fin, which is the most important factor, for, both by its surface area and position, a correcting turning moment is obtained from it.

Lateral and directional stability are inter-related. A roll is followed by a yawn and *vice versa*, and the study of the two cannot be separated.

Control.—It is no doubt necessary that an aeroplane should be stable but that is not enough. It is also necessary to control the machine to force it to take any desired position, or to correct any tendency of the machine to wander from any desired path. When the pilot desires to bring about such changes he has at his disposal three movable control surfaces which are operated from the cockpit by means of cables or rods : (*a*) *the elevator*, (*b*) *aileron* and (*c*) *rudder*.

(a) **Longitudinal Control and the Elevators.**—Longitudinal control is the control of pitching and is obtained by the elevators which are flaps hinged behind the tail plane by which the angle at which the machine is flying can be altered and thus the nose of the machine can be raised or lowered as desired. Elevators are operated by means of the control column (also called *Joystick*) situated in front of the pilot's seat. By pulling the Joystick backwards, the elevators are raised by which action the aeroplane begins to ascend, and the opposite action takes place by moving the Joystick forward.

(b) **Lateral Control and the Ailerons.**—Lateral control is the control of rolling of the lateral axis and is obtained by the ailerons which are flaps hinged at the rear of the main wings near each wingtip. They are connected together so that when one flap is depressed, the other on the opposite wingtip is raised. When a machine has been tilted through an angle laterally by an wind gust, the pilot rights the aeroplane by depressing the ailerons. Thus by the aid of the ailerons the aeroplane may be *banked*, that is, the machine may fly with one wing lower than the other. The ailerons are operated by moving the control column by the hand or sometimes by a control wheel like the steering wheel of motor car.

The linkages of the control surfaces are so designed that the controls may be moved instinctively from the pilot's cockpit when any manœuvre is desired.

The elevators and ailerons are moved by a single control column in the pilot's cockpit. By pushing the control column to the left, the right-hand aileron is lowered and the right-hand wings is lifted up ; while at the same time the left-hand aileron is raised and the left wing dips down. Thus the whole aeroplane is banked to the left. This control is required to be instinctive.

(c) **Directional Control and the Rudder.**—It is the control of yawing or rotation about the normal axis, and is obtained by the rudder, which is a vertical flap hinged on the rear of the fin. This is operated by a *rudder bar* in the cockpit and worked by the pilot's feet. On pressing the right foot forward, the rear of the rudder will be moved to the right and the aeroplane will turn to the right and so on. The function of the rudder is to keep the machine in its correct course, and it is also used in conjunction with the ailerons for *turning* the machine.

In general, the movement by the rudder will give rise to a side force on the fin, the movement of the elevator will produce a force on the tail plane, while the movement of the aileron increases or reduces the lift on the wing, as the aileron is pulled down or pulled up.

It should be noted that in each of the above cases the control surfaces are placed as far as possible from the centre of gravity of the machine so as to provide sufficient leverage to alter its position.

(d) **Engine.**—Besides the above control units, the engine is also considered as another unit, the primary function of which, from the point of view of control, is to vary the height at which the machine is flying for a given angle of attack, speed, etc.

257. Stability and Control :– The difference between stability and control should be clearly noted. Stabilising devices, such as the tail plane and fin, restore the aeroplane to its original path of flight after a disturbance has occurred while, on the other hand, the pilot uses the control surfaces, such as the elevators, etc to manœuvre the machine into any desired position ; but the change of altitude will be resisted by the inherent stabilising devices. The control surface should therefore be effective enough to overcome the action of the stabilising devices.

Stability and Trim.—When an aeroplane is in trim it will continue to fly without change of direction or altitude, even when the pilot takes his hand off the controls provided it has the necessary stability. But if the aeroplane is not in trim it will either go slowly up or down, and this want of trim can be corrected by the use of small auxiliary flaps, called *trimming tabs*— hinged to the trailing edge.

258. Manoeuvres :—The various manœuvres which an aeroplane may be required to perform are given below—

(1) Take-off and Landing.—In *take-off*, the throttle of the engine is oppened, and the machine moves over the ground gaining speed, while the pilot depresses the elevators, thus raising the tail. The machine then rises up attaining the minimum speed to be sustained in air.

Landing is done by bringing down the speed of the aircraft until it is brought into contact with the ground. Landing may be slow or fast.

(2) Gliding.—In this the engine is throttled down until the speed of the engine is just sufficient to keep the engine going. Now the thrust T disappears and the aircraft must be kept in equilibrium by the forces of lift, drag, and weight only, *i.e.* the total reaction, or the resultant of the lift and drag must be exactly equal and opposite to the weight. The angle between the path of the glide and the horizontal is called the *gliding angle* which is the same as the angle between the lift and the total reaction.

(3) **Climbing.**—In order to make a climb, the pilot holds the control column backward to have the angle of attack between the normal and stalling values.

(4) **Banking.**—Banking is accomplished by moving the ailerons over, so that one wing drops and the other rises. In this the lift force, in addition to lifting the machine, supplies a component towards the centre of the turn, so that a large force is obtained for pulling the machine, into a circular path and setting it down to the steady condition.

Besides these, other different manoeuvres done by expert pilots are as follows—(5) *Side slip* ; (6) *Ceiling* ; (7) *Loop* ; (8) *Spin* ; (9) *Roll* ; (10) *Zoom* ; (11) *Nose-dive.*

259. High Altitude Flying :—It has already been pointed out that with the increase of altitude the density, pressure and temperature of the atmosphere all decrease, and these cause important modifications in the forces acting on the aircraft. The effect of decrease of the density of air may be summarised as follows—(*a*) Decrease in lift and drag ; (*b*) Falling-off in the power of the engine ; (*c*) Decrease in airscrew thrust.

(*a*) Lift and drag depend on the density of air. At higher altitudes the density of air is considerably less than that at ground level and so the lift can no longer balance the weight of the aircraft. It is necessary, therefore, either to increase the speed of the aircraft, having the angle of attack the same, to obtain sufficient lift to balance the weight or to increase the angle of attack which in turn will increase the drag.

There is, however, a limit to the possible increase of speed as it depends on the power of the engine which is also limited, and further there is also a limit to the increase of the angle of attack, as, we know, when this is made too great, the lift will decrease instead of continuing to increase, or, in other words, there will be stalling of the machine.

(*b*) As the pressure of air decreases with height, the weight of petrol-air mixture taken into the cylinder of the engine for combustion is reduced and so there is considerable falling-off in the power of the engine. This may be remedied to a certain extent by *super-charging*, i.e. by forcing the mixture into the cylinder with a pump. But ultimately the atmospheric pressure becomes so small that, with all existing engines, there is a height at which the power begins to fall off in spite of the super-charger, and we find that sooner or later a height is reached which cannot be exceeded. Thus the maximum height to which an aeroplane can fly depending on the construction, design, and weight and engine power, is called the **ceiling** of the aeroplane.

(*c*) In rarefied air the airscrew-thrust is sufficiently reduced even when the engine and propeller revolutions per minute are sufficiently increased. In such cases variable pitch airscrew are usually employed to compensate for the loss, to some extent.

In the stratosphere the temperature is nearly $-60\ F.$, and at this low temperature all metal joints become leaky, rubber becomes brittle, pipe lines freeze, and so on, unless special precautions are taken.

Again the low temperature and low pressure at high altitudes affect the comfort of the pilot and other passengers. For the low temperature, heavy warm clothing (woollen, preferably leather cloth) garments are essential and the cabin should be electrically heated. For oxygen deficiency in the lungs, oxygen is supplied from cylinders But at sufficiently high altitudes, the pressure in the lungs becomes so low that the oxyen deficiency may finally endanger life. The cabin requires, therefore, to be properly sealed and pressurised to maintain the standard pressure inside.

At heights more than 10,000 ft. symptoms such as drowsiness, breathlessness, muscular weakness, etc. becomes pronounced, and at about 25,000 ft. they become dangerous. Apparatus for the artificial administration of oxygen is always necessary for high altitude flights, and with its aid flying upto heights of about 35,000 ft. may be safely undertaken.

Besides this, discomfort or pain in the ear is often felt by the pilot due to changing atmospheric pressure. Against these disadvantages, one should remember that the weather conditions remain fairly constant at high altitudes. So high altitude flight is smooth and safe, which has made it popular from the commercial point of view.

Questions (Chapter XIII)

1. What are meant by 'stream-line' flow and 'stream-line' body ? What is the importance of stream-lining all parts of an aircraft which are exposed to airflow ?

2. Write short notes on *any three* of the following :—(*a*) Airships ; (*b*) Aerofoil ; (*c*) Stream-line and Turbulent flow ; (*d*) Stalling.

3. What is meant by aerofoil ? How does the aerofoil of an aeroplane determine the efficiency of it ?

Explain fully how the wings of an aeroplane support it high up in the air. Indicate the forces that act on the machine.

4. Write a note on the flight of an aeroplane indicating the part played by the more important portions of it.

5. Describe what happens when a flat plate moves through air, and explain why aeroplane parts are stream-line shaped.

6. Explain what is meant by 'stream-line' flow. Describe an experiment to demonstrate the deformation of stream-line by an obstacle.

Discuss the flow air past a flat plate moving through air with a high velocity with its plane inclined at a small angle to the direction of motion. Show how a lifting force is produced on the plate and explain how it varies with the angle of incidence of the plate.

7. What are cambered wings in an aeroplane? Explain their action. Also explain with neat diagrams the actions of the tail, elevator, fin and rudder.

8. Explain with the aid of neat diagrams the action of (*i*) elevator, (*ii*) fin and (*iii*) rudder of an aeroplane.

9. What is a 'cambered wing'? Draw a neat diagram of a section of it perpendicular to its span, and indicate in it the length known as the 'chord', the 'upper camber,' and the 'lower cambér'. What are the essential points that are kept in view in its construction?

10. What are the functions of—(*a*) the propeller; (*b*) the wings; and (*c*) the ailerons in an aeroplane? Is there any limit to the height to which an aeroplane can ascend? Give reasons.

11. What is a stream-lined body? Describe the structure of an aeroplane wing and discuss the factor upon which the lifting efficiency depends.

12. Write notes on (*a*) stream-line and turbulent flow, (*b*) lift and drag. (*c*) aerofoil and (*d*) airscrew.

13. How do you get the 'lift' that supports an aeroplane in the air and what is the corresponding 'wing-drag'?

Define coefficients of 'lift' and 'drag' and prove that in horizontal flight (when lift must be equal to the weight of the machine.)

$V = \sqrt{\frac{2W}{dsk_L}}$, where V is the velocity of the machine, s the area of the wing, k_L the coefficient of lift, and d the density of air supposed uniform.

14. Explain what you understand by 'lift' and 'drag'. Illustrate graphically how the ratio of the two varies with the angle of incidence of the aerofoil. Explain the action of airscrew.

15. Write short notes on the variations of 'lift' and 'drag' with the angle of incidence.

16. A stream-line body having a frontal area of 1 sq. ft. moves through air with a speed of 180 m.p.h. Calculate the 'drag' on the body assuming the value of 'drag' coefficient to be 0·04 and the density of air 0·056 lb. per cu. ft.

[*Ans.*: 2·424 lbs.-wt.]

17. Define 'Centre of Pressure'. How does the C.P. of an aerofoil move with the increase of the angle of attack from 0° to 20°?

18. What are the factors on which the 'lift and drag' of an aircraft depend?

19. Criticise the following statements—(*a*) 'Lift' increases as the angle of attack of the wing increases; (*b*) lift is always vertical; (*c*) 'lift and drag' are affected only by air speed and angle of attack.

20. Draw a neat sketch of an aeroplane showing its essential parts and explain fully the control system in it. (*cf.* C. U. 1963)

21. Draw a sketch showing the four principal forces acting on an aeroplane in normal horizontal flight.

22. What is an airscrew ? Explain how it gives the forward motion to an aeroplane.

23. Write a short note on the airscrew and explain clearly how it propels an aeroplane through air.

24. Describe the parts of an aeroplane which ensure its stability in all possible modes. Illustrate, by neat sketches, the mechanisms to control its motion in various directions and indicate how the pilot manipulates them in taking a turn

25. How can you distinguish between stability and control ? Name the axes about which pitching, rolling and yawing of an aircraft take place. Which control is used to produce each motion ?

26. Compare the flight of an aeroplane with that of a kite in air. Explain how an aeroplane maintains its stability during flight.

27. At a certain speed of normal horizontal flight of an aeroplane the ratio of its lift to drag is 7·5 to 1 What were the values of 'lift, thrust and drag' when there is no force on the tail plane ? The weight of the aeroplane is 3500 lbs.

[*Ans.* : lift=3500 lbs.-wt. ; thrust=467 lbs.-wt. ; drag=467 lbs.-wt.]

28. What is the true air speed on an aeroplane at a certain height weighing 60,000 lbs. and having a wing area of 1300 sq. ft. The lift coefficient is 0·5 and the density of air at that height is 0·056 lb. per cu. ft.

Calculate also 'thrust' and 'drag' when the value of L/D ratio is 8.

[*Ans.* : Speed=222 m.p.h. ; thrust=7500 lbs.-wt. ; drag=7500 lbs.-wt.]

29. Write notes on any four of the following—(*a*) stream-line flow, (*b*) Bernoulli's law, (*c*) stalling, (*d*) rolling and (*e*) pitching.

30. Can an aeroplane fly without wings ? Can an aeroplane fly in a vacuum ? Give reasons for your answer.

31. (*a*) Show with suitable illustrations the nature of airflow and pressure distribution over an aerofoil.

(*b*) What are the forces acting on an aeroplane in normal horizontal flights ? Discuss the conditions of equilibrium of these forces in relation to the flight of the aeroplane. (C. U. 1967)

CHAPTER XIV

UNITS AND DIMENSIONS

260. Units :—Any physical quantity is measured in terms of a small part of it. This small part is called the unit of measurement of the quantity. The units of length, mass and time are called the *fundamental units*. They are so callled because the units of other physical quantities can be derived from them. The units of area, volume, force, energy, etc. derived from the fundamental units are called the *derived units*. The units are based on international standards which are established, constructed and maintained for the purpose. The two common systems of units that are used in all scientific and engineering measurements are the C.G.S. (Centimetre-Gramme-Second) and the F.P.S. (Foot-Pound-Second) systems.

For the facility of large-scale practical measurements particularly in engineering and in order to get all the so-called practical units of energy, power, etc. directly as the corresponding units of the system, a new system of units known as M.K.S. units has been introduced. Here the unit of length is one metre $(M)=10^2$ cms., that of mass is one kilogram $(K)=10^3$ gms. and that of time is one second (S). The unit of force in M.K.S. system is *newton* where 1 newton = unit mass × unit acceleration,

$$= 1\text{Kg.}\times 1 \text{ Metre per sec.}^2 = 10^3 \times 10^2 \text{ gm.} \times \frac{\text{cm.}}{\text{sec.}^2}$$

$$= 10^5 \text{ dynes.}$$

Thus 1 unit of energy in M.K.S. system $= 1$ newton × 1 metre

$$= 10^5 \text{ dynes} \times 10^2 \text{ cm.} = 10^7 \text{ ergs.}$$

$$= 1 \text{ Joule.}$$

261. Dimensions and Dimensional Equations :—Whenever a physical quantity involves the measurement of a length we symbolise it by L, and when it involves the measurement of a mass or a time, we symbolise it by M or T respectively. These symbols L, M and T indicate the type of measurements involved in order to get these quantities. The *dimensions* of a quantity indicate the *powers* to which the fundamental units of length, mass and time must be raised so as to represent it. The *dimensional equation or formula* of a physical quantity represents an expression showing its relation to the fundamental units of measurement. A numerical quantity is dimensionless as it is unrelated to the fundamental units.

The actual equations representing different physical phenomena must be homogeneous from the dimensional stand-point of view, *i.e.*, the dimensions of the left side of the equation must be the same as the dimensions of the right-hand side. This property, generally known as the *principle of homogeneity of dimensions*, sometimes help us to predict the equations governing different physical phenomena even before we know them by actual experiments.

As an example, if we want to find the nature of variation of time period t of a simple pendulum with its length l, the mass of the bob m and the acceleration due to gravity g, we can easily put,

$t=f(m, g, l)$ when f represents an unknown function.

Let this function be represented as $K(m^x g^y l^z)$ where x, y, z are the powers to which m, g and l should be raised in the final expression and K is a dimensionless constant.

$$\therefore \quad t=Km^x g^y l^z, \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

Now the dimension of t is T, that of m and l are respectively M and L and that of g (acceleration in cms./sec.^{2}) is $L/T^2 = LT^{-2}$.

$$\therefore \quad T = KM^x(LT^{-2})^y L^z = KM^x T^{-2y} L^{y+z}.$$

Since this equation should be homogeneous dimensionally, we can equate the powers of the corresponding dimensions.

$$\therefore \quad \begin{aligned} x&=0. \\ -2y&=1. \\ y+z&=0. \end{aligned}$$

Solving we get, $x=0$, $y=-\frac{1}{2}$ and $z=\frac{1}{2}$

$\therefore$ Putting in (1),

$$t = Km^0 g^{-\frac{1}{2}} l^{\frac{1}{2}}$$

$$= K\sqrt{\frac{l}{g}}.$$

Thus t is independent of m which is corroborated by experiments. The above method of ***dimensional analysis***, as it is generally called, gives us no information of dimensionless quantities like K which is actually 2π in this case, as we know experimentally. This can, however, be determined by a single experiment.

The dimensional analysis also enables us to test the equations derived mathematically by checking the dimensional homogeneity of both sides. The successful use of dimensional analysis, however, involves a thorough and wide knowledge of Physics. A correct result depends on the correct choice of variable and this selection is guided mainly by physical analogy and intuition. Sometimes, in course of analysis, dimensional constants, like the constant of gravitation, may creep in, It is rather a very difficult job, requiring subtle considerations, to decide whether such constants are to be introduced as additional variables or not.

262. Dimensional Formulae for different Physical quantities with their C.G.S. units :—

$[\text{Area}]=[\text{Length}\times\text{Breadth}]=[\text{Length}]^2=L^2.$ (Unit :—cm².)

$[\text{Volume}]=[\text{Length}\times\text{Breadth}\times\text{Height}]=[\text{Length}]^3=L^3.$

(Unit :—cm.³)

$$[\text{Angle}]=\left[\frac{\text{Arc}}{\text{Radius}}\right]\text{ radians or }\left[\frac{\text{Arc}}{\text{Radius}}\times\frac{180}{\pi}\right]^0$$

$$=\left[\frac{L}{L}\right]\text{ or, }\left[\frac{L}{L}\times\text{ a number }\right]=L^0\text{ (no dimension)}$$

(Unit :—degree or radian).

$$[\text{Velocity}]=\left[\frac{\text{Distance}}{\text{Time}}\right]=\frac{L}{T}=LT^{-1}$$ (Unit :—cm. per sec.)

$$[\text{Acceleration}]=\left[\frac{\text{Velocity}}{\text{Time}}\right]=\frac{LT^{-1}}{T}=LT^{-2}$$ (Unit : – cm. per sec.²)

$[\text{Force}]=[\text{Mass}]\times[\text{Acceleration}]=M\times LT^{-2}=MLT^{-2}.$

(Unit :—dyne or gm.-cm. per sec.²)

$[\text{Energy}]$ or $[\text{Work}]=[\text{Force}]\times[\text{Distance}]=MLT^{-2}\times L=ML^2T^{-2}$

(Unit :—erg or dyne-cm.)

$$[\text{Density}]=\frac{[\text{Mass}]}{[\text{Volume}]}=\frac{M}{L^3}=ML^{-3}$$ (Unit :—gm. per cm.³)

$$[\text{Specific gravity}]=\frac{[\text{Density of substance}]}{[\text{Density of water}]}=\frac{ML^{-3}}{ML^{-3}}=M^0L^0$$ (No dimensional formula. No unit)

$$[\text{Frequency}]=\text{Number of vibration per sec.}=\frac{[\text{Number}]}{[\text{Time}]}=T^{-1}$$

(Unit :—vibr. per sec.)

[Co-efficient of expansion]

$$=\frac{[\text{change in length or area or volume}]}{[\text{original length or area or volume}][\text{Change in temp.}]}$$

$$=\frac{[\text{Number}]}{C}\text{ (where }C=\text{change in temp.)}$$

$$=C.^{-1}$$ (Unit :—per °C)

[Surface Tension] = Force per unit length of the boundary line of a dimensio.

liquid surface $=\dfrac{[\text{Force}]}{[\text{Length}]}=\dfrac{MLT^{-2}}{L}=MT^{-2}$. (Unit :—dynes/cm.) ;

or, $[\text{S. T.}]=$ Energy per unit area $=\dfrac{[\text{Energy}]}{[\text{Area}]}=\dfrac{ML^2T^{-2}}{L^2}\,MT^{-2}$.

$[\text{Stress}]=$ Force per unit area $=\dfrac{[\text{Force}]}{[\text{Area}]}=\dfrac{MLT^{-2}}{L^2}\,ML^{-1}T^{-2}$

(Unit :—dynes/cm.^{2})

$$[\text{Strain}]=\frac{\text{change in length or any other relevant quantity}}{\text{original length or the quantity itself}}=\text{a number.}$$

(no unit).

$\therefore$ $[\text{Modulus of elasticity}]=\dfrac{[\text{stress}]}{[\text{strain}]}=\dfrac{ML^{-1}T^{-2}}{\text{a number}}\,ML^{-1}T^{-2}$.

(as a number has no dimension)
(Unit :—dynes/cm.^{2}).

$$[\text{Poisson's Ratio}]=\frac{[\text{Lateral strain}]}{[\text{Longitudinal strain}]}=\frac{\text{a number}}{\text{a number}}\ \text{(no dimension).}$$

$[\text{Moment of Inertia}]=\dfrac{[2\times K.E.]}{[\omega]^2}$, (where $K.E.=$ kinetic energy and $\omega=$ angular velocity),

$$=\frac{[2\times\text{work}]}{[\text{angle/time}]^2}=ML^{+2}T^{-2}\times T^2=ML^2.$$

($\because$ The numeral 2 and the angle are dimensionless) (Unit :—gm.-cm.^{2}.)

$[\text{Radius of Gyration}]=\left[\dfrac{I}{M}\right]^{\frac{1}{2}}$, ($\because$ $I=ML^2$)

$$=\left(\frac{ML^2}{M}\right)^{\frac{1}{2}}=L.$$ (Units :—cm.).

$[\text{Moment of Momentum}]=[I][\omega]=ML^2\times\dfrac{\text{angle}}{T}=ML^2T^{-1}$.

($\because$ angle is dimensionless). (Unit :—ergs. sec.)

$[\text{Gravitational Constant}]=\dfrac{[\text{Force}]}{[m_1m_2/d^2]}=\dfrac{MLT^{-2}\times L^2}{M^2}=M^{-1}L^3T^{-2}$.

(Unit :—C.G.S. units or dynes/gm.^{2}/cm.^{2})

[Coefficient of Viscosity] = Tangential stress per unit velocity gradient

$$=\frac{\text{Stress}}{[\text{Velocity}]/[\text{Distance}]}=\frac{ML^{-1}T^{-2}\times L}{LT^{-1}}=ML^{-1}T^{-1}.$$

(Units : Poise or dynes/cm.^{2} per unit vel. grad.)

[Intensity of Gravitational field]

$$=\text{Force per unit mass}=\frac{[\text{Force}]}{[\text{Mass}]}=\frac{MLT^{-2}}{M}=LT^{-2},$$

(Unit :—dynes/gm.)

[Gravitational Potential] = work done per unit mass

$$=\frac{[\text{Work}]}{[\text{Mass}]}=\frac{ML^2T^{-2}}{M}=L^2T^{-2}.$$

(Unit :—ergs/gm.)

263. The uses of Dimensional Equations :—The important uses of the dimensional equations may be given as follows :—

(a) To check the accuracy of an equation representing a physical phenomenon.—As has already been referred to, the actual equations must be homogeneous from the dimensional stand-point of view, *i.e.* the dimensional equation or formula for the left-hand side of the equation must be equal to the dimensional formula for the right-hand side. To illustrate this let us take equation representing the velocity of sound in an elastic medium, which is written as,

$$V=\sqrt{\frac{E}{\rho}},$$

where V = the velocity of sound, E = coefficient of elasticity of the medium (either volume or longitudinal) and ρ = density of the medium.

Dimensions of V LT^{-1} = Dimemsions of the left-hand side ;

Dimensions of $E=ML^{-1}T^{-2}$;

Dimensions of $\rho=ML^{-3}$.

$$\therefore \text{ Dimensions of the right-hand side}=\sqrt{\frac{ML^{-1}T^{-2}}{ML^{-3}}}$$

$$=\sqrt{L^2T^{-2}}=LT^{-1}.$$

Hence the dimensions of the right-hand side and left-hand side are identical. Thus the equation is found to be correct.

(b) To predict the equations governing the different physical phenomena.—This method of derivation depends on a knowledge of the various factors which control a particular physical quantity. The nature of variations of these factors should also be known. As has been shown in general discussions on dimensions and dimensional equations, the formula for the time period of a simple pendulum can be easily derived by the method. In this method also the main principle that is to be applied is the principle of homogeneity of dimensions. A few more examples given at the end of this chapter will illustrate this method of predicting the equations clearly.

(c) To change one system of units into another.—When the dimensional formula of a physical quantity is known, its numerical value while changing from one system of units to another can be found out. If the dimensional formula of a quantity $[M^xL^yT^z]$ and if its numerical value is n_1 when the fundamental units are M_1, L_1, and T_1, then its numerical value n_2 on the other system of units M_2, L_2, and T_2 will be given by the relation,

$$n_2[M_2{}^xL_2{}^yT_2{}^z]=n_1[M_1{}^xL_1{}^yT_1{}^z].$$

$$\therefore\ n_2=n_1.\left(\frac{M_1}{M_2}\right)^x.\left(\frac{L_1}{L_2}\right)^y.\left(\frac{T_1}{T_2}\right)^z.$$

As an example if we want to convert a velocity of 5 ft. per sec. to the $C.G.S.$ system of units, we shall have to start with the dimensional equation of velocity, which is given as LT^{-1} or $M^0L^1T^{-1}$.

$$\therefore\ n_2=n_1\left(\frac{M_1}{M_2}\right)^0\left(\frac{L_1}{L_2}\right)^1\left(\frac{T_1}{T_2}\right)^{-1}.$$

$$L_2=1\text{ cm.}=\frac{1}{12\times2{\cdot}54}\text{ft.}$$

Time in both the units is the same and $n_1=5$.

$$\therefore\ n_2=5\left[\frac{1}{1/(12\times2{\cdot}54)}\right]^1[1]^{-1}$$

$$=5\times12\times2{\cdot}54\text{ cm. per sec.}$$

Examples

1. The time of oscillation T of a small drop of liquid under surface tension depends upon the density d, radius r and surface tension S. Prove dimensionally that $T=\sqrt{\frac{dr^3}{S}}$

Ans.: Since T depends on d, r and S, let us suppose that $T=Kd^xr^yS^z$.

But the dimension of d are ML^{-3}, those of S are MT^{-2} and the dimension of r is L,

$\therefore$ $[T]=K[ML^{-3}]^x.L^y.(MT^{-2})^z=K.M.^{x+z}L^{y-3x}T^{-2z}$.

$\therefore$ Equating the powers, $x+z=0$, $y-3x=0$, and $-2z=1$.

Hence $z=-\frac{1}{2}$, $x=\frac{1}{2}$ and $y=3/2$.

$$\therefore\ T=Kd^{\frac{1}{2}}r^{\frac{3}{2}}S^{-\frac{1}{2}}=K\sqrt{\frac{dr^3}{S}}.$$

2. The value of the acceleration due to gravity is 32 ft./sec.^{2} What will be its value if the unit of length is a mile and that of time is a minute?

(Agra U. 1951)

Ans. : Here $g = 32$ ft./sec.^{2}.

The dimensions of acceleration $= LT^{-2} = M^0 LT^{-2}$

Let n_2 be the numerical value of g in the new units. n_1 as given above, is 32.

$\therefore \quad n_2 = n_1 \left(\frac{M_1}{M_2}\right)^0 \left(\frac{L_1}{L_2}\right)^1 \left(\frac{T_1}{T_2}\right)^{-2}$, where $L_2 = 1760 \times 3$ ft. and $T_2 = 60$ secs.

$$\therefore \quad n_2 = 32 \left(\frac{1}{1760 \times 3}\right) \left(\frac{1}{60}\right)^{-2} = \frac{32 \times 60 \times 60}{1760 \times 3} = \frac{240}{11} = 21\frac{9}{11}$$

miles per min^2.

3. *If the excess pressure inside a soap bubble over that outside depends on the surface tension S of the soap solution and the radius r of the bubble, show that $p = K\frac{S}{r}$.*

Ans. : From the conditions of the problem, $p = KS^x r^y$.

The dimensions of pressure p are $ML^{-1}T^{-2}$, those of S are MT^{-2} and r has the dimension of L.

$\therefore \quad ML^{-1}T^{-2} = K(MT^{-2})^x L^y = KM^x L^y T^{-2x}$.

Equating the powers of the corresponding quantities from two sides, we get,

$$x = 1, \; y = -1.$$

Hence, $\quad p = K\frac{S}{r}$.

4. *If the frequency (n) of a stretched string depends on its mass per unit length (μ), its length (l) and the tension applied (F), show that $n = \frac{K}{l}\sqrt{\frac{F}{\mu}}$.*

Ans. : From the given conditions, $n = K\mu^x l^y F^z$;

n has dimensions, T^{-1} ;

μ has dimensions, ML^{-1} ;

F has dimensions, MLT^{-2}.

$\therefore \quad T^{-1} = K(ML^{-1})^x L^y (MLT^{-2})^z = KM^{x+z} L^{-x+y+z} T^{-2z}$.

$\therefore \quad x + z = 0, \quad -x + y + z = 0$ and $z = \frac{1}{2}$.

$\therefore \quad x = -\frac{1}{2}$ and $y = -1$.

$$\therefore \quad n = K\mu^{-\frac{1}{2}} l^{-1} F^{\frac{1}{2}} = \frac{K}{l}\sqrt{\frac{F}{\mu}}.$$

5. *Assuming that the mass M of the largest stone that can be moved by a flowing river depends only upon v, the velocity, ρ, the density of water and on g, show that M varies with the sixth power of the velocity of flow.*

(*P. U. 1944 ; Aligarh U. 1948*)

Ans. : From the given conditions, $M = Kv^x \rho^y g^z$

v the velocity has dimensions LT^{-1}. ρ, the density has dimensions, ML^{-3} and g, the acceleration due to gravity, has dimensions LT^{-2}.

$$\therefore \quad M = K(LT^{-1})^x (ML^{-3})^y (LT^{-2})^z$$
$$= KM^y L^{x-3y+z} T^{-x-2z},$$

$\therefore \quad y = 1$; $x - 3y + z = 0$ and $x + 2z = 0$;

or, $\begin{cases} x + z = 3, \\ x + 2z = 0, \end{cases} \quad \therefore \quad x = 6$ and $z = -3$.

$\therefore \quad M = Kv^6 \rho / g^3$.

$$\therefore \quad M \text{ is proportional to } v^6.$$

Thus the mass varies as the sixth power of the velocity of flow.

6 *If the critical velocity of flow* (v) *of a liquid through a narrow tube depends on viscosity* (η), *the density* (ρ) *of the liquid and radius* (r) *of the tube, show that* $v=\dfrac{K\eta}{\rho r}$.

Ans. : From the given conditions, $v=K\eta^x\rho^y r^z$.

The dimensions of v, η, ρ and r are given by,

LT^{-1}. $ML^{-1}T^{-1}$, ML^{-3} and L respectively,

$$\therefore \quad LT^{-1}=K(ML^{-1}T^{-1})^x(ML^{-3})^y.(L)^z$$

$$=KM^{x+y}L^{-x-3y+z}T^{-x}.$$

So, $x+y=0$, $-x-3y+z=1$ and $x=1$.

$\therefore \quad y=-1$ and $z=-1$. Hence $v=K\eta\rho^{-1}r^{-1}=K\eta/\rho r$.

7. *Express* $4{\cdot}2\times10^7$ *ergs in M.K.S. system of units.*

Ans. : We know,

$$n_2=n_1\left(\frac{M_1}{M_2}\right)^x\left(\frac{L_1}{L_2}\right)^y\left(\frac{T_1}{T_2}\right)^z$$

Here $n_1=4{\cdot}2\times10^7$.

The dimensions of work (ergs) is ML^2T^{-2}.

$$\therefore \quad n_2=n_1\left(\frac{M_1}{M_2}\right)\left(\frac{L_1}{L_2}\right)^2\left(\frac{T_1}{T_2}\right)^{-2}.$$

Here $M_2=1\ Kg=10^3$ gms. $=10^3M_1$; $L_2=1$ metre $=10^2$ cms. $=10^2\ L_1$; $T_2=T_1$.

$$\therefore \quad n_2=4{\cdot}2\times10^7\left(\frac{1}{10^3}\right)\left(\frac{1}{10^2}\right)^2(1)^{-2}$$

$$=4{\cdot}2\times10^7\times10^{-7}=4{\cdot}2 \text{ M.K.S. units } (\textit{Joules}).$$

8. *Find the number of dynes in a force which acting on 15 kg. for 1 minute produces a velocity of 3·6 kms./sec.*

Ans. : Let x dynes be the required force.

$$\therefore \quad x \text{ dynes}=x \text{ gm. cm. sec.}^{-2}=15 \text{ kg. min.}^{-1}\times3{\cdot}6 \text{ km. sec.}^{-1}$$

$$(\because\ P=mv/t)$$

$$\text{Hence } x=15\times3{\cdot}6\times\frac{kg}{gm}\times\frac{km.}{cm.}.\ \frac{sec.^{-1}}{sec.^{-2}}.\ \frac{1}{60 \text{ sec.}}$$

$$=15\times3{\cdot}6\times1000\times10^5\times\frac{1}{60}=9\times10^7.$$

264. Limitations of Dimensional Equations : -Though the method of dimensional equations is very useful for the solution of various problems, it is not possible to apply this method to all sorts of problems.

(a) In the case where a relation involves a constant whose dimensions are not known to us owing to our lack of knowledge of the exact nature of that constant, the method of dimensions fails. For example, let us consider the force of attraction between two masses (m_1 and m_2) which depends on these attracting masses and the distance (d) between them.

$$\therefore \quad \text{Force}=Km_1{}^xm_2{}^yd^z.$$

$$\text{Hence, } MLT^{-2}=KM^{x+y}L^z.$$

This equation fails because -2 dimension in time on the left-hand side cannot be equated to zero dimension of time on the right-hand side.

(b) The method fails when there are more than three possibilities of variation of a quantity because the dimensional formulae can supply only three equations for these more than three unknowns, each one of which relates to one possibility of variation.

(c) When the dimensional relations involve unknown indices whose number exceeds the number of equations supplied.—As an example let us take the case of the final velocity (v) of a body starting with a velocity (u) and moving with a uniform acceleration (f). Let v be acquired after t seconds.

$$\text{Then } v = Ku^x f^y t^z.$$

$$\begin{aligned}\text{Hence, } LT^{-1} &= K(LT^{-1})^x (LT^{-2})^y T^z \\ &= KL^{x+y} T^{-x-2y+z}.\end{aligned}$$

$$\therefore \quad x+y=1 \text{ and } x+2y-z=1.$$

But these two equations cannot be solved for x, y and z. Hence the method fails here.

Note - Over and above the limitations of the method described, in some cases, the dimensional formula of a vector quantity like the moment of a force may be found to be identical with that of a scalar quantity like energy. But as the two physical quantities are fundamentally of different nature they should never be taken as quantities of the same kind inspite of their dimensional identity.

Questions (Chapter XIV)

1. What do you understand by fundamental and derived units? Discuss the basis of dimensional equations in physical problems and illustrate with an example. (Utk. U. 1951)

2. Deduce the dimension of (a) the coefficient of viscosity and (b) the constant of gravitation G.

Obtain a formula for the time of swing of a simple pendulum from a knowledge of the dimensions of the physical quantities involved. (Punj. U. 1936)

3. Explain what you mean by the dimensions of a physical quantity; calculate the dimension of Young's modulus.

Assuming the period of vibration of a tuning fork depends upon the length of the prong and on the density and Young's modulus of the material, find by the method of dimensions, a formula for the period of vibration.

$$\left[Ans : \ ML^{-1}T^{-2} ; \ t \propto l\sqrt{\frac{D}{Y}} \right]$$ (C. U. 1950)

4. Find the number of dynes in the force which acting on 1 cwt. for one minute produces a velocity of one mile per hour.

Given 1 ft.=30.5 cms. and 1 lb.=453 gms.

[Ans. : 37870 dynes.] (C. U. 1952)

SOUND

CHAPTER I

PRODUCTION AND PROPAGATION OF SOUND

1. **Definition of Sound** :—Sound is a kind of sensation received by means of the ears and carried to the brain which is responsible for the perception of hearing. The external cause which produces such sensation is a form of energy.

Acoustics is that branch of Physics which deals with the study of the nature, propagation and perception of sound.

The subject of acoustics may be conveniently divided into a number of subdivisions :—

(*a*) General sound which includes production, propagation and the theoretical study of the nature of sound.

(*b*) Musical sound which mainly deals with its musical aspects and the sound produced by the musical instruments.

(*c*) Physiological sound which deals with the production and perception of sound by human beings.

(*d*) Technical sound deals with the technical aspects and applications of sound.

2. **Sound is produced by the vibratory motion of a material body** :—

Whenever any sound is produced, on tracing its origin it will be found that it is due to the vibratory movement of a material body. The vibrations may in some cases be too rapid to be seen by our naked eyes but we can feel their existence by touching the source. When air is blown through a whistle a nail is struck by hammer, or ammunition explodes in a gun, we have instances where sounds are produced by the matter in motion.

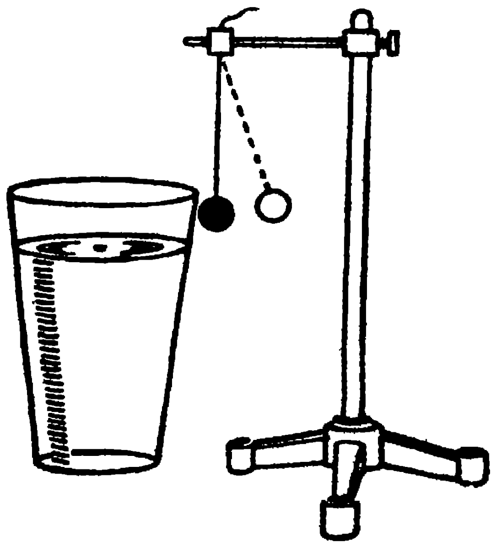


Fig. 1

When we strike a metal vessel with a piece of matter we hear a sound, and the indistinctness of the outline of the vessel shows that it is vibrating. By touching the body the vibrations are stopped and the sound also is stopped at the same time.

Pour water in a wide-mouthed thin-walled glass tumbler until it is almost full and keep a pith ball suspended by a fine thread in touch with the rim of the vessel (Fig. 1). On bowing the edge of the tumbler with a violin bow, ripples will be produced in the water, and the pith ball will be observed to jump forward by receiving a series of shocks from the rim on coming in contact with it, proving that the vessel is in a state of vibration.

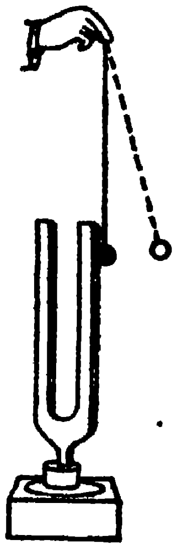


Fig. 2—Tuning fork.

A most commonly used apparatus for the production of sound vibration is a *tuning fork.* It is a *U*-shaped steel bar provided with a handle at the bend of the *U* (Fig. 2) and is made to vibrate by striking one of its prongs on a hard cushion. Its special quality is that it produces a sound of a single frequency.

If a sounding tuning fork is brought into contact with a pith ball suspended by a thread the pith ball will be thrown into vibration.

The other common sources for producing sound vibrations are the stretched string struck, plucked or bowed, the organ pipes and the reed instruments etc.

On examining the string of a sounding violin it will be found to have blurred outline due to its to-and-fro vibratory motion, which can be detected by placing a *V-shaped* paper rider on the string.

Thus a body must be made to vibrate in order to emit sound, but, even when it is vibrating, the sound cannot be received or heard unless the mechanism of the ear also vibrates. We receive sound by the vibrations of a membrane in the ear, called the *ear-drum,* and these vibrations are transmitted to the brain and interpreted there as sound.

3. Audibility limits :—Audibility means the easy and clear perception of sound by ear. The sensitivity of the ear to a sonorous vibration depends both on the intensity and frequency.

The intensity of a pure tone which can just be perceived by the ear is known as the *threshold of audibility*. The ear can detect higher intensities up to a maximum value which is known as the *thresho'd intensity of feeling* beyond which the sensation changes to one of pain and does scarcely help hearing.

The average range of frequencies to which the ear responds varies from about 20 to 20,000 vibrations per second at an average pressure (10 dynes/cm.^{2}) of the sound waves. This range defines the *audibility limits of frequencies*. The audibility limits vary from person to person and the range of frequencies diminishes as a man grows older. They also vary if the pressure of the sound or its intensity rises or falls.

The vibrations of frequencies smaller than the lowest audibility limit are known as *Infrasonic* vibrations and those of frequencies exceeding the upper limit of audibility are known as *Ultrasonic* or *supersonic* vibartions.

4. Propagation of Sound (a material medium necessary) :— In order that sound may be heard, the disturbance from the source must be carried to the ear through a space. This space is spoken of as the *medium*. Air is the usual medium through which sound travels, but it can also pass through any other material medium provided it is elastic and continuous. Thus an observer placing his ear against a continuous iron rail can hear distinctly even slight taps, given on the metal, several hundred yards away. The ticking sound of a watch placed at one end of a table is heard clearly by applying the ear to the other end of it. Again a diver inside water distinctly hears any sound produced in the water. ***Sound cannot, however, travel through a vacuum*** and in this respect, ***it differs from light which can easily pass through a vacuum***. Sound requires a ***material*** medium for its propagation.

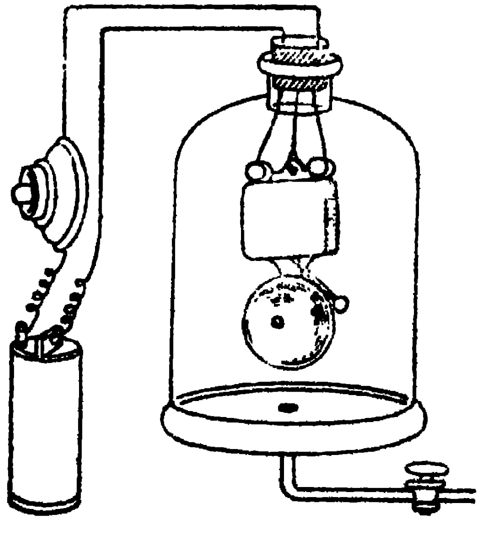


Fig. 8

That sound requires a material medium for its propagation and cannot travel through a vacuum may be demonstrated by the following experiment.

An electric bell (Fig. 3) is placed inside the receiver of an air-pump and worked by a cell placed outside the jar. The bell is suspended inside the receiver by means of a hook passing through a rubber stopper fitted tightly into the neck. The sound of the bell is distinctly heard as long as there is air inside the receiver ; if the air is gradually pumped out, the sound grows fainter and fainter and finally becomes quite inaudible. On re-admission of air, the intensity of sound increases again.

It must also be noted that for the propagation of sound, not only the medium must be a material one but also it should be elastic and continuous. Inelastic substances are not able to transmit sound to a great distance as the energy is dissipated very quickly. Again, non-continuous substances, such as saw dust, felt, etc. are bad conductors of sound.

5. **Propagation of Sound** :—It has been already mentioned that a vibrating source emits sound. A continuous and elastic material medium is always necessary for the propagation of sound which is ultimately received by a suitable receiver. Let us now examine the method by which sound is actually propagated through air. Suppose a body is struck. As a result of this, every particle, constituting the body begins to vibrate—that is, to move to-and-fro to a nearly equal distance on both sides of its mean position of rest. During this state of vibration, each of the extreme particles of the vibrating body in contact with air, at the time of moving to-and-fro between its extreme positions, strikes the line of air particles in contact with it and starts them moving to-and-fro. These air particles in their turn strike the particles beyond them, set up similar vibrations in them, and this goes on from particle to particle. In this way a chain of vibrations is set up from the sounding body ; each particle on the way begins to vibrate when it is struck by its neighbour, and in its turn strikes its next neighbour, until the vibrations reach the membrane of the ear of the listener. The motion of the membrane is communicated to the brain by the mechanism of the ear and perception of the sound is caused.

Mechanism of Propagation.—Suppose a tuning fork is vibrating, Let B be the position of one prong of it when at rest, and C and A, the two extreme positions of it on either side of B (Fig. 4).

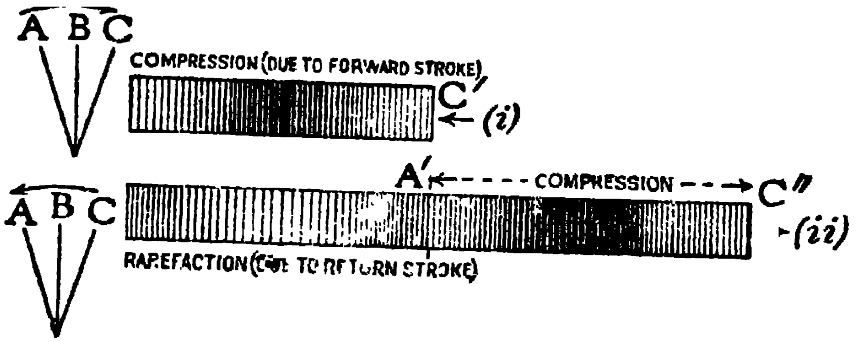

A B C
COMPRESSION (DUE TO FORWARD STROKE)
C′
(i)
A′
COMPRESSION
C″
A B C
(ii)
RAREFACTION (DUE TO RETURN STROKE)

Fig. 4—Propagation of Sound waves.

The time taken by the prong to move from one extreme position to the other and back again to the first position, i.e. from A to C and back, is called the **period of vibration.**

Now, as the prong is moving from A towards C, it presses the air particles in front of it, which in turn press the particles next to them, and this pressure is passed on to the successive layers of the medium. So, considering the effect of the movement of the prong upon a column of air on the right-hand side, it will be seen that, by the time the prong reaches C, the air particles between A and some point C' [Fig. 4(i)] will be compressed, and a pulse of *compression* will move forward (with the velocity of sound). During the return movement when the prong moves back from C to A, it tends to leave a partial vacuum behind it, due to which the layer in contact, being relieved of pressure, expands on the side of the prong and the pressure is consequently diminished. Each succeeding layer acts in the same way and a *rarefaction* pulse is handed on from layer to layer, travelling *forward* in the same direction and with the same velocity as that of the compression pulse. This goes on up to the time the prong takes to reach A. During the time taken by the prong to travel from C to A, the compressed pulse also travelled onwards and occupied a region $A'C''$ [Fig. 4(ii)] equal to AC' in [Fig. 4(i)], which is now occupied by the rarefied pulse as given by AA' [Fig. 4(ii)]. So in a complete period of vibration of the prong, the disturbance travels up to C'', one half of which ($A'C''$) is occupied by a *compressed pulse*, and the other half (AA') by a *rarefied pulse. A compressed pulse followed by a rarefied pulse together forms a complete sound wave.*

The amount of compression or rarefaction is not, however, equal at all points in the complete wave. The reason is that the energy communicated by the prong to the air at any instant depends on the velocity of the prong which varies from instant to instant in course of a period of vibration. The velocity of the prong being maximum at the mean position and zero at the extreme positions, the compression or the rarefaction is also maximum in the middle and zero at the ends of a zone of compression or rarefaction, as shown in Fig. 4(*ii*).

If the *displacements* of the particles lying along the line of propagation at any given instant of time be plotted as the ordinate against their *distances* as abscissa, the graph, which is known as the *displacement-distance* curve, assumes the form of a wave. *The wavelength is the distance covered by one compression pulse and a rarefaction pulse together, i.e. the distance through which the disturbance travels in one period of vibration of the source.*

It is, however, to be noted that each particle in the medium of propagation during a periodic time of its vibration passes through all the phases of displacement as depicted in one wavelength in a sound wave ; while if the particles are considered at the same instant of time, they only successively differ in phase from one to the next.

When we say that 'sound travels in the form of waves', it does *not imply that the sound travels in a wavy path* but it indicates that if the displacement of any particle in the medium is plotted for a periodic time, or the displacements of the various particles in the path of propagation considered at the same instant of time are plotted against their distances, the curve obtained assumes a wavy form.

When a body is sounded in a homogeneous medium, alternate pulses of compression and rarefaction start out in succession in all directions travelling with the same velocity. These pulses are like so many spherical shells of equal thickness spreading out with an expanding radius with the passing of time (Fig. 5). They are analogous to the circular waves caused around a stone thrown into a calm sheet of water. Here a series of circular waves having alternate depressions and elevations are generated. They appear and disappear in a periodic time in succession. The depressions are called the *troughs* and the elevations the *crests.* They also spread out with an expanding radius till they reach the shore. The trough and the crest respectively correspond to the maximum rarefaction and maximum compression states in the propagation of sound through a medium.

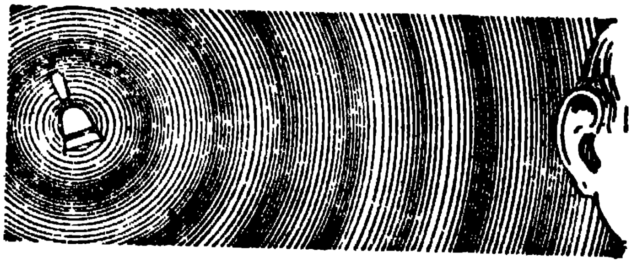


Fig. 5– Sound waves caused by a vibrating Bell.

The difference between the two cases is that a particle on a sheet of water is displaced up and down at right angles to the path of propagation of the disturbance which travels along the surface towards the shore whereas a particle in the medium of propagation of sound is displaced to-and-fro along the same path in which the sound travels. That is, water waves are **transverse,** whereas sound waves are **longitudinal.** It should, however, be noticed that neither in water waves nor in sound waves, the medium is bodily shifted from one place to the other along with the disturbance. In both the cases the particles of the medium only vibrate about a mean position in a plane either perpendicular to or along the direction of propagation and it is the energy of the wave or the disturbance itself that is transferred from one position to the other with the help of these particles.

6. Representation of a Sound wave :—Let a series of dots in Fig. 6(*a*) represent a row of undisturbed particles of air. When a sound wave passes along this row, the particles in certain positions of the row will, at a given instant, come closer (*i.e.* compressed), and in certain other portions be drawn apart (*i.e.* rarefied) as represented in Fig. 6(*b*).

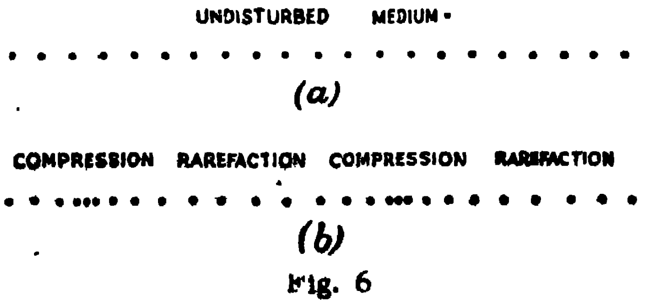


Fig. 6

CHAPTER II

WAVE MOTION

7. Wave Motion :—Every one is familiar with the circular waves which are produced when a stone is thrown into still water. The waves consisting of a series of *crests* and *troughs* travels outwards from the centre of disturbance in gradually widening circles. But if some pieces of cork, or bits of paper, floating on the disturbed water, are carefully watched, it will be found that the floating objects, and therefore, the particles of water, are only moving up and down, but they do not travel outwards with the waves. It should be noted also that they rise and fall, not together but in succession, one after the other, showing that when the waves pass over water each separate particle of the medium must perform the same movement, not simultaneously, but each one a little later than the one preceding it. **It is the wave-form which travels forward,** while every particle of the water moves up and down about its own mean position of rest. Similarly, when a wave crosses a cornfield, the tips of the corn blades do not move forward ; the form of the wave only moves forward. The vibratory motion of a series of particles in a medium as referred to above gives rise to a *wave motion*.

(a) Transverse and Longitudinal Waves.—

When a wave motion passes through a medium in such a way that the vibrations of the particles of the medium are at right angles to the direction of propagation of the waves, the wave is called a **transverse wave.**

When a wave motion passes through a medium in such a way that the vibratory motion of the particles of the transmitting medium is along the same line as the line of propagation of the waves, it is called a **longitudinal wave motion.**

In a transverse wave, as is set up in a stretched string when plucked, struck or bowed, a particle of the string displaced at right angles to the length of the string drags also its neighbouring particles at right angles due to its elastic connections with them. So the transverse wave results. Thus for the propagation of a transverse wave a medium having a considerable cohesion between molecules is necessary such that it can sustain tangential stresses. In other words, *a medium possessing a shearing elasticity is essential for the propagation of a transverse wave.* Ether is supposed to possess a high modulus of rigidity, and solids are the best media for the transverse wave propagation. Gases possess no shearing elasticity and as such cannot have any transverse vibration set up in them.

In a longituidnal wave a particle of the medium displaced in the direction of propagation pushes the particle in front of it, which in its turn again pushes the next one and so on. Thus here due to alternate compression and rarefaction, the volume is also altered and as such volume elasticity is involved in the process.

Now sound wave may be propagated through solids, liquids and gases. But the *gaseous medium possesses only volume elasticity* but no elasticity of shear as the constituent molecules of a gas have practically no cohesion between them. So the *sound waves which can be propagated through air must be longitudinal in nature.* Evidently the solids and liquids possessing considerable volume elasticity, the transmission of sound through them is in no way contradictory to the longitudinal nature of the sound waves.

Sound waves in air or any other medium, which comprise of pulses of compression and rarefaction, are *longitudinal* while radiant waves in ether, such as heat waves and light waves, are *transverse.* The electromagnetic waves used in wireless telegraphy and telephony are also instance of transverse wave motion.

(b) Progressive Waves.—

The longitudinal sound waves in air or in any other medium as well as the transverse waves like the water waves, or heat (or light) waves are characterised by the fact that a particular state of motion in each case is handed on from one part of the medium to the other with the passing of time and the wave form travels outwards with a definite velocity. That is why the general name for these waves is **progressive waves**. In a progressive wave motion, it is the disturbance that moves forward and not the particles of the medium. As will be apparent from the subsequent treatments the movement of each neighbouring particles of the medium begins a little later than its predecessor in a progressive wave. So there is always a difference in the state of motion (*i.e.* phase) between two neighbouring particles.

(c) Representation of transverse and longitudinal Wave motion.—In Fig. 7(*a*). *AB* represents a row of particles transmitting a transverse wave. As the wave passes, each individual particle of the medium will move up and down one after another at right angles to the line *AB* (as shown by the double-headed arrows) along which the wave is propagated.

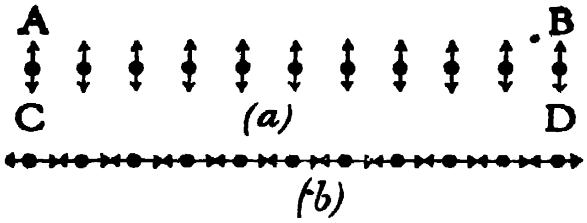


Fig. 7—Illustration of Transverse and Longitudinal Wave motion.

When a longitudinal wave passes such a row of particles, each particle will vibrate to-and-fro about a mean position along the line of propagation *CD* [Fig. 7(*b*)], and such motion of the particle will take place one after another in succession. The dot represents the mean position and the two arrows on either side the to-and-fro motion.

(d) Demonstration of wave motion.—

(i) Longitudinal Waves.—The propagation of longitudinal waves can be conveniently illustrated by a spiral spring suspended horizontally by threads from two parallel bars. *AB* and *A'B'*, as shown in Fig. 8. On pushing the end *A* of the spring suddenly forward the nearest turns are compressed and the compression is seen to move forward along the coil with a certain velocity towards the other end, each turn moving *forward* a little when the compression reaches it. This represents the state of the layer of air particles when a wave of compression travels through it. Again, if the end *A* be suddenly pulled outwards, the end turns will be separated from each other and this state of rarefaction, as we call it, is seen to be travelling along the coil to

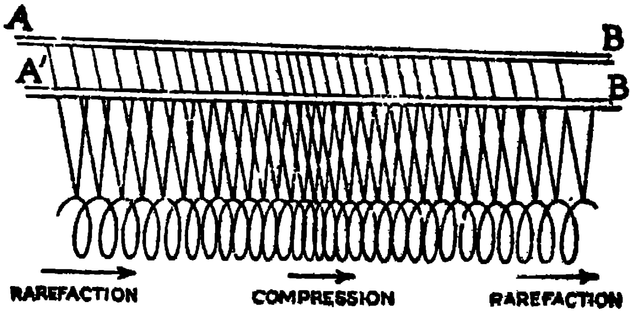


Fig 8

the farther end, each turn of the spiral moving *backward* a little when the extension reaches it. This represents the state of rarefaction travelling through air.

Thus if one end of the coil be alternately pushed forward and pulled outwards in a periodic manner, longitudinal wave motion of compression and rarefaction will be seen to travel along the spiral with a constant velocity. Each turn of the spring executes a to-and-fro movement in the line of propagation of the pulse, but it is not bodily transferred from one position to another. In the same way, at the time of propagation of sound through air, the *particles* of the air only move about their mean positions of rest, and are not *bodily transferred from one plane to another*. **It is the wave form, or a succession of compressed and rarefied pulses, that travels forward.** For this reason a blast of air is never felt to spread outwards even in the case of the loud report of a cannon.

(ii) Transverse Waves.—Fig. 9 illustrates a popular model for demonstrating transverse waves. A number of straight and parallel rods, each of which carries a small ball at the top, are placed at equal spaces apart in the same vertical plane in a stand. Each rod rests on an eccentric wheel and passes through a hole provided for it in a crosspiece held horizontally by the stand. All the eccentric wheels have a common spindle which can be rotated by a handle. When the handle is turned continuously, each ball undergoes a periodic up-and-down motion while a wave form travels from one ball to the next onwards from one side of the frame to the other as shown in

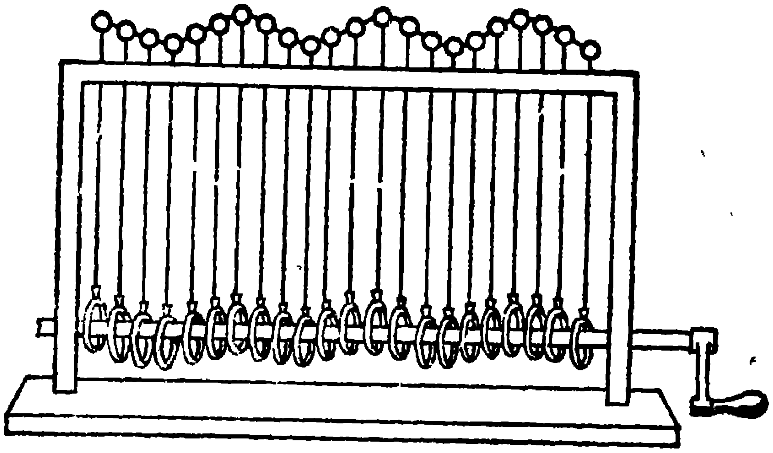


Fig. 9—Demonstration of Transverse Waves.

the figure. The motion of each ball being transverse to the line of propagation of the wave form, the waves produced are known as transverse waves.

8. Graphical Representation of a Sound Wave :—In a transverse wave, the movements of the particles and the line of motion of the waves are mutually at right angles to each other and they can be represented in the ordinate and abscissa respectively. But in a longitudinal wave, as in the case of sound, the displacements of the particles take place along the path of propagation of the wave and so a similar representation, as in the case of transverse wave, showing the *actual position* of displacements along the line of propagation, will result in tracing out a straight line along the line of propagation.

But if the *displacements* of the particles are shown in the ordinate against their *mean positions* in the line of propagation represented as abscissa, for the same instant of time, a very valuable graph is obtained, which is known as the *displacement-distance curve* or simply the *displacement* curve for the longitudinal wave. Since the distance traversed by a sound wave moving with a constant velocity is directly proportional to the time of traverse, the *displacement-time* curve will also be of the same form as the displacement-distance curve. For each particle, at its mean postion of vibration, a perpendicular is to be drawn to the line of propagation proportional to its displacement at the same instant of time. The perpendicular is to be drawn *above* the line of propagation if the particle moves to the *right* at the instant considered and *below* the line if it moves to the *left* at that instant. The displacement curve traced out in this way reveals all the properties, *e.g.* velocity, acceleration, state of compression or rarefaction, etc. of the particles in the medium.

In Fig. 10(*i*), a vibrating tuning fork *F* (which, it should be noted, always emits a simple harmonic type of sound wave) and a horizontal column of air in front transmitting the emitted sound are shown, where the points *A*, *C*, *E*, indicate layers of maximum *compression* and *B*, *D* indicate layers of maximum *rarefaction* considered at the same instant of time. Since the pressure variation at the maximum compression region is also minimum and that at the maximum rarefaction region is maximum, we may suppose that the sound waves actually consist of a series of alternate high pressure and low pressure regions. Or, *in other words, sound waves may be as well considered to be a type of pressure waves.*

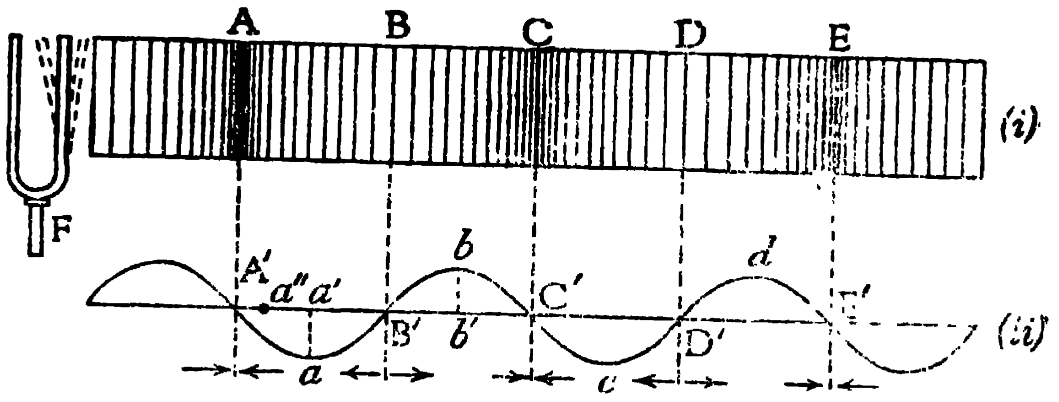


Fig. 10

Fig. 10(*ii*) depicts the conditions of all the air particles in the horizontal column above at some instant of time when the longitudinal waves emitted by the tuning fork passes onward, *i.e.*, from the left side to the right side of the figure. That is $A'aB'bC'c$ represents the displacement curve for the wave, where the line $A'B'C'D'$...gives the line of propagation of the wave.

To determine the actual position of a particle on $A'B'C'$...draw a normal to it at the undisplaced position of the particle to meet the displacement curve $A'a\ B'b$.... (1) If the normal is above $A'\ B'\ C'$... the displacement is towards the right-hand side and if below, the displacement is towards the left-hand side ; (2) the length of the normal is proportional to the displacement ; (3) the slope of the *displacement curve* at any point gives *the velocity of the particle* at the actual position of the particle corresponding to the point ; (4) the rate of change of slope represents *the acceleration of the particle.*

It should be noted that the slope of the *displacement curve* decreases as we pass from A' to a and at a it is zero. From a it increases again till it becomes maximum at B'. But the *rate of change of slope* increases from A' to a and becomes maximum at a. Then it decreases from a to B' where it is zero. So the velocity of a particle which is proportional to the slope decreases from A' to a' and is zero at a'. Then it increases again and becomes maximum at B'. The acceleration which is proportional to the *rate of change of slope*, however, increases from A' to a' and is maximum at a'. Then it decreases again as we pass on to B' and at B' it is zero.

Then actual position of a particle whose undisplaced position is, suppose, a' is given by a'', where $a'\,a'' = a'a$ and to the left of a', for the normal is below the line of propagation $A'B'C'$...

The displacement diagram $A'\,a\,B'\,b$...also clearly shows the states of compression and rarefaction of medium of propagation. For the particles to the right of A' and C' are displaced to the left while those to the left of them are displaced to the right, as have been shown by the arrows. So A' and C' are places of *maximum compression.* The particles to the right of B' are displaced to the right and those to the left of B' are displaced to the left and so B' is a place of *maximum rarefaction.* In the immediate neighbourhood of a' the particles on both side of it are equally displaced to the left, while in the neighbourhood of b' the particles on both sides are equally displaced to the right. This shows that a', b', etc. are places of *normal pressure.*

9. Some Important Terms :—

Frequency.—The number of complete vibrations made by a vibrating body in one second is called the frequency of the vibration. Thus, if n denotes the frequency of vibration, and T the periodic time of a vibrating body, we have $nT=1$; or, $n=1/T$.

Amplitude.—It is the maximum distance to which a vibrating body moves from its mean position of rest, *i.e.*, its maximum displacement during a vibration. In Fig. 124, General Physics, BC or BD is the amplitude of the oscillating bob of the pendulum. In Fig. 11, FE or GH is the amplitude.

Phase.—The phase of a vibrating particle at any instant is the state of the particle in regard to its position and direction of motion in the path of vibration at that instant. *Two particles moving exactly in the same way are said to be in the same phase* ; that is, particles which are at the same distance from their positions of rest, and are moving in the same direction, are said to be in same *phase.* Thus anything by which the direction of motion and displacement of a vibrating particle at an instant can be specified, will be a measure of its phase at that time.

A water particle A (Fig. 11) at the highest point of the crest of a water wave is in the same phase with the particle B at the highest point of the next crest, and no other particle between these two positions are in the same phase.

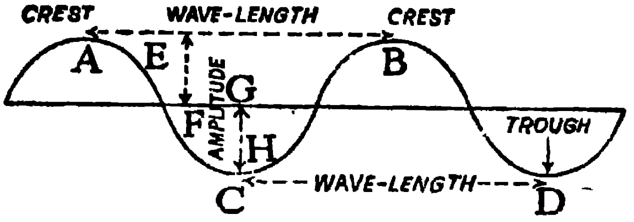


Fig. 11

Phase may be expressed in three ways—(*i*) *By the fraction of the period that elapses after the vibrating body passes through some standard position, say, the mean position of rest, in a given direction.*

Thus in Fig. 124, General Physics, the phase of the oscillating bob at C in the direction BC is expressed by $\frac{1}{4}T$, and at D in the direction BD by $\frac{3}{4}T$, when B is its mean position of rest.

(*ii*) *By the angle* (as θ in Fig. 12) *traced out by the generating point with reference to either of the co-ordinate axes* (*vide* Art. 12). Thus, the phase of the vibrating particle M is denoted by the angle θ (Fig. 12) traced out by the generating point P rotating along the circumference of the circle. Again, it will be observed that the phases 90° and 450° are the same, while phases 90° and 270° are opposite to each other.

(*iii*) *The phase of two points on a wave are also expressed by their path difference, i.e. by the fraction of a wavelength.* In Fig. 11, A and B are in the same phase the path difference being one wavelength and A, C are in opposite phases, their difference in path being half the wavelength.

Wavelength.—It is the distance through which the wave motion travels in the time taken by the vibrating body or any of the particles of the medium of propagation, to make one complete vibration. It can also be defined as the *least distance* between two particles in the same phase of vibration.

In the case of a *transverse wave* the *wavelength* is the distance between one *crest* (or *trough*) and the next *crest* (or *trough*) as AB or CD in Fig. 11. In the case of a *longitudinal wave* it is the length occupied by a pulse of *compression* together with a pulse *of rarefaction*, as AC or BD in Fig. 10.

Wave-front, Spherical waves and Ray.—The wave-front is defined as the envelope or trace drawn through all the points on a wave which are exactly in the same condition as regards displacement and direction of motion, *i.e.* in the same phase. Thus, a surface drawn along the crest of a water wave is a wave-front and also a surface drawn along troughs would be another wave-front.

In a homogeneous medium a wave generated at a point travels out in all directions around the point with the same velocity. At any instant of time, the wave motion lies upon the *surface of a sphere* whose centre is the generating point and radius equal to the product of the velocity and time. Such waves set up in a homogeneous medium due to the vibration of a point or a small sphere in it are called *spherical waves.* On this sphere the particles are all in the same phase of motion. This equi-phase surface is the wave-front at the time. At a very long distance from the source of disturbance, the spherical surface, over a limited region may be treated as *plane.* So the wave-front may be taken as plane, if the source of disturbance is at a very long distance. A normal drawn to the wave-front at any point indicates the direction of propagation of the wave and is known as a *ray.*

Period.—The period of vibration is the time taken by a vibrating body to execute one complete vibration.

10. Velocity of Sound Waves :—It is measured by the distance travelled over by a sound wave in one second. If λ denotes the wave-length of a sound wave and n the frequency of vibration, then, in one second there will be n complete vibrations and for each vibration the wave travels forward through a distance λ. Therefore the total distance travelled in one second $= n\lambda$. Hence, if V be the velocity of propagation of the wave, we have, $V = n\lambda$.

Otherwise, $\text{velocity} = \dfrac{\text{distance travelled}}{\text{time taken}}$, *i.e.* $V = \dfrac{\lambda}{T} = n\lambda$, $(\because nT = 1)$.

Example

A body vibrating with a constant frequency sends waves 10 cms. long through a medium A and 15 cms. long through another medium B. The velocity of the waves in A is 90 cms. per sec. Find the velocity of the waves in B.

Ans. Let V be the velocity of the wave in B. Since velocity = frequency × wave-length, we have $90=n\times 10$ where n is the frequency of vibration. $\therefore$ $n=9$ per second. Again, for the medium B, $V=n\times 15$ (n being constant in both the cases) $=9\times 15=135$ cms. per sec.

11. Simple Harmonic Motion :—If a motion is repeated at regular intervals of time, the motion is said to be **periodic.** Thus the motion of a particle, continuously moving round a circle, or an ellipse, in constant time, is said to be *periodic* and in this sense, the motion of the earth round the sun is periodic.

A vibratory or oscillatory motion is a periodic motion that reverses in direction. It has a position of rest at which the reversal in direction takes place. The motion of a pendulum is oscillatory.

The simplest type of vibratory motion is that executed along a straight line by a particle moving to-and-fro. If this vibratory linear motion be such that the acceleration of the moving particle is always directed towards a fixed point in its path and is always proportional to the displacement of the particle from that fixed point, the motion is called a Simple Harmonic Motion (also written as S.H.M.).*

To understand the nature of a particle executing simple harmonic motion, let us imagine a particle P (Fig. 12) moving round a circle with uniform speed. The particle P is called the *generating point* and the circle $XYX'Y'$ round which it moves is known as the *circle of reference.*

Let PM be a perpendicular dropped from P on any fixed diameter XX' of the circle. Now as P moves once round the circle in the direction of the arrow and makes a complete revolution, the foot M of the perpendicular, PM, moves to-and-fro along the diameter

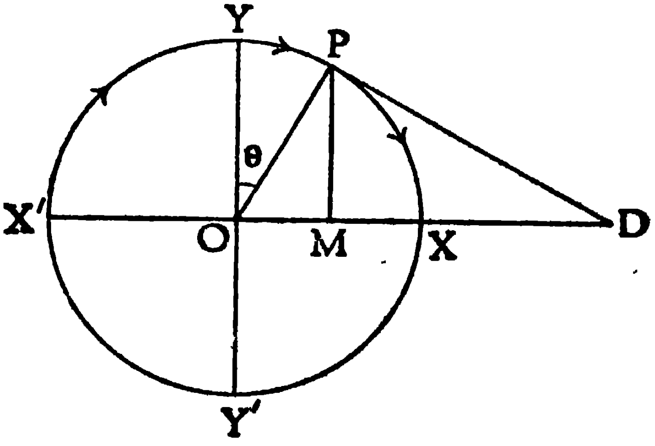


Fig. 12

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* For details of simple harmonic motion, composition of S.H.M.'s and the displacement curves see the chapter on Simple Harmonic Motion in General Physics (Chapter VI).

XX' from the starting point M up to X, then back to X' and therefrom once again to the starting point M.

This to-and-fro movement of M about O along XX' continues as P moves round the circle with a uniform speed. It can be proved that the acceleration of M is always directed to the mean position O and is proportional to its displacement measured from O. The motion of M is thus a simple harmonic motion. So, **if a point moves with a constant speed along the circumference of a circle, and if a second point moves along a fixed diameter of the circle so as to be always at the foot of the perpendicular drawn from the first point on the said diameter, then the motion of the second point is Simple Harmonic.**

12. Equation of a Simple Harmonic Motion :—Let P be a point which is travelling in the direction of the arrow round the circumference of a circle ($XX'P$) of radius OP $(=a)$ with a uniform speed, and let XOX' and YOY' be two diameters of this circle at right angles to each other (Fig. 12). Let T be the period, *i.e.* the time for one complete revolution of P, and ω its angular velocity, *i.e.* the angle through which the radius OP revolves in 1 second. Then

$$\omega T=2\pi\ ; \quad \text{or,} \quad T=2\pi/\omega \qquad \ldots \qquad \ldots \quad (1)$$

As P moves round the circle, the point M, the foot of the perpendicular drawn from P on XOX', executes a S.H.M., and the frequency of vibration of M is the same as that of the point P. Hence the frequency of M is $n=1/T$.

Let the time be counted from the instant when M is passing through its mean position O in the positive direction (*i.e.* from left to right, when it is crossing the line (YOY'). Let t be the time which has elapsed since M was last at O, *i.e.* time taken by OP to make an angle θ with OY. The angle θ is called the *phase* of the vibrating particle M at that instant.

Then, $\theta=\omega t$. We have, $OM/OP=\cos\ POM=\cos\ (90^\circ-\theta)=\sin\theta$.

$\therefore$ The displacement of P (i.e. OM) is $x=OP\sin\theta=a\sin\theta$ ... (2)

$$=a\sin\omega t=a\sin\frac{2\pi}{T}t$$

$$=a\sin 2\,\pi nt, \text{ where } n \text{ is the frequency.}$$

N.B.—If time is recorded from an instant when the generating point is P on the left of Y (*i.e.* M is also on the left of O) to such an extent that the generating line OP makes an angle α with OY, then $\theta = \omega t - \alpha$. That is, $x = a \sin(\omega t - \alpha)$. This $-\alpha$, which is the *phase at the commencement of time, is called the* **epoch.** The *sign* of the *epoch* may be positive or negative depending upon the *position* of the particle from which the time is measured.

The greatest value of $\sin \theta$ is unity, hence the maximum value of x is a, which is therefore, the **amplitude of vibration.** Thus the displacement has a positive maximum value at X when $\theta = 90°$ and a negative maximum value at X' when $\theta = 270°$.

The displacement of a body executing a S.H.M. is always given by an equation like (2).

13. Velocity and Acceleration in S. H. M. :—

Velocity.—The velocity of M at any instant along XX' is the same as the component of the velocity of P parallel to XX' (Fig. 12). Let PD be the tangent at P, meeting $X'X$ at D. The linear velocity of P at any instant is equal to v and is along the tangent PD. The component of v parallel to XX', i.e. in the direction $OD = v \cos PDO = v \sin POD = v \sin(90° - \theta) = v \cos \theta$. ... (3)

Thus, the velocity of M is zero at X, where $\theta = 90°$ $(\cos 90° = 0)$, and also at X', where $\theta = 270°$. The velocity is a maximum at O, where $\theta = 0°$, and $\cos \theta = 1$ (the maximum value of $\cos \theta$), and also it is a maximum in the negative direction where $\theta = 180°$; and after a complete swing, when $\theta = 360°$, the velocity is again a maximum in the positive direction. Thus, *in one complete oscillation* the velocities of M are *zero at the ends* of the swing *i.e.* at X and X' and *maximum* when passing *through the origin O*. At O the velocity of M is parallel, and so equal, to that of P.

Acceleration—The *generating point* P moving with constant speed round O has an acceleration v^2/a directed towards O, a being the radius of the circle of reference (Fig. 12). The acceleration of M is the component of the acceleration of P along OX. Hence the direction of the acceleration f of M is towards O and is given by,

$$f = \frac{v^2}{a} \cos POM = \frac{v^2}{a} \sin \theta \qquad \ldots \quad (4)$$

But because v is the linear velocity of P, which describes the distance $2\pi a$ in time T, we have $2\pi a = vT$.

So, $$v=\frac{2\pi}{T}.\ a=\omega a\ ;\ \text{or, } v^2=\omega^2 a^2.$$

$\therefore$ From (4), acceleration f of $M=\omega^2 a \sin\theta=\omega^2\times$ *displacement*...(5)

Hence, $$\frac{\text{acceleration of } M}{\text{displacement of } M}=\omega^2=\frac{4\pi^2}{T^2}=\text{a constant.}$$

Thus, *when a particle is describing a* S.H.M., *the ratio of the acceleration to the displacement is constant* ; *that is, when a particle M executes a S.H.M., its acceleration is proportional to its displacement OM and is directed towards a fixed point O in the line of vibration.*

The acceleration of M depends upon the sine of an angle just as the displacement does, and so the maximum and minimum values of acceleration occur exactly at times as those of displacement.

14. Characteristics of S.H.M. :—(*i*) The motion is periodic, (*ii*) It is a vibratory (to-and-fro) motion ; (*iii*) The motion takes place in a straight line ; (*iv*) *The acceleration of the body executing a S.H.M. is proportional to its displacement and is directed towards a fixed point in the line of vibration.*

15. The Displacement curve of a S.H.M. :—The displacement x of a particle executing a simple harmonic motion is given by the equation $x=a\sin\omega t$. If we plot a curve to show the relation between x and t, the curve will be a *sine curve*. Fig. 13 represents the displacements curve of a point M starting from O and moving with S.H.M. along YOY' due to the point P moving from X with a uniform speed along the circumference of the circle having O as centre in the direction XY as shown in the figure. Divide the circumference into any number of equal parts, say, eight, and draw straight lines through the points of divisions P, Y, P', etc. parallel to XOX'. If AB represents the period T divide it into 8 equal parts. The time $(T/8)$ taken by P to

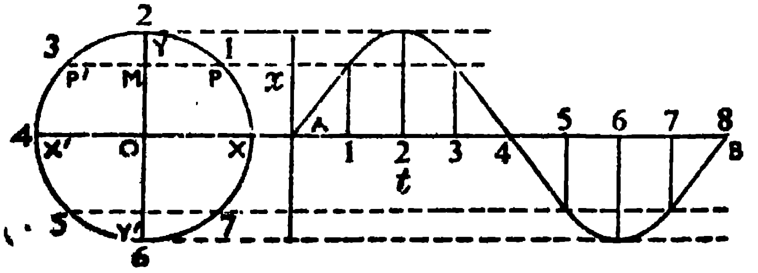

Fig. 13

move through each part of the circumference will then be represented by each division of AB. Draw ordinates at the point 1, 2, 3, etc. that are equal to the displacements OM, OY, etc. In plotting the distances, the points below O should be taken as of sign opposite to those above O. Now joining the tips of these ordinate lines, the displacement curve is obtained which is identical with the well-known ***sine curve.***

N.B.—Each particle in the medium transmitting a longitudinal sound wave executes a S.H.M. with time. So the time-displacement curve for each particle in the medium will also be a sine curve. The displacement, however, is in the line of propagation of the sound. The motion of the succeeding particles lying on the line of propagation reckoned at the same instant of time will differ in phase from particle to particle. If the ***displacements*** of the particles at the same instant are plotted as the ordinate against their ***distances*** as ***abscissa*** (though they are in the same straight line), the graph will also be a sine curve. This type of displacement curve is known as the ***distance-displacement*** curve.

16. Examples of S.H.M. :—The to-and-fro movement of one prong of a vibrating tuning fork, the movement of a point in a stretched string when the string is plucked sideways, and also the motion of the bob of a simple pendulum oscillating with a small amplitude, are some familiar examples of Simple Harmonic Motion.

17. Importance of Simple Harmonic Motion :—The Simple Harmonic Motion is of great importance in the study of sound since a vibration of this type gives only the sensation of a pure tone. Any other kind of vibration gives rise to a compound note which is composed of two or more simple tones. The importance of a tuning fork in sound lies in its unique property of giving a pure tone when sounded. All other known sources of sound give out, when sounded, complex notes which contain a number of tones. So when a sound of single frequency is required, a suitable tuning fork is used.

18. Sound is a Wave Motion :—Sound is produced by the ***vibration*** of a sounding body, and the assumption that it is conveyed to the ear by means of waves is based on the consideration that the characteristics of wave propagation do also apply to the case of transmission of sound.

(1) A wave takes time to travel from one place to another.

Sound also takes time to travel from one place to another, *i.e.* it has definite velocity.

(2) A wave requires a medium to pass through. Sound also requires an elastic medium to pass through.

The medium as a whole does not move but only allows the sound to pass through it.

(3) Waves are, reflected or refracted obeying definite laws.

Sound is also reflected or refracted according to the same laws.

(4) Two sets of waves meeting each other at the same place of a medium at the same time may destory the effect of each other under certain conditions. This is the phenomenon of **Interference.**

Sound also shows interference, as in the phenomena of beats, stationary vibrations, etc.

(5) Sound can bend round an obstacle. Moreover, sounds of different acuteness or pitch show this effect by different amounts. The phenomenon is known as **diffraction**. Diffraction is possible owing to the wave .character of sound. Since sound of different acuteness have different wavelengths, the amount of diffraction caused by them should be different.

(6) A wave of condensation started from a source has actually been photographically detected by R. W. Wood. The reality of ***secondary wavelets,*** first conceived by Huygens in his wave theory, has thus been proved.

(7) The phenomenon of polarisation is shown by transverse waves only. Light waves being transverse shown the phenomenon of polarisation but the fact that sound waves fail to show the phenomenon of polarisation prove that the vibration in this case is longitudinal and not transverse.

18 (a). Photography of Sound waves :—

Töepler in 1867 suggested the possibility of photographing the sound waves taking help of the inhomogeneity produced in the transparent air medium near a spark gap producing a sound. This method is known as the **schlieren** (or spark shadow) **method**. Many subsequent workers in the field applied this method to photograph the sound waves. In this method light from a spark gap G is focussed by an achromatic lens L of good quality on to the edge of a screen [Fig. 13(a)]. Another spark gap S producing sound is placed between the lens and the screen and a camera behind the screen is focussed on to S. The sound spreads out from S in a spherical pulse. The pulse is a thin layer of air having a very high density and refractive index (due to compression) relative to the surrounding air. In the absence of the sound wave, the field of the camera remains dark even when the gap G sparks, because direct light is cut off by the screen. But if sound is produced at S, light from G is deviated by the highly compressed shell of air (produced by the acoustical wave-front) and appears on the camera plate as a bright line on its dark background. For the success of the experiment it is necessary to produce sound in S, *a fraction of a second earlier than the light spark in G*. This is so done by having the two gaps S and G in the same circuit with a condenser C in parallel with the gap G. The light spark is then delayed for a while depending on the value of the capacitance.

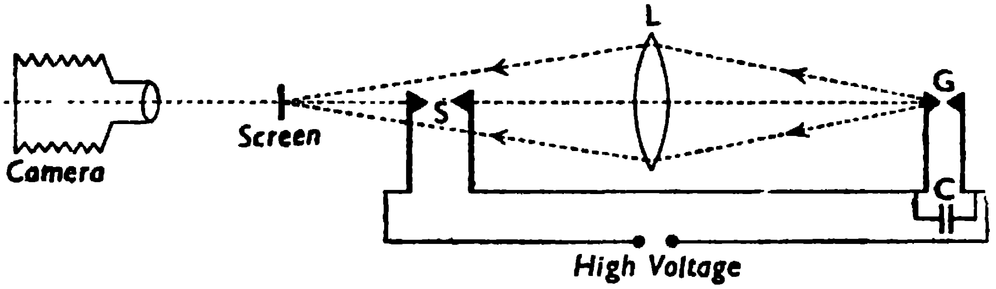

Fig. 13(a)

The method has also been used to demonstrate the reflection of a sound pulse. This method with certain modifications has been adopted by Davis and Kaye in the National Physical Laboratory for directly photographing the sound waves. By this method photographs of shock-waves due to projectiles moving at supersonic speeds and of explosion waves from the mouth of a tube have also been successfully taken.

19. Expression for Progressive Wave Motion :—Assuming the motion of any particle on a progressive wave to be simple harmonic, the displacement of the particle at any instant t is given by either, $y=a \sin \omega t$ or $y=a \cos \omega t$, where a represents amplitude of vibration of the particle (*i.e.* the amplitude of the wave), ω = circular frequency (or the pulsatance or the angular velocity) of the wave $=2\pi n$, n being the frequency of the vibrating particle and ωt is the phase of that particle. Evidently in the case of a transverse wave, the displacement, y, will be normal to the direction of propagtion and in the case of a longitudinal wave it will be along the direction of wave propagation. In the subsequent treatments the wave is supposed to be propagated along the X-axis and in the case of a longitudinal progressive wave the displcement, y, which is along the X-axis should not be confused with the displacement along the co-ordinate axis Y.

In the deduction of above displacement equation we suppose that vibrating particle starts its motion from the mean position. But if a particle starts vibrating from a distance x from the mean position in an exactly similar way as above, its phase is retarded by an initial phase or epoch corresponding to this displacement x. Since for a displacement of λ, the phase changes by an angle 2π, the phase change corresponding to the displacement x is $\alpha=\frac{2\pi x}{\lambda}$. Hence the phase of the particle or the wave at the instant t is given by $(\omega t-\alpha)=\left(\omega t-\frac{2\pi x}{\lambda}\right)$.

So the displacement equation at the instant is

$$y=a\sin(\omega t-\alpha)=a\sin\left(\omega t-\frac{2\pi x}{\lambda}\right) \qquad \ldots \quad \ldots \quad \ldots \quad (1)$$

(Considering only the sine type of the wave)

Again as $\omega=\frac{2\pi}{T}=2\pi n$ and $n\lambda=V$, the velocity of propagation of the wave, we can write,

$$y=a\sin\left(\frac{2\pi}{T}t-\frac{2\pi x}{\lambda}\right)=a\sin 2\pi\left(\frac{t}{T}-\frac{x}{\lambda}\right) \qquad \ldots \quad \ldots \quad (2)$$

$$\text{Or, } y=a\sin\left(\frac{2\pi V}{\lambda}t-\frac{2\pi}{\lambda}x\right)=a\sin\frac{2\pi}{\lambda}(Vt-x) \qquad \ldots \quad \ldots \quad (3)$$

$$\text{Or, } y=a\sin\left(\omega t-\frac{2\pi}{V/n}x\right)=a\sin\left(\omega t-\frac{2\pi nx}{VT}\right)$$

$$=a\sin\left(\omega t-\frac{\omega x}{V}\right)$$

$$=a \sin \omega \left(t - \frac{x}{V} \right) \quad \ldots \quad \ldots \quad (4)$$

Equations (1), (2), (3) and (4) represent only different forms of expression for the displacement of a particle on a progressive wave. They represent as well the different forms of equation of a progressive wave either longitudinal or transverse.

If, instead of the sine form, we start with the cosine form of the displacement equation, exactly similar equations can be obtained, the sine functions in (1), (2), (3) and (4) being only replaced by the cosine functions.

Notes :—(A) In order to show that the wave represented by the above expression is *progressive* in nature, let us consider equation (3). Suppose we consider the form of the wave after an interval δt. So t is to be replaced by $(t+\delta t)$. Again as V is the velocity of the wave, the distance x is to be replaced by $(x+V\delta t)$. Putting these values in eqn. (3), we get, $y=a \sin \frac{2\pi}{\lambda} [V(t+\delta t)-(x+V\delta t)]=$

$a \sin \frac{2\pi}{\lambda}(Vt-x)$ again. So the disturbance at a place is repeated at a time δt later and at a position $V.\delta t$ farther away from the source. So the form of the wave progresses as it does with time. Hence the wave is progressive in nature.

(B) The equation deduced above represents a progressive simple harmonic wave in which no energy is dissipated during its propagation through the medium. But when an actual wave is propagated through a medium, its amplitude will gradually diminish due to the resisting effect of the viscous medium. A part of the energy will also be lost due to the alternate heating of the compressed layers. Absorption of energy in the medium always plays an important part in decreasing the amplitude of vibration. Due to the above reasons the amplitude a in the above expressions of the progressive wave will be modified as ae^{-bx} where b is a constant. Thus the wave gradually attenuates as a result of this gradual decrease of amplitude. Such a wave whose amplitude gradually attenuates is known as an **attenuated or damped progressive wave.**

20. Velocity and acceleration of a particle on the progressive wave :—

The displacement equation of a vibrating particle at a distance x from the mean position at an instant t, is given by,

$$y=a\sin\omega\left(t-\frac{x}{V}\right).$$

Hence the rate of change of displacement or the velocity of the same particle at that instant is given by,

$$v=\frac{dy}{dt}=a\omega\cos\omega\left(t-\frac{x}{V}\right)\quad\ldots\quad\ldots\quad(1)$$

This velocity (v) of the vibrating particle (*particle velocity*) is quite different from the velocity of propagation V of the wave (*wave velocity*), the latter being solely dependent on the nature of the medium, whereas $\frac{dy}{dt}$ is, as apparent from eqn. (1), a function of t and x.

The slope of displacement curve is given by,

$$\frac{dy}{dx}=-\frac{a\omega}{V}\cos\omega\left(t-\frac{x}{V}\right)\quad\ldots\quad\ldots\quad\ldots\quad(2)$$

Thus from (1) and (2),

$$\frac{dy}{dt}=-V\frac{dy}{dx}.$$

So the *velocity of the vibrating particle on a progressive wave is V times the negative slope of the corresponding displacement curve.*

Again the acceleration of the vibrating particle at the instant, t, is given by,

$$\frac{d^2y}{dt^2}=-a\omega^2\sin\omega\left(t-\frac{x}{V}\right)$$

$$=-\omega^2\times y\quad\ldots\quad\ldots\quad\ldots\quad(3)$$

The curvature of the displacement curve can be related to the above acceleration in the following manner. The curvature, we know, is given by $\frac{d^2y}{dx^2}$ (neglecting minute terms in the expression ; *vide* any standard book on Differential Calculus)

$$\therefore \text{ From (2), } \frac{d^2y}{dx^2} = -\frac{a\omega^2}{V^2}\sin\omega\left(t-\frac{x}{V}\right)$$

$$= -\frac{\omega^2}{V^2}y \qquad \dots \qquad \dots \qquad \dots \quad (4)$$

Thus from eqns. (3) and (4), we have

$$\frac{d^2y}{dt^2} = V^2\frac{d^2y}{dx^2} \qquad . \qquad \dots \qquad .. \quad (5)$$

Hence the *acceleration of the vibrating particle is V^2 times the curvature of the displacement curve.*

The equation (5) derived from the above simple considerations represents the *differential equation of a wave motion.*

21. Energy of Progressive Wave :—If we consider *unit volume* of the medium through which a progressive wave is transmitted, its mass is equal to the density, ρ of the medium. Since the velocity of the particles of the medium is $\frac{dy}{dt} = a\omega\cos\omega\left(t-\frac{x}{V}\right)$, the kinetic energy per unit volume of the medium at any instant, t, is given by,

$$W_k = \tfrac{1}{2}\rho.\left(\frac{dy}{dt}\right)^2 = \tfrac{1}{2}\rho a^2\omega^2\cos^2\omega\left(t-\frac{x}{V}\right) \qquad \dots . \quad (1)$$

In a simple harmonic motion, the sum of the potential and kinetic energies of the vibrating body is constant throughout the motion and at the extremities of the path of vibration the energy is wholly potential whereas at the mean position it is wholly kinetic. So the maximum value of either of these energies represents the total energy at any position.

So the maximum kinetic energy per unit volume,

$$W_k\ (\text{max.}) = \tfrac{1}{2}\rho a^2\omega^2,\ \left[\because \cos^2\omega\left(t-\frac{x}{V}\right)(\text{max.}) = 1\right]$$

$$= \text{the total energy per unit volume.}$$

This is commonly known as the *energy density* of the medium.

22. Intensity of sound wave :—*The intensity of a sound wave is measured by the average rate of flow of energy through a unit area placed normal to the direction of propagation of the sound wave.* To calculate the intensity of a sound wave let us consider a cylinder (Fig. 14) in the medium of cross-sectional area unity and of length V, where V is the velocity of the sound wave, the axis of the cylinder being parallel to the direction of propagation. The energy associated with the wave that crosses the section at A at the beginning of a particular second will reach B at the end of that second. During this interval of one second the whole cylinder AB will be filled with sound energy. Since the volume of the cylinder is $1\times V=V$ and the energy density is $\frac{1}{2}\rho a^2\omega^2$, the total energy contained in AB is $\frac{1}{2}\rho a^2\omega^2 V$. This energy has evidently crossed the section at A in one second. So this is the rate of flow of energy through a unit area of the medium which is known as the ***intensity*** of a sound wave or its ***energy current***.

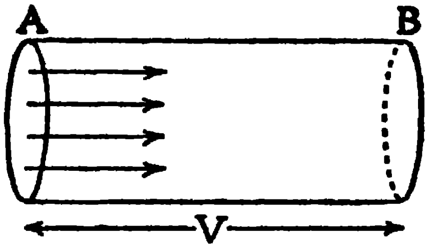

Fig. 14

Thus intensity $I=\frac{1}{2}\rho a^2\omega^2 V=2\pi^2 n^2\rho a^2 V$. ($\because \omega=2\pi n$).

Hence the intensity is directly proportional to (1) the density of the medium, (2) the wave-velocity, (3) the square of the frequency and (4) the square of the amplitude of the wave.

Pressure of Sound Waves.—Whenever sound waves are incident on a solid wall they get reflected from the same and at the same time they exert a definite pressure on the wall. If the isothermal condition prevails during this process of impingement of the waves, the pressure exerted is found to be equal to the energy density. Thus $p=\frac{I}{V}$ where I=intensity of the sound wave and V its velocity. As the process is not generally isothermal as has been suggested by Laplace, if the adiabatic condition is supposed to hold good, the pressure of sound waves is found to be equal to,

$$p=\frac{\gamma+1}{2}\cdot\frac{I}{V},$$

where γ is the ratio of the two specific heats of the gaseous medium.

Questions (Chapters I & II)

1. Explain the mechanism of the propagation of sound waves. How do you demonstrate that a medium is necessary for the propagation of sound ?

[cf. Utk. U. 1953]

2. Clearly distinguish between a transverse and a longitudinal wave. How do you infer that sound waves are longitudinal in nature ?

[*cf.* Del. U. 1958 ; **Nag. U. 1951**]

3. Outline briefly the properties of longitudinal waves. [E.P.U. 1953]

4. Distinguish between transverse and longitudinal waves. Which of these waves cannot exist in a gas and why ? [*cf.* Bom. U. 1954]

5. Deduce a mathematical expression for a progressive wave.

Calculate the acceleration of a vibrating particle on this wave.

How can you show that the wave equation, you deduce, indicates a progressive wave ?

6. Calculate the energy density of a progressive wave motion. Hence find an expression for the intensity of sound.

7. Define the intensity of sound and clearly bring out the factors on which it depends.

8. Show that the particles velocity $\frac{dy}{dt}$ in the case of a plane progressive wave is given by $\frac{dy}{dt} = -C\frac{dy}{dx}$. Hence derive the differential equation of a wave motion. [Raj. U. 1951]

9. What reasons are there for believing that sound is conveyed by wave motion ? [Utk. U. 1951 ; C. U. 1953]

CHAPTER III

REFLECTION AND REFRACTION OF SOUND

23. Sound and Light Compared :—When a disturbance occurs in open air, sound waves proceed radially outwards in all directions from the source as the centre, just as light radiates out from a centre in all directions around it. But there is a fundamental difference between the methods of their propagations. Sound is propagated in the form of longitudinal waves, whereas light is propagated in the form of transverse waves. The term *rays of light* is used to express the directions in which light waves proceed from a source. Similarly, any line, along which a sound wave is propagated, may be called a *sound ray*. These terms are, however, only a convenient way of representation and have no reference to the actual modes of propagation. Light waves are reflected from plane and spherical surfaces obeying certain laws ; sound waves are also *reflected* according to the same law—*viz*. that the angles of incidence and reflection are equal and that the incident and reflected rays and the normal at the point of Incidence are in the same plane ; but conditions under which reflections of these two waves take place are widely different because the lengths of light waves and the lengths of sound waves are of widely different orders. It must also be marked that light can travel through vacuum whereas sound waves require a material medium for their transmission.

Under favourable conditions sound waves can also be *refracted* like light-waves, and there may be also *interference* due to two waves of sound as due to two appropriate waves in the case of light. Moreover, the light rays being transverse in nature can be *polarised* whereas the longitudinal sound waves can under no circumstances be polarised.

Light from a luminous source is usually complex being composed of light rays of simple colours mixed up in some proportions. Sounds emitted by common sources are also complex. The quality of a sound depends upon the number of simple tones present in the sound, their order, and also on their relative intensities. The colour of light, say, red or blue, depends upon the frequency of the waves produced ; similarly, the pitch of a sound depends on the frequency of vibration produced.

Sound waves are detected by the *auditory* nerves of the ear while light waves are detected by the optic nerves.

24. **Reflection of Sound** :—In order that appreciable reflection of a wave may take place from any surface, the area of the surface should be fairly large in comparison with the wavelength of the wave incident on it. Sound waves are much larger than light waves. The lowest audible note has got a wavelength of about half-an-inch and the highest audible note has got a wavelength of 32 ft.—for example the wavelength corresponding to the note *C* is nearly 4 ft. whereas the wavelengths of visible light are included between 16 and 30 millionths of an inch. Consequently, it is evident that larger surfaces are required for complete reflection of sound waves than are required for light waves. On the other hand, the sound waves being larger do not require the reflecting surface to be so smooth as may be required for light waves. For this reason, a brick wall, a wooden board, a row of trees, or a hillside, all serve as reflectors of sound waves. The following experiment will illustrate the reflection of sound waves.

(1) **Reflection at a Plane Surface.**—Fix a large plane wooden board *AB* vertically and place a long hollow tube T_1 with its axis pointing to some point *C* on the board making a definite angle with the plane of the board (Fig. 15). Now place another similar tube T_2 with its axis pointing towards *C*. Hold a small watch just in front of the tube T_1 and put your ear at the end of the reciving tube T_2 which is turned with the point *C* as centre in all possible positions till the sound of the watch appears maximum, a board *S* being placed between the tubes to cut off the direct sound.

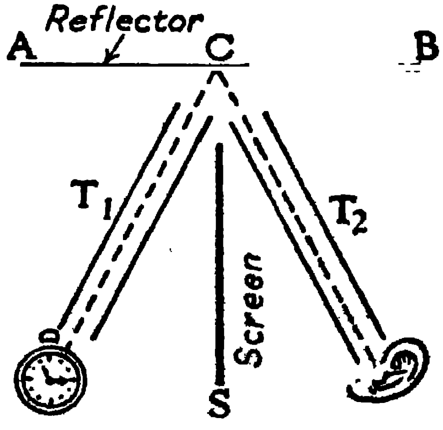

Fig. 15—Reflection of Sound

It will be found that sound obeys the same laws of reflection as light, viz.—

(*i*) The angle of reflection is equal to the angle of incidence; that is, the axes of T_1 and T_2 make equal angles with the normal to AB at C.

(*ii*) The reflected sound ray, the incident ray, that is the axes of T_1 and T_2, and the normal at the point C of incidence on the board lie in one plane.

(2) **Reflection by Concave Surface.**—Two large concave spherical mirrors M and M' are placed co-axially on a table facing each other (Fig. 16). A watch is placed at the focus of one of them, M. The sound waves proceeding from the watch being reflected from the first mirror will fall on the second mirror, and will be converged at the focus of M', where the sound waves can be received by the ear E by means of a funnel tube. The ticking of the watch will be distinctly heard at the focus, and it will be practically inaudible at other points, or at the same point, by displacing the mirror a little.

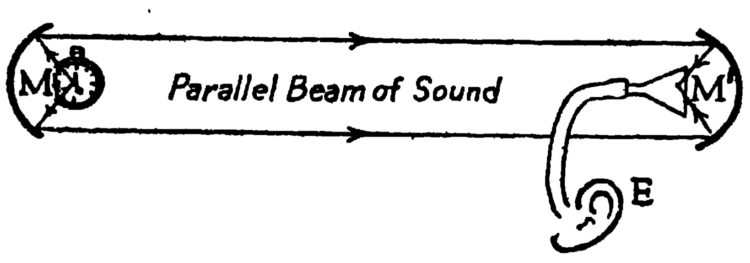

Fig. 16

25. **Practical Examples** :—The principle of reflection of sound is applied in *speaking tubes, ear-trumpets, doctors' stethoscopes*, etc. In these cases the sound waves are reflected repeatedly from side to side of the tubes (Fig. 17). Here the sound waves cannot spread. So the energy of the waves, instead of being distributed through a rapidly

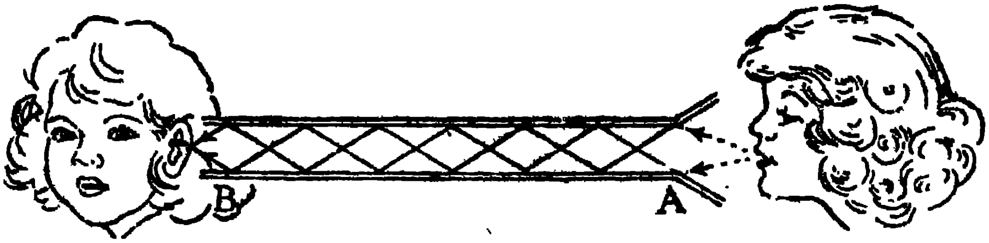

Fig. 17—Reflection in a Tube.

increasing space, remains more or less confined within the limits of the tubes, and so an ear, placed at the distant end, can hear the sound distinctly.

Reflection in an Auditorium.—Sometimes the rooms and halls of buildings with arched ceiling serve as reflectors of sound waves. The walls of large halls also often reflect the sound waves which interfere with the words of a speaker, and the effect is confusing. This may be avoided by the hanging up of screens and curtains which are bad reflectors for sound waves. The interference is also avoided to a certain extent when the hall is filled with an audience whose bodies serve to damp the sound, and for this reason it is often easier to speak before a large audience than in an empty hall. On the other hand, it has been found that in the open air, where there is no echo, it is rather difficult to make oneself heard to a large crowd, and this is not so in a big hall as a certain amount of reflection helps in increasing the volume of sound. It has been practically seen that the effect is better when echo is heard nearly about 2 seconds after the original sound.

In churches there is often a concave reflecting board above the pulpit which reflects the sound made by the preacher down to the congregation.

If a source of sound is placed at the focus of a parabolic reflector, the sound rays are rendered parallel whereby they can reach great distances.

It is known to everyone that the hollow of the hand held at the back of the ear in a curved way serves to concentrate the sound waves and thus helps one to hear a distant sound.

26. Echo :—*When sound returns back after reflection from an obstacle, it is called an echo.* A speaker's own words at a place are often repeated by reflection from a distant extended surface, such as a distant cliff, a row of buildings, a row of close trees, etc. The phenomenon is known as an *echo* and is a very familiar example of the reflection of sound waves. A sound made near a wall, or a hill-side will be reflected and heard as two distinct sounds, provided the distance between the observer and the reflecting surface is large enough to allow the reflected sound to reach him without interfering with the direct sound. The impression of a sound persists for about $\frac{1}{10}$th of a second alter the exciting cause ceases to exist. This period may, therefore, be regarded as *the period of presistence of sensation of sound.* Taking the velocity of sound in air to be 1100 ft. per second approximately, a sound wave can travel 110 ft. in $\frac{1}{10}$th of a second. So, in order that the echo of a sound may be distinctly heard, the reflecting surface should be at a distance not less than 55 ft. from an observer, so that the reflected sound wave may reach the ear of the observer not earlier than $\frac{1}{10}$th of a second after the first sound is heard.

Velocity of Sound by the Method of Echo.—*By means of echoes it is possible to obtain a rough estimate of the velocity of sound.* Suppose you stand some hundreds of yards from a hill and try to find out the time between a shout and its echo. If you are 500 yds. from the hill and the echo comes back in 3 seconds, the sound has travelled twice 500 yds. or 3000 ft. in 3 seconds and therefore has travelled 1000 ft. in a second. So the velocity of sound is 1000 ft. per second.

Series of Echoes.—Suppose a person at A is placed between two reflectors B and C situated at a distance of 330 ft. from each other, so that the distance AB is 110 ft. and AC, 220 ft. Now if a pistol is fired at A, the wave travels to B, is reflected and comes back to A in $\frac{2\times110}{1100}=\frac{2}{10}$ second. This reflected wave then travels to C and comes back to A in $\frac{2\times220}{1100}=\frac{4}{10}$ second after the first echo, *i.e.* $\frac{6}{10}$ second from the beginning. The wave again travels to B and is reflected. This goes on.

But in the beginning the sound wave also directly travels to C, and comes back to A, after reflection in $\frac{4}{10}$ second. It then goes to B and comes to A, $\frac{2}{10}$ second later and so on. So we get a series of echoes resulting from B, in $\frac{2}{10}$, $\frac{6}{10}$, $\frac{8}{10}$, $\frac{12}{10}$, $\frac{14}{10}$ etc. second, and another series of echoes resulting from C in $\frac{4}{10}$, $\frac{6}{10}$, $\frac{10}{10}$, $\frac{12}{10}$, $\frac{16}{10}$, etc. second.

Articulate Sounds.—In the case of *articulate* sounds, however, the distance of the obstacle should be at least twice, that is, 110 ft. instead of 55 ft. as observed above. It is so, because a person cannot pronounce more than 5 syllables distinctly in one second and the ear also cannot recognise them if more than 5 syllables are pronounced in one second. If a person pronounces *a*, he takes $\frac{1}{5}$th of a second for it by which time the sound can travel through 220 ft. taking the velocity of sound to be 1100 ft. per second. So, an echo will be heard only if the reflecting surface be at least at a distance of 110 ft. from the observer. If the person pronounces any 5 syllables, say, *a*, *b*, *c*, *d* and *e*, and if the reflecting surface be at a distance of 110 ft. then he will hear the echo of the first syllable just as he is about to pronounce the second syllable *b*. Similarly, the echoes of *b*, *c*, *d*, would come to him by $\frac{1}{5}$th of the second just as he is about to pronounce the next one. So only the echo of the last syllable will be distinctly heard. This echo which enables us to hear only one syllable distinctly is called a **mono-syllabic** echo. If the reflecting surface be at a distance of two or three times, the echo will be **di-syllabic** or **tri-syllabic** and so on. Evidently, if the distance be n times 110 ft., then the echo of the last n syllables can be heard. Echoes which enable us to hear two or more syllables are sometimes called **poly-syllabic** echoes.

Examples

(1) *A man stationed between two parallel cliffs fires a gun. He hears the first echo after two seconds and the next after 5 secs. What is his position between the cliffs and when he hears the third echo?*

Ans. Let V be the velocity of sound in air, x the distance of one of the cliffs from the man and y the distance of the other cliff. Then, if the first echo be heard after two seconds, $2=\frac{2\times x}{V}$; or, $V=x$.

The sound wave will also be reflected by the other cliff and come back after 5 seconds. $\therefore$ $5=\frac{2\times y}{V}$; or $V=\frac{2}{5}y$, $\therefore$ $x=\frac{2}{5}y$; or, $\frac{x}{y}=\frac{2}{5}$.

That is, the position of the observer divides the distance between the cliffs in the ratio of 2 : 5.

The third echo will be heard 7 seconds after the firing of the gun for the sound wave reflected from either of the cliffs will be reflected from the other cliff and take 7 seconds to come to the man.

(2) *An engine is approaching a tunnel surmounted by a cliff, and emits a short whistle when half a mile away. The echo reaches the engine after $4\frac{1}{2}$ seconds. Calculate the speed of the engine assuming the velocity of sound to be 1100 ft. per second.*

Ans. Let A be the first position, B the second position when the echo of the whistle is heard and C the position of the cliff.

Then, $AC=\frac{1}{2}$ mile$=2640$ ft. In $4\frac{1}{2}$ seconds the distance to be travelled by sound$=1100\times\frac{9}{2}=4950$ ft. $\therefore$ The distance $(AC+CB)=4950$ ft.

So $BC=4950-2640=2310$ ft. and $AB=(2640-2310)=330$ ft. This distance is travelled by the train in 4½ secs.

$\therefore$ Speed of engine$=\frac{330\times 2}{9}$ ft. per sec.$=50$ miles per hour.

27. Echo Depth Sounding :—The phenomenon of reflection of sound has been applied in measuring the *depth of the sea*. For this purpose a *hydrophone* H (an under-water microphone) is placed under water at a depth d near one end of the ship and a small under-water charge C of some explosive is placed at the same depth d near the other end. Two sounds are received when the charge is fired, the sound of explosion coming direct to the hydrophone and the echo of it coming a little later by reflection from the sea-bed, the receiver being suitably screened from the direct blow of the explosion. The instants of reception of the two sounds by the hydrophone are automatically recorded on a rotating drum by means of a stylus which begins to operate electrically just at the time of explosion. From the knowledge of the speed of rotation of the drum the time intervals t_1 and t_2 secs., between the two receptions and the explosion are found. Now if V_w be the velocity of sound in water, the distance between HC is $V_w\times t_1$ and the distance $HB=BC=x=V_w\times\frac{t_2}{2}$. Hence the true depth D at the position will be given by

$$D=d+\sqrt{x^2-\left(\frac{V_w\times t_1}{2}\right)^2},$$

as is evident from the geometry of Fig. 18(a).

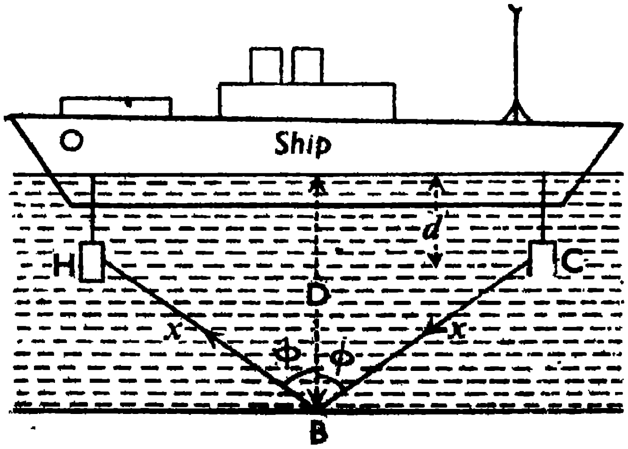

Fig 18(a)

An instrument constructed on the above principle known as a **fathometer** is used for depth sounding in oceans. Echo of radio waves is used to explore the ionised layers of the upper atmosphere.

28. Echo Prospecting :—This is nothing but a modified method of echo sounding developed to find the depth of the mineral strata below the surface of the earth. In this method compressional waves (like those in earthquakes) are set up inside the hard layer of the earth's crust by exploding some detonator just at the surface of this hard stratum at a known depth below the surface of the earth. The first set of these waves reach the detecting binaural geophones (small seismographs) direct through the surface layer. Other sets of waves partly reflected from the softer geological strata below and partly refracted from the upper layers are received by similar geophones placed at different positions. From the records of these geophones valuable information regarding the position of these mineral strata, their density and approximate nature etc. can be obtained.

To understand the principle of echo-prospecting let us consider a surface of separation *AB* between a harder upper stratum and a softer geological lower stratum [Fig. 18(*b*)]. Let V_1 and V_2 be the velocities of sound in the two media respectively. Let a compressional wave be emitted just on the surface of the harder stratum at *O*. At *AB* the waves from *O* will be partly reflected and partly refracted according to the angle of incidence. The critical angle in this case is given by $\theta_c = \sin^{-1}\frac{V_1}{V_2}$. If the wave is incident at angle less than θ_c, it will be refracted at *T* (say) and will be wholly or partly reflected at the interface *CD* according to the nature of the layer below. Let *OTQRS* be the path of this compression ray. Let a second ray *OP* incident on *AB* at *P* at an angle greater than θ_c be totally reflected in the direction *PS* so as to reach the same point *S* at which the ray *OT* was received. If the thickness of strata *OA* and *AD* be l_1 and l_2, then the time to traverse the path *OPS* is $2l_1/V_1\cos\theta_1$ and that for the path *OTQRS* is $2l_1/V_1\cos\theta + 2l_2/V_2\cos\phi$, where $\sin\theta/\sin\phi = \frac{V_1}{V_2}$.

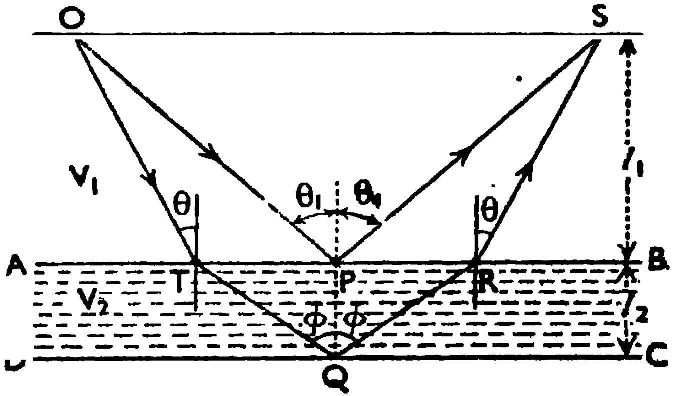

Fig. 18(*b*)

If S is near O the wave reflected from the surface AB will reach S earlier than the ray reflected from CD. But if S is shifted far away from O, the ray *via* Q reaches earlier (because of the fact that the velocity of the sound or earthquake waves in the lower oil-bearing strata is greater than near the surface, as has been shown by Jeffreys). In actual practice a geophone at S receives three pulses one direct along OS after an interval OS/V_1 and the other two as indicated above. By shifting the position of S, the equality of time of reception of the first impulse and that of the third impulse is secured. Sometimes simultaneous reception of the second and third impulses is also ensured. Then equating the mathematical expressions for these time intervals (as shown above), l_1, l_2, V_1, V_2 can be calculated, the angles θ_1, θ being obtained with the help of binaural geophones. These data give valuable information regarding the geological strata.

29. Harmonic Echo :—The reflected sound (*i.e.* echo) depends predominantly on the nature and dimension of the reflector. It is often found that the echoes are not faithful reproductions of the original note particularly when it is a complex one. If the reflector be small, the reflected waves contain more of the higher frequencies of the complex note. This class of echoes where the upper components of a complex note predominate were investigated by Lord Rayleigh in 1873 and were named *harmonic echoes* by him. He found that in this reflected sound, the intensity varies inversely as the fourth power of its wavelength and as such the phenomenon is similar to the scattering phenomenon in light. So it is sometimes called the *scattering of sound.*

30. Musical Echo :—When a single pulse of vibration (as a clap of the hand) is successively reflected from a series of equispaced and identical obstacles, a succession of reflected pulses originate at regular intervals and form a sort of musical note. This is called a *musical echo.* A single clapping of hand near a fence of stakes or poles will produce such a musical echo because these stakes being almost similar in size and equispaced the clap will be reflected from each of them in succession at regular intervals of time. So the man hearing this succession of reflected sounds will have a perception of a musical ony in it.

31. Whispering Gallery :—In the circular gallery in the dome of St. Paul's Cathedral in London, a whisper is transmitted round the surface of the gallery and is perfectly audible all around near the wall. This and similar other galleries are known as '*whispering galleries*. Rayleigh explained this phenomenon of transmission of whispers to be due to a continued creeping of the sound round the gallery horizontally along the arc in front of the whisperer. Raman and Sutherland by conducting experiments in St. Paul's Cathedral also corroborated Rayleigh's view point. Sabine ascribed the efficacy of the different whispering galleries to the inward slope of the wall which partly prevents the dissipation of sound from the roof of the building and keeps most of the waves down to the level of the gallery.

32. Phase change[*] in the Reflected Longitudinal Waves :—Whenever a longitudinal wave passing through one medium meets another medium of different density, it will be partly reflected, *but the type of the reflected wave will depend upon the density of the second medium*. This can be understood from the following illustration :—

Reflection at a rigid surface.—Let a number of light and heavy steel balls be arranged in one line, the light balls representing the particles of a lighter medium and the heavy balls, those of a denser medium. If a forward push be given to one of the lighter balls; it will strike the next ball, which in turn will strike its neighbour, and in this way, energy will be handed on from one to the other until the last light ball strikes a heavy ball After the impact, the light ball will rebound and strike a ball just behind it, and thus set up a *reflected pulse* backwards. It should be noticed that at the time of proceeding forward, one ball was pressing against another and it appeared as if a compression wave was moving onwards.

After the impact, also, the same process is repeated backwards. Therefore the nature of the pulse is not changed. Similar thing happens in the case of longitudinal sound waves. *When such a wave meets a fixed end or the surface of a denser medium, a wave of compres-*

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[*] It should be noted that the phase change here relates to the pressure of the wave. The displacement actually suffers a phase reversal at a rigid wall and at a free boundary no phase change occurs to it.

sion is reflected back as a wave of compression, and a wave of rarefaction is reflected back as a wave of rarefaction,—that is, in reflection from a fixed and rigid surface, there is no change of phase between the incident and the reflected wave of compression or rarefaction.

Reflection at a yielding surface.—Now, in the above experiment, if a forward impulse be given to one of the heavy balls, the direction of motion of the heavy ball after impact with a light ball will remain the same, *i.e.* forward. But the second ball, being lighter than the striking heavy ball, after impact, will move with greater speed, so it will create *rarefaction* behind it. Consequently, *in the case of a longitudinal wave meeting a less dense medium, the reflected wave suffers a reversal of type ; a compresed wave is reflected back as a rarefied wave, and vice versa, or in other words, the reflected and the incident waves will be in opposite phases.*

If both ends of a spiral are free, the pulse of condensation travelling to the other end is reflected along the same path as a pulse of rarefaction. So also a pulse of rarefactions returns as a pulse of condensation. This phase change can also be shown from the consideration of stationary wave (*vide* Chap. IV).

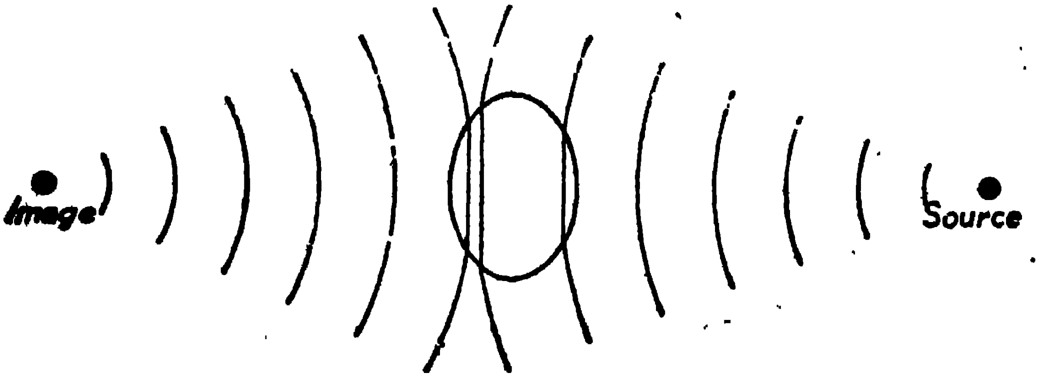

Fig. 19

33. Refraction of Sound :—When sound waves cross the boundary separating two media in which the velocity of transmission is different, they are refracted obeying the same laws of refraction as for light. The refraction of sound may be demonstrated by taking a lens-shaped India-rubber balloon filled with any gas, say, carbon dioxide, whose density is different from that of air. In case the gas filling the balloon is of higher density, the lens will behave as a converging one. But if the density of the gas in the balloon be less than that of air, the balloon, though apparently convex in shape, will behave as a diverging lens. Fig. 19 shows the conjugate foci positions of a sound source and its image formed by a converging acoustical lens of the above nature.

Rubber prism filled with various gases may also be used for determining acoustic refractive indices of those gases. This refractive index, as will be seen from Huygens' principle, is given by,

$$\mu = \frac{\text{velocity of sound in air}}{\text{velocity of sound in the gas}}.$$

(a) Effect of Temperature—As the density of air changes due to the change of temperature and so also the velocity of sound, it follows that the change of temperature of air causes refraction of sound waves. During the day time the lower layers of air are at a higher temperature than those higher up. So the sound waves, as they travel, will be refracted upwards, *i.e.* their line of advance will be bent away from the ground and hence the intensity at a distance will be diminished due to this effect. On the other hand, at night time when the lower layers are colder than those above, the bending of the line of advance will be towards the ground and the intensity will be increased. So, in this case, sound from a longer distance will be heard much more clearly than in day time. When the air above is hotter, the air layers higher up in the atmosphere will move faster and hence the plane wave-front *WF* of sound will be distorted. The upper part of *WF* will move in hot air rather more quickly than the lower part. So the wave-front after a time takes the new position *W'F'* [Fig. 19(*a*)] having a convexity upwards. So the lines of acoustic flux are directed towards the ground.

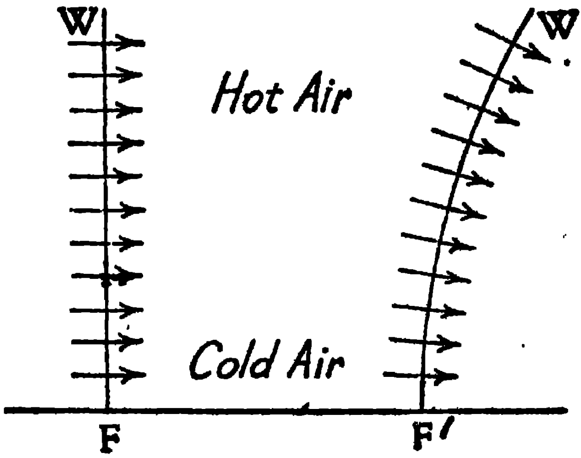

Fig. 19(*a*)

(b) Effect of Wind.—A sound wave travels a longer distance in the direction of the wind than against it.

This is due to refraction of sound. Each vertical column of air on the earth's surface move during a strong wind, through a greater distance at the top than at the bottom. When the sound moves in the direction of the wind, the velocity of sound is ***augmented*** more in the upper layers than in the lower layers of such a column (Fig. 20). The direction of propagation of the sound being normal to the column, the sound bends downwards, *i.e.* there is a concentration of sound near the surface of the earth.

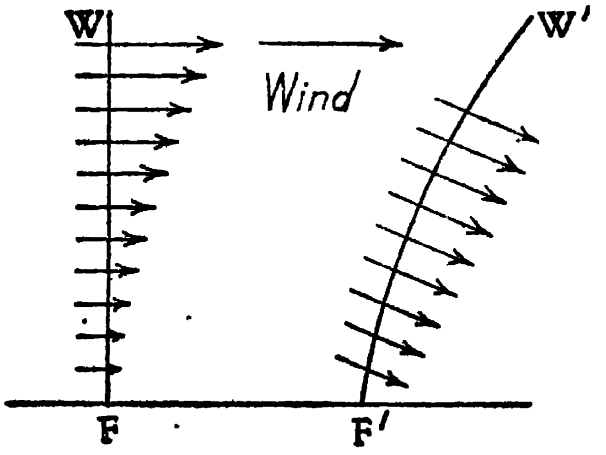

Fig. 20

If *WF* (Fig. 20) represents the wave-front (plane) of a sound wave and the wind blows in the same direction as the propagation of sound, the upper portion of the wave-front travels quicker due to wind velocity and the wave-front in the position *W'F'* becomes convex upwards. So the sound rays bend downwards.

When wind blows against the sound velocity of the sound ***is diminished*** more in the upper layers of the air than in the lower layers of each column of air. So the sound is refracted upwards. The nature of the change in the shape of the wave-front and the corresponding change in the direction of the rays are evident from Fig. 21.

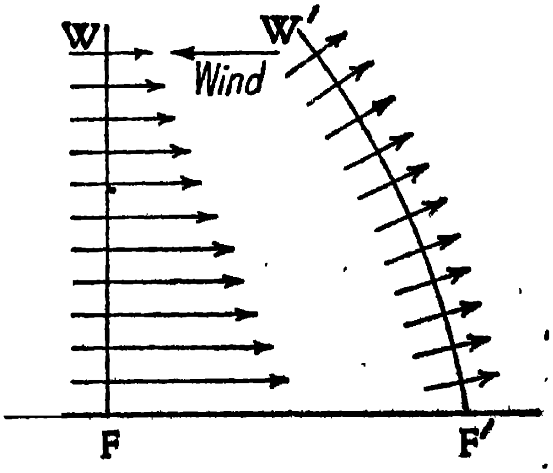

Fig. 21

33(a). Propagation of a sound ray in the upper atmosphere :—

We know that the upper atmosphere is divided into two distinct regions namely *Troposphere* and *Stratosphere.* The Troposphere extends upto a height of 12 Kilometres. In this region the temperature falls at the rate of 6°C per Kilometres ascent. In the Stratosphere the temperature remains fairly constant (at about −53°C up to a certain height, about 17 Km.) above which it begins to increase. So when a sound ray travels higher and higher up in the atmosphere its velocity continuously diminishes in the Troposphere till it reaches the Stratosphere. If the ray proceeds obliquely, it will bend towards the normal in each refraction at the successive layers of gradually decreasing temperatures. In the Stratosphere the inclination at first remains constant due to the constancy of temperature of the region At still higher altitudes the temperature increases and as a result the velocity of the sound ray also increases. So the ray now begins to bend away from the normal *i.e.*, towards the horizontal surface and will finally become parallel to the horizontal at a certain height. Then it bends downwards and finally comes back on to the surface of the earth. If, however, a ray proceeds normally upwards, there is no bending due to refraction. But the ray will be considerably deprived of its energy contents. Moreover, in the upper atmosphere the pressure being very small, the mean free paths of the particles are of the same order as the vibration amplitude of the sound waves. So a large fraction of the sound energy will be scattered by these particles in all directions. So by gradual scattering the sound waves die out completely and do not return back to the earth's surface.

34. Total internal Reflection of sound :—Sound waves are also found to be totally reflected in a denser medium in course of its passage from an acoustically denser to a rarer medium when the angle of incidence at the interface is greater than a limiting angle. This limiting angle is generally known as the *critical angle* for the two media under consideration. The phenomenon is exactly similar to the corresponding optical phenomenon. In the case of a total reflection the entire energy of the sound wave is turned back in the same medium and none penetrates into the second. The total internal reflection takes place when the angle of incidence, i, is greater than the critical angle, θ_c where $\sin\theta_c = \frac{1}{\mu}$, μ being the refraction index of the denser medium with respect to the lighter one.

$$\therefore \sin\theta_c = \frac{\text{Velocity in the denser medium}}{\text{Velocity in the rarer medium}} = \frac{V'}{V}.$$

If we consider an interface of air and water, air is acoustically a denser medium than water, V being much greater than V'. V (in water) is 1440 metres per sec. while V' (in air) is only 340 metres per sec.

$$\therefore \sin\theta_c = \frac{340}{1440}.$$

$$\text{Hence } \theta_c = 13{\cdot}5^\circ.$$

This explains why the utterances of the bathers with their mouth close to the water surface can be clearly heard on the shore. The phenomenon has its important applications in echo prospecting wherein sound waves totally reflected internally from geological strata can be clearly detected from convenient directions.

35. Huygens' Principle and explanation of Reflection and Refraction :—

If S (Fig. 22) be a sound source, the disturbance produced at S according to the wave theory spreads out uniformly in an isotropic medium with equal velocity in all directions. So the wave-front at any instant will be the surface of a sphere drawn with S as the centre and Vt as the radius where t is the time and V the velocity of propagation. Let AB (Fig. 22) represent the trace of such a wave-front at a particular instant on which every point is in the same state of disturbance *i.e.* AB is the wave-front. At any later instant of time, similarly let the disturbance lie on the trace A_1B_1. Evidently the wave has travelled from AB to A_1B_1 in the interim period. Huygens considers the mechanism of such propagation as follows : Each particle of the medium in the wave-front AB becomes a new centre of disturbance from which spherical waves, called secondary wavelets, spread out

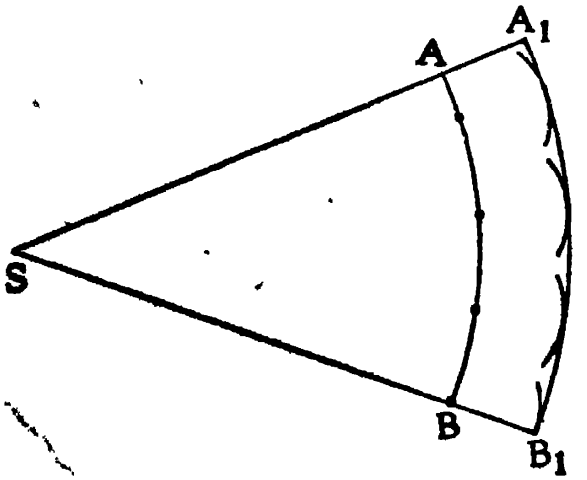

Fig. 22

with the same velocity as in the case of the wave travelling from S to AB, provided, of course, the medium remain the same. The envelope of these secondary wavelets, *i.e.* the spherical surface whose trace A_1B_1 touches all these spheres, is the new wave-front at the end of the period envisaged. The wave travels outwards from A_1B_1 to succeeding fronts as it did from AB to A_1B_1. Evidently the wave-front at any point moves in the direction of the normal to the wave-front at that point.

(a) **Reflection.**—According to Huygens' method of construction the laws of reflection of sound are deduced below by taking the case of reflection of a plane wave-front at a plane surface.

Let XY be a reflecting surface and AF be a plane wave-front travelling in the direction CA and incident obliquely on XY (Fig. 23). As each point of AF reaches XY in succession, the point struck becomes a centre of disturbance from which spherical waves start. In the time t taken by F to reach E, the wave from A will travel a distance $= Vt$, where V is the velocity of propagation of the original wave-front. Now with the centre A and radius AB $(\doteq Vt)$ describe a sphere. From E draw a plane EB tangential to the sphere and at right angles to the plane of the paper. Then EB is the reflected wave-front travelling in the direction AB.

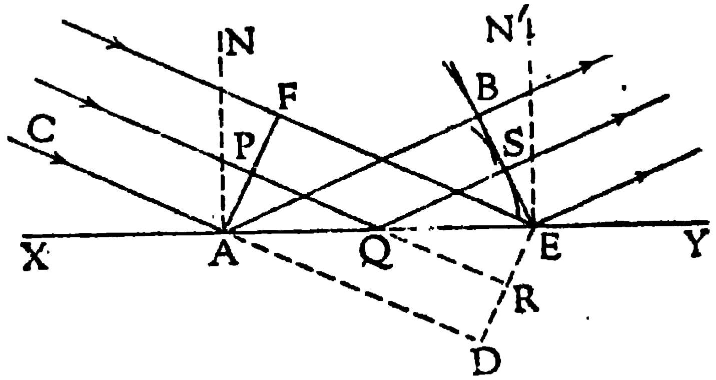

Fig. 23—Reflection on the Wave Theory.

Now in the triangles AEB and EFA, $AB = FE = Vt$, AE is common and $\angle ABE = \angle EFA = 90°$. $\therefore$ The two triangles are equal in all respects.

Hence $\angle AEB = \angle EAF$ ... ... (1)

Here CA, which is perpendicular to the incident wave-front and which is the direction of the incident wave, represents an *incident sound ray* at the point of incidence A, and AB, the direction in which the reflected plane wave travels, represents a *reflected sound ray*.

To prove that the angle of incidence is equal to the angle of reflection, draw AN perpendicular to XY at A. Because AN is perpendicular to the plane AE and AB is perpendicular to the plane EB, we have,

$$\angle AEB = \angle NAB;$$

and because AN and CA are perpendicular to AE and AF respectively we have,

$$\angle EAF = \angle NAC.$$

$$\text{But from (1), } \angle AEB = \angle EAF.$$

$$\therefore \angle NAC = \angle NAB.$$

Thus *the angle of incidence is equal to the angle of reflection.*

Again, the incident ray CA and the reflected ray AB lie in the same plane as the normal AN which is in the plane of the paper. Thus both the laws of reflection are deduced from the wave theory.

(b) **Refraction.**—Let XY be the trace of a plane surface separating two acoustical media, of different densities (Fig. 24), and AB be a plane wave-front travelling in the first medium in the direction EA with velocity V. When the wave-front AB meets the refracting surface XY obliquely at A, the particle at A is disturbed and according to Huygens it becomes a new centre of disturbance from which the generated wavelets spread out into the second medium with a velocity V', say.

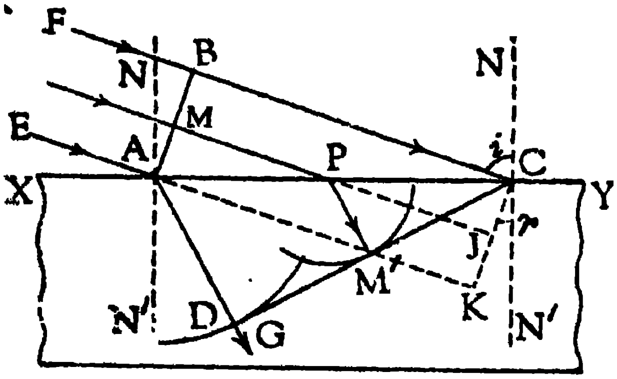

Fig. 24—Refraction on the Wave Theory.

In the time t, say, taken by B to reach C, a point on the separating surface XY, let AD be the distance traversed by the wavelet from A in the second medium. Now with centre A and radius $AD(=V't)$ describe a sphere. From C draw a plane tangential to this sphere and at the right angles to the plane of the paper. Then CD represents the refracted wave-front and it travels in the direction AG. Thus AG, which is perpendicular to the wave-front CD, represents the *refracted sound ray*, and similarly EA, which is perpendicular to the wave-front AB in the air, represents *the incident sound ray.*

Now because EA is perpendicular to the plane AB and NA is perpendicular to AC, $\angle EAN = \angle BAC$; and $\angle DAN' = \angle ACD$ ($\because$ AN' and AD are perpendiculars to AC and DC respectively).

If the $\angle$s EAN and DAN' are denoted by i and r respectively, we have,

$$\frac{\sin i}{\sin r} = \frac{\sin EAN}{\sin DAN'} = \frac{\sin BAC}{\sin ACD} = \frac{BC}{AC} \div \frac{AD}{AC}$$

$$= \frac{BC}{AD} = \frac{Vt}{V't} = \frac{V}{V'} = \text{a constant.}$$

Thus, *the sine of the angle of incidence bears a constant ratio to the sine of the angle of refraction*, and this verifies **second law of refraction**. This constant is called the refractive index μ of the second medium with respect to the first. The first law of refraction follows from the fact that EA, AD and NAN' all lie in the plane of the paper.

Questions (Chapter III)

1. Deduce the laws of reflection and refraction of sound from Huygens' principle.

How do you demonstrate the reflection and refraction of sound in the laboratory ?

2. Explain the formation of echoes. Why is succession of echoes sometimes heard ?

What do you mean by a harmonic echo ?

3. Write notes on :—

(1) A whispering gallery, (2) Musical echo, (3) Echo sounding, (4) Echo prospecting, (5) Total internal reflection of sound.

4. Describe experiments to show that sound waves are reflected, refracted and can also interfere. (Bom. U. 1941)

5. Write notes on phase change of reflection. (C. U. 1956)

6. Explain the production of echo.

A man fires a gun on the sea shore in front of a line of cliffs. A man standing 300 feet away from the gun and equidistant from the cliffs noticed that the echo takes twice as long to reach him as does the direct report. Find by calculation or graphically the distance of the gun from the cliffs. (C. U. 1929)

[*Ans.* 259·8 feet]

7. A cannon is placed 550 yards from a long perpendicular line of smooth cliffs. An observer at the same distance from the cliffs hears the cannon shot 4 seconds after he sees the flash. If the velocity of sound is 1100 ft. per second when he will hear the echo from the cliffs ?

[*Ans.* 1 sec. after hearing the direct report]

8. Explain with figure how the sound waves are refracted due to the wind and also due to the variation of temperature.

9. An observer at a certain distance from a cliff notes that the interval between the sound he makes and its echo is 3 secs. He walks 550 ft. nearer and the corresponding interval reduces to 2 secs. Find (*a*) the velocity of sound, (*b*) the observer's original distance from the cliff. (C. U. 1966)

[*Ans.* (*a*) 1100 ft./sec. ; (*b*) 1650 ft.]

10. Explain the production of echo. Explain how this can be utilised in measuring the depth of the sea. (C. U. 1968)

CHAPTER IV

INTERACTION OF SOUND WAVES

36. Principle of Superposition and Interference of Sound :— Whe two systems of waves travel through the same medium simultaneously, the actual disturbance at any point of the medium at any instant is the resultant of the component disturbances produced by the waves separately, *i.e.* the *actual displacement of a particle at any point in the medium is the vector sum of the displacements which the waves would separately produce.* This is known as the **principle of superposition.** It holds good also for other vector quantities like velocity, acceleration, etc. associated with the waves but not for the scalar quantities like intensity, energy etc. The principle gives us a ready method of finding the resultant effect of displacement, velocity or acceleration at a point in the medium when two waves simultaneously reach there. This resultant is obtained by applying the vector law. If the compressions of the two waves arrive simultaneously at the same point, *i.e.* if they are in the *same phase*, then they will combine to produce large compressions ; and similarly two rarefactions arriving at the same point at the same instant will intensify the rarefaction. But if the two waves are exactly *similar*, and if conditions are such that the compressions of one wave fall upon the rarefactions of the other, *i.e.* if they are in *opposite phase*, then they completely annul one another and the result will be the absence of any disturbance in the medium at that place at that instant and the *two sound waves*, in such a case, *will produce silence.* This is the **principle of interference of sound.**

By dropping two stones into a pond simultaneously at two neighbouring points, two sets of ripples are produced and when these ripples meet one another, a definite *interference pattern* is observed. Some lines can be seen along which the water particles are undisturbed and there are other intermediate lines along which a maximum disturbance occurs. Similarly, for sound waves, the compressions of one set may serve to neutralise rarefactions of another set at some points of a medium and to reinforce the compressions of the other set at other points of the medium.

37. Stationary Waves :—*When two sets of progressive waves, having the same amplitude and period, but travelling in opposite directions with the same velocity meet each other in a confined space, the result of their superposition is a set of waves, which only expand and shrink but do not proceed in either direction. These waves are called stationary waves.* They are so called because they remain confined in the region in which they are produced and are non-progressive in character. Moreover, the nature of vibration at each position alone such a wave is fixed.

Stationary vibrations may be longitudinal as well as transverse in character. In the case of an organ pipe the longitudinal waves travelling from one end of it get reflected from the other end and travel back. These direct and reflected waves, *identical in character* but *opposite in direction* of travel, have also the *same velocity* and so they produce longitudinal stationary waves within the organ pipe. When a string stretched on a sonometer between two bridges is plucked, the transverse vibrations travel along the string and being reflected from the bridges travel in the opposite direction with the same velocity. The direct and the reflected disturbances are identical in character, but travelling in opposite directions produce stationary vibrations transverse in character which remain confined within the string between the bridges.

Unlike in a progressive wave, here the particles in the confined space lying along a line do not successively pass through similar movement, but each particle vibrates in a simple harmonic manner with an amplitude which is fixed for it. The amplitude is minimum at equidistant fixed positions along the confined space, *i.e.* the particles at such positions are permanently almost at rest. Such positions are called **nodes.** From one node to the next, the amplitude of vibration of the successive particles gradually increases to a maximum (double of the maximum for each constituent wave) midway between the two nodes, and then decreases to a minimum at the next node, but the particles between the two consecutive nodes are always vibrated in the same phase. The positions of maximum amplitude are called the

antinodes. If the displacement is positive in the region between two consecutive nodes, the displacement is negative in the region between

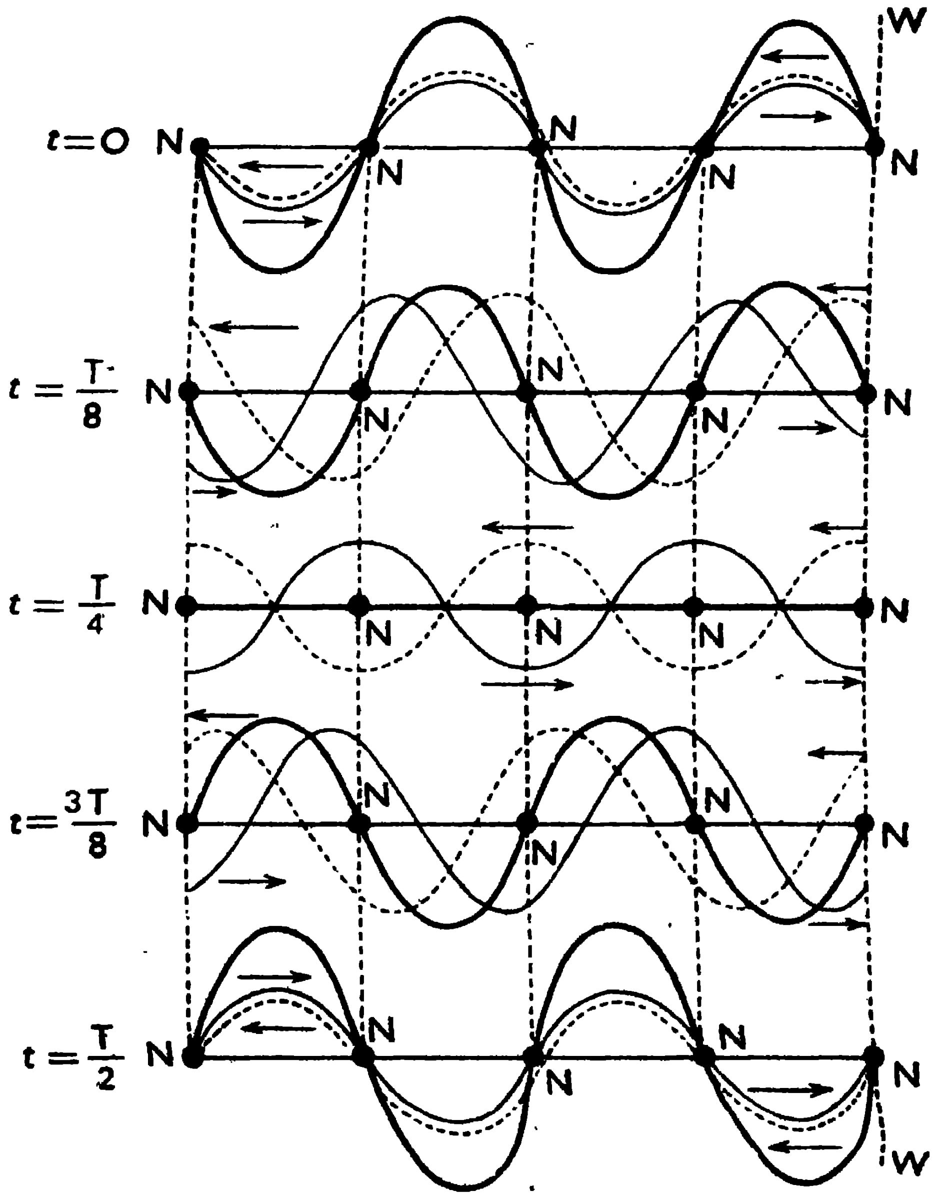

Fig. 25 —Formation of Stationary waves

the next two nodes, *i.e.* the particles between two consecutive nodes differ in phase by 180° from the particles between the next two consecutive nodes. The positions of the nodes and antinodes are fixed and their features are invariable. This distance between two consecutive nodes (or between two consecutive antinodes) is equal *to half the wavelength* of either of the two superposing waves.

(a) Graphical Representation of Stationary Waves.—

In the first diagram of Fig. 25 is shown graphically the superposition of two identical transverse simple harmonic progressive waves travelling in opposite directions the displacements of which are in phase at all points. The full curve represents the resultant wave obtained by adding the ordinates, *i.e.* the displacements of the two dotted curves. The second diagram in the figure shows the two waves and their resultant, at a time $\frac{1}{8}T$ later than the first ; that is, each wave has advanced one-eighth of a wavelength (λ), one $\frac{1}{8}\lambda$ to the right and the other $\frac{1}{8}\lambda$ to the left. The third diagram shows the waves $\frac{1}{8}T$ later than the second, *i.e.* $\frac{1}{4}T$ later than the first and one of the dotted curves has moved $\frac{1}{8}\lambda$ to the right farther than the preceding one and the other $\frac{1}{8}\lambda$ to the left farther than the preceding one. The dotted curves exactly neutralize one another and the resulting disturbance is represented by a straight line. Similarly, the fourth and the fifth diagrams represent the waves and their resultants respectively after times $\frac{3}{8}T$ and $\frac{1}{2}T$. By taking time $\frac{5}{8}T$, $\frac{6}{8}T$, etc., it will be seen that the same changes are produced in the reverse order.

Note that the points of the full curves marked N through which dotted vertical lines pass are always at rest. These points are the *nodes*. The points midway between the node are the ***antinodes*** or loops. These are points of maximum disturbance. The resultant disturbance simply shows a change of form from instant to instant as given by the full curve, but there is no forward or backward motion of the wave as a whole. Such waves in which the positions of the nodes and antinodes are fixed are called **Stationary waves.**

(b) Mathematical analysis of Stationary or Standing waves.—

When two progressive waves of identical period (hence frequency) and amplitude travel along the same straight line but in opposite directions the resultant wave will be stationary in character. This can be analytically shown in the following way.

The equation of displacement of a progressive wave moving in the positive direction of x is represented by,

$$y_1 = a \sin \omega\left(t - \frac{x}{V}\right) \qquad \ldots \qquad \ldots \quad (1)$$

The equation of a similar progressive wave of same period and amplitude but moving in the negative direction of x is given by,

$$y_2 = a \sin \omega\left(t + \frac{x}{V}\right) \qquad \ldots \qquad \ldots \quad (2)$$

If these two waves are made to superpose, the resultant displacement at an instant t is given by,

$$y = y_1 + y_2 = a \sin \omega\left(t - \frac{x}{V}\right) + a \sin \omega\left(t + \frac{x}{V}\right)$$

$$= 2a \cos \frac{\omega x}{V} \sin \omega t$$

$$= 2a \cos \frac{2\pi x}{\lambda} \sin \omega t \qquad \ldots \qquad \ldots \quad (3)$$

$$= 2a \cos \frac{2\pi x}{\lambda} \sin \frac{2\pi t}{T}. \qquad \ldots \qquad \ldots \quad (4)$$

The equation (3) or (4) evidently represents a ***stationary wave***. That this resultant wave is stationary is evident from the fact that here y varies as the sine function of t and as the cosine function of x and as such if t is changed by Δt, it is not possible to change x by an amount independent of t so that y once again takes up the same value as before. So the nature of displacement at a particular instant is stationary and the disturbance is no longer a progressive one.

In either of the above equations the sine function containing t is evidently periodic in nature and hence the amplitude of the resultant waves is given by $2a \cos \frac{2\pi x}{\lambda}$. Hence if $x = \frac{\lambda}{4}, \frac{3\lambda}{4}, \ldots, \frac{(2n+1)\lambda}{4}$ *i.e.* when x is an odd multiple of $\frac{\lambda}{4}$ the amplitude vanishes. Hence here, $y = 0$ for all values of t. These are the positions of the **nodes** occurring at intervals of distance equal to $\frac{\lambda}{2}$.

Again when $x = 0, \frac{\lambda}{2}, \lambda, \ldots \frac{(2n)\lambda}{4}$, *i.e.*, when x is zero or any even multiple of $\frac{\lambda}{4}$, the amplitude is maximum either in the positive or in the negative direction (*i.e.* $\pm 2a$). Hence $y = \pm 2a \sin \frac{2\pi t}{T}$.

These positions spaced midway between the nodes correspond to the positions of the *antinodes* where the maximum amplitude is twice that of the constituent progressive waves, *i e.* either $+2a$ or $-2a$.

38. Velocity and Acceleration of a particle on the Stationary Wave :—Since the displacement of a particle on a stationary wave is given by,

$$y = 2a \cos \frac{\omega x}{V} \sin \omega t,$$

the *velocity* of the vibrating particle is,

$$\frac{dy}{dt} = 2\,a\omega \cos \frac{\omega x}{V} \cos \omega t.$$

The *acceleration* of same particle is given by,

$$\frac{d^2y}{dt^2} = -2a\omega^2 \cos \frac{\omega x}{T} \sin \omega t.$$

The slope of the displacement curve of the stationary wave,

is,
$$\frac{dy}{dx} = -\frac{2a\omega}{V} \sin \frac{\omega x}{V} \sin \omega t$$

and its curvature is
$$\frac{d^2y}{dx^2} = -\frac{2a\omega^2}{V^2} \cos \frac{\omega x}{V} \sin \omega t$$

$$= \frac{1}{V^2} \cdot \frac{d^2y}{dt^2}.$$

$$\therefore \frac{d^2y}{dt^2} = V^2 \frac{d^2y}{dx^2}$$

Note : Change of Phase by Reflection :—

Let the displacement, y_1, of the incident wave be,

$$y_1 = a \sin \omega\left(t - \frac{x}{V}\right).$$

and that of the wave reflected from an yielding or rigid surface and moving in the opposite sense be,

$$y_2 = ma \sin \omega \left(t + \frac{x}{V}\right)$$

where m is an unknown factor arising out of reflection.

At $x = 0$, the resultant displacement, $y = y_1 + y_2$

$$= (1+m)\, a \sin \omega t \qquad \ldots \quad (1)$$

Again if we consider a length dx of the medium of unit cross-sectional area wherein increased pressure Δp causes a slight change in displacement $-dy$, we can write the volume **strain** due to a **stress** Δp to be $-\frac{dy}{dx}$.

Hence the volume elasticity is $E = -\Delta p \Big/ \frac{dy}{dx}$;

$$\text{or, } \Delta p = -E \frac{dy}{dx}.$$

$\therefore$ For the incident wave $\Delta p_1 = \frac{Ea\omega}{V} \cos \omega \left(t - \frac{x}{V}\right)$

and for the reflected wave $\Delta p_2 = -\frac{Eam\omega}{V} \cos \omega \left(t + \frac{x}{V}\right)$.

So when $x = 0$, total pressure change, $\Delta p = \Delta p_1 + \Delta p_2$

$$= (1-m)\frac{Ea\omega}{V} \cos \omega t \qquad \ldots \qquad \ldots \quad (2)$$

At a rigid surface, $y = 0$. $\therefore$ $m = -1$ and $\Delta p = \frac{2Ea\omega}{V} \cos \omega t$.

But at an yielding surface, $\Delta p = 0$, $m = 1$ and $y = 2a \sin \omega t$.

Thus on reflection from a rigid wall, there is a change of phase π of the displacement but not of pressure and due to reflection from a free end the displacement suffers no change of phase but the pressure does.

Pressure Variations at nodes and antinodes of a Stationary Wave :—

As the variation of pressure Δp in a sound wave is given by $\Delta p = -E\frac{dy}{dx}$ (Art. 38. Note), and y, the displacement in a standing wave is given by $y = 2a \cos\frac{2\pi x}{\lambda} \sin \frac{2\pi}{T}t$, by correlating these relations we get the pressure variations at the *nodes* and *antinodes* of a progressive wave.

$$\text{Here, } \frac{dy}{dx} = -\frac{4\pi a}{\lambda}\sin\frac{2\pi x}{\lambda}\sin\frac{2\pi}{T}t.$$

$$\therefore\ \Delta p = \frac{4\pi aE}{\lambda}\sin\frac{2\pi x}{\lambda}\sin\frac{2\pi}{T}t.$$

So the amplitude of this pressure variation is $\frac{4\pi aE}{\lambda}\sin\frac{2\pi x}{\lambda}$.

At the nodal points, $x = \frac{\lambda}{4}, \frac{3\lambda}{4}, \ldots\ldots \frac{(2n+1)}{4}\lambda$; hence, $\sin\frac{2\pi x}{\lambda} = 1$.

$\therefore$ The amplitude of pressure variation at the nodes is $4\pi aE/\lambda$ and is maximum at these points.

At the antinodes, $x = 0, \frac{\lambda}{2}, \lambda, \ldots \frac{n\lambda}{2}$; hence, $\sin\frac{2\pi x}{\lambda} = 0$.

Hence the amplitude of pressure variation is minimum at these points.

39. Progressive and Stationary Waves compared :—

Progressive Waves	*Stationary Waves*
(*i*) All particles of the medium execute periodic motions about their mean positions, and have identical motions (with the distance from the source increasing, the amplitude will, however, decrease gradually from one particle to the next).	(*i*) All particles of the medium (except at some equidistant points) execute periodic motions having amplitudes which are fixed for them. From a definite particle along the line of propagation the amplitude increases gradually from a minimum to a maximum at some other definite particle and then decreases in the same fashion to a minimum again. This is repeated throughout. The points where the amplitude is minimum are called ***nodes*** and the points (midway between the nodes), where the amplitude is maximum, are called ***antinodes***. The period of motion for the particles is the same as that of the component waves.

Progressive Waves	*Stationary Waves*
(*ii*) The wave travels onward with a definite velocity and the disturbance is transferred from particle to particle in the direction of propagation.	(*ii*) The wave is not bodily transferred from one part of the medium to another ; and the compressions and rarefactions or the crests and troughs, in the case of longitudinal waves or transverse waves as the case may be, merely appear and disappear without progressing in either direction.
(*iii*) The movement of one particle begins just a little later than its predecessor, or, in other words, the phases of the particles change continuously from one particle to the next.	(*iii*) At any instant all the particles in any one segment, *i.e.* between two consecutive nodes or antinodes, are in the same phase, but the particles in two consecutive segments are in opposite phases.
(*iv*) Each particle of the medium in turn goes through a similar movement, *i.e.* similar changes of pressure, density, etc. as a complete wave passes through it and is restored to its initial condition after the periodic time.	(*iv*) The particles at nodes undergo maximum change of pressure and density while those at antinodes undergo minimum change of pressure and density throughout a periodic motion.
(*v*) In a complete vibration there is no instant when all the particles are stationary.	(*v*) Twice in each complete vibration all the particles are at rest at the same moment.

40. Interference of Sound :—As has been already mentioned that if two sound waves of same amplitude and frequency are made to superpose, under certain conditions the resultant sound will be intensified while under certain other conditions they will produce silence. This phenomenon is known as ***interference of sound***. The phenomenon is similar to that in optics with the difference that in optical interference the interfering trains must have their origins in the same source whereas in acoustical interference this condition of coherence is not essential and interference may take place even between sounds emitted from different sources. But the amplitude and frequency conditions must be nearly satisfied.

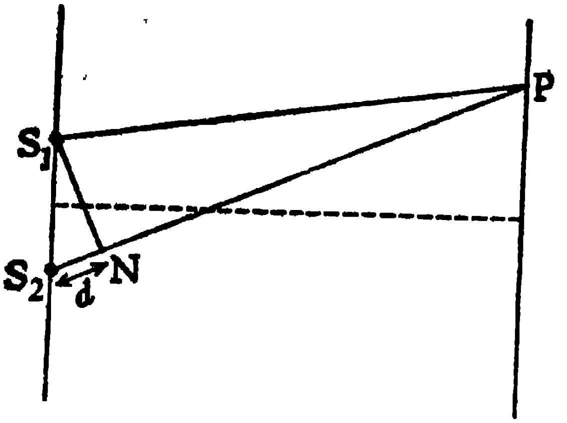

Fig. 26

Let S_1 and S_2 (Fig. 26) be two sources emitting sound waves of the same frequency and amplitude. When the waves from S_1 and S_2 reach the point P, they are to traverse unequal paths. So a path difference d is introduced between them where $d = S_2N = S_2P \sim S_1P$. Hence a phase difference $\phi = \frac{2\pi d}{\lambda}$ is introduced between the two waves (displacement type). If this phase difference be zero or even multiples of π, the two may be supposed to be in the same phase and as a result the two waves will reinforce each other. Again if this phase difference be an odd multiple of π, the two waves meet in opposite phases at P and hence they produce almost a silence there.

So the condition of reinforcement, is,

$$\phi = 0 \text{ or } 2n\pi \text{ where } n \text{ is zero or an integer,}$$

and that of destructive interference is

$$\phi = (2n+1)\pi \text{ where } n \text{ is zero or an integer.}$$

Conditions for Interference of two Sounds.—

In order that two wave trains may successfully interfere the following conditions must be fulfilled by them :—

(1) The component waves must have the same frequency and amplitude.

(2) The two waves should be similar with regard to their intensities.

(3) The displacements caused by them must be in the same line.

41. Experimental Demonstration of Acoustical Interference :—

(a) Quincke's tube.—Two separate sources producing waves satisfying the conditions for interference cannot be realised in practice. That is why, in practice, the waves from a single source are divided at a point and made to reunite again at some other region after travelling paths of different lengths. **Quincke** based his arrangement on this principle, and his apparatus consisted of a mouth-piece *A* connected to the two limbs *B* and *C* which combine again into one tube *EF* against which the ear is placed (Fig. 27). *D* is a sliding tube, by drawing it in or out the length of the path *ACDE* can be suitably altered. A vibrating tuning fork *T* is held at *A* and the resulting sound at *F* is heard. When the sliding tube is at *D*, the paths *ABE* and *ACDE* are equal so that the two waves passing through them meet in the same phase at *F* and produce a maximum sound. The path *ACDE* is then increased by drawing out the sliding tube *D* until a position D_1 is obtained when a minimum sound is produced. The difference in path between *ACDE* and ACD_1E is half the wavelength. By further drawing out the tube *D* from D_1 to D_2, again a maximum sound is obtained, the shift so made being equal to half the wavelength again. Thus the full wavelength of the sound used is obtained.

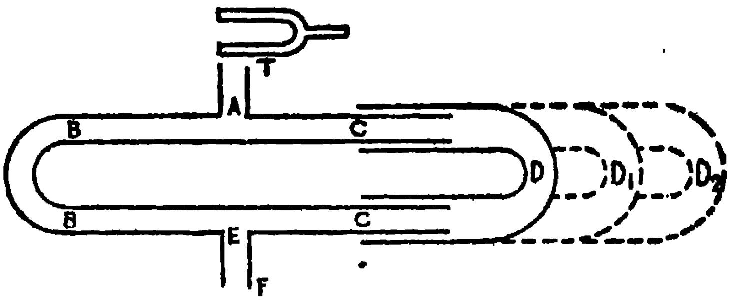

Fig. 27

(b) The formation of stationary waves.—

This is another way of demonstrating the interference of sound. In instruments like organ pipes, sonometers, etc. stationary waves are formed due to interference between the incident and the reflected waves. At certain positions the displacements of these two waves are always in opposite phases and the resultant displacements are zero there. These are the nodes of the stationary wave. At the antinodes the two displacements are sometimes in the same phase and sometimes in the opposite phases. So here the displacement varies from a maximum to zero according as the phase difference permits.

(c) Interference of Sound with a Tuning Fork.—

When the two prongs *a* and *b* of the tuning fork are nearest, together they compress the air between them and hence set up a rarefaction on the other side of each prong. Exactly $T/2$ earlier, where T is the period of vibration of the fork, the effect was just the opposite. A rarefaction existed between the receding prongs while a compression was formed in front of each of them. The alternate compressions and rarefactions created between *a* and *b* advance in a direction perpendicular to *ab* and those created on the other sides of the prongs spread out along *ab* (Fig. 28). These sound waves interfere and produce silence along the directions shown by the dotted lines. They may be detected by slowly rotating the fork near the ear when 4 distinct positions (*i e*. along the ***lines of silence***) of minimum sound can be exactly located. If an air column resonant with the fork is taken and the fork slowly rotated, minimum sound will be heard when the lines of silence pass along the air column.

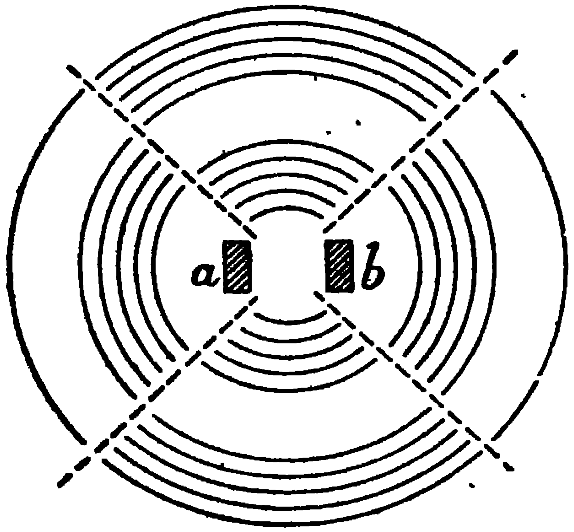

Fig. 28

42. Beats :—When two sounds preferably of the same type and intensity but with slightly different frequencies are produced together, a fluctuation of loudness (waxing and waning of sound) occurs at any place in the neighbourhood of the sources of sound due to the mutual interference of two notes. In the resulting sound-wave the component waves periodically reinforce each other at some instant of time and destroy each other at some other instant of time and so the

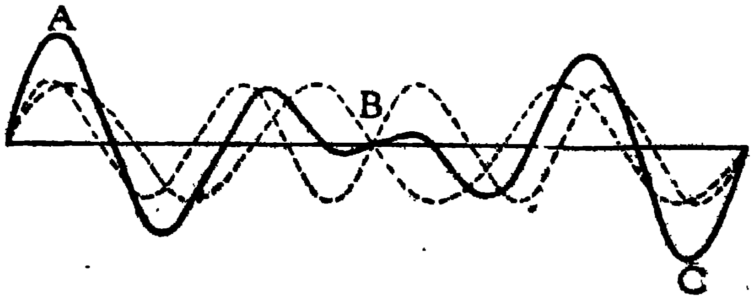

Fig. 29—Formation of Beats.

sound heard possesses a characteristic throbbing or beating effect. *This phenomenon is known as* **beats.**

(a) **Graphical method.**—In Fig. 29, the dotted curves represent two wave systems (arranged on the same axis) produced by two vibrating forks of slightly different frequencies. At the beginning of a given second, the two forks are swinging together so that they simultaneously send out condensations, and the result of the two condensations will produce a double effect upon the ear (as at *A* in Fig. 29). But as the frequencies of the forks differ, the subsequent effects upon the ear is represented by the continuous curve, which is the result of combining these two wave systems, and is obtained by finding the vector sum of the separate displacements, as time passes.

It is evident from the nature of the continuous curve that its amplitude varies in a periodic manner, being maximum at *A* and *C*, and minimum at *B*, due to which there is a periodic change in the intensity of the sound heard. At *A*, when the vibrations are in the *same phase*, the resultant displacement is the sum of the displacements of the two component waves, and at *B* these are in *opposite phases*, and the resultant displacement is given by their difference. As the loudness depends upon the amplitude of vibration, the sound heard for small intervals corresponding to instants *A* and *C*, is the loudest when the amplitudes are maximum, and it is minimum at *B* when the amplitude is minimum. Such fluctuations of loudness of the sound are known as **beats.**

Suppose two tuning forks having frequencies 256 and 257 per second respectively, are sounded together. If, at the beginning of a given sound they vibrate in the *same phase* so that the compressions or rarefactions of the corresponding waves reach the ear together, the sound will be strengthened. Half a second later when one makes 128 and the other $128\frac{1}{2}$ vibrations, they will be in the *opposite phases, i.e.*, a compression of one wave will unite with a rarefaction of the other and will tend to produce silence. At the end of one second they will again be in the *same phase* and the sound will be augmented, and by this time, one fork will gain one vibration over the other. Thus, in the resultant sound the observer will hear the maximum of loudness at every interval of one second. Similarly, a minimum of loudness will be heard at an interval of one second. As we may consider a single beat to occupy the interval between two consecutive maxima or minima, *the beat produced in the above case is one in each second.* It is evident, therefore, that when two sounds of nearly the same vibration frequencies are heard together, the number of beats per second is equal to the *difference* of the frequencies of the two vibrating sources. Thus, if n_1 and n_2 $(n_1 > n_2)$ be the frequencies of the two sources then the number of beats per second is equal to $(n_1 - n_2)$. Thus, *the number of beats heard per second is numerically equal to the difference in frequencies of the two sounds.*

(b) **Analytical method.**—Suppose two sounds of slightly different frequencies are produced together. Let the smaller of the two frequencies be n_1 and the other greater than it by n. Assuming that they start with the same phase, displacements produced by the two wave systems at a point at some instant of time t will be given by,

$$y_1 = a \sin 2\pi n_1 t, \text{ and } y_2 = b \sin 2\pi(n_1 + n)t.$$

By the principle of superposition, the resultant displacement will be given by,

$$\begin{aligned} y = y_1 + y_2 &= a \sin 2\pi n_1 t + b \sin 2\pi(n_1 + n)t \\ &= \sin 2\pi n_1 t(b \cos 2\pi\, nt + a) + b \cos 2\pi n_1 t. \sin 2\pi nt. \end{aligned}$$

This equation represents a wave equation which may be condensed into the form $y = F \sin (2\pi n_1 t + \alpha)$, where F is its amplitude and α, the epoch. The values of F and α can be found by comparing the two equations and equating the coefficients of $\sin 2\pi n_1 t$ and $\cos 2\pi n_1 t$. That is,

$$F \cos \alpha = b \cos 2\pi nt + a, \text{ and } F \sin \alpha = b \sin 2\pi nt.$$

By squaring both sides and adding,

$$\begin{aligned} F^2 &= b^2 \sin^2 2\pi nt + b^2 \cos^2 2\pi nt + 2ab \cos 2\pi nt + a^2 \\ &= a^2 + b^2 + 2ab \cos 2\pi nt \qquad \ldots \qquad \ldots \quad (1) \end{aligned}$$

Also,

$$\tan \alpha = \frac{b \sin 2\pi nt}{b \cos 2\pi nt + a} \qquad \ldots \qquad \ldots \quad (2)$$

It is evident from (1) the amplitude of the resultant wave varies with time. It assumes maximum and minimum values as follows—

when $t = \frac{1}{2n}$, $\cos 2\pi nt = -1$, $F = a - b$ (minimum).

similar is the case when $t=\frac{3}{2n}, \frac{5}{2n}, \frac{7}{2n}$, etc. ;

when $t=0$, $\cos 2\pi nt=1$, $F=a+b$ (maximum) ;

also when $t=\frac{1}{n}$, $\cos 2\pi nt=1$, $F=a+b$ (maximum).

Similar will be the case when $t=\frac{2}{n}, \frac{3}{n}, \frac{4}{n}$......etc.

Thus in an interval of $\frac{1}{n}$ second, two maxima and an intermediate minimum take place. Similarly, it can be shown that between two minimum sounds, a maximum occurs in a period of $\frac{1}{n}$ sec. So *the number of beats* (two successive maxima or two successive minima produce one beating effect) *per sec. is equal to the difference of the frequencies*, $(n_1+n)-n_1=n$. If the two wave trains have identical amplitudes, *i.e.* if $a=b$, the amplitude of the resultant wave varies from 0 to $2a$. The resultant wave, $y=F\sin(2\pi n_1 t+\alpha)$, has a frequency n_1 and its phase difference α with the original vibration is not a constant but changes with t.

43. Tuning Instruments :—It should be *remembered* that beats can be heard only when **frequencies** of the notes are nearly equal to each other ; if their difference is greater than 15 or 16, separate beats cannot be heard and a discordant unpleasant noise is the result. It is for the above reason that musical instruments are **tuned** by means of **beats.** If beats are heard between the first overtone of a lower note and its octave, say in the case of a piano or organ, it is a sure test that the instrument needs tuning. Beats are not heard when the frequencies of the two sounds are exactly equal.

44. Determination of the Frequency of a Fork by the method of Beats:—Two forks having nearly the same frequency are mounted on sounding boxes and sounded together. The number of beats in any time is counted by means of a stop watch, and from this, the number of beats per second is determined, which is equal to the difference of the frequencies of the forks. By knowing the vibration frequency of one of them, that of the other can be determined. *To know whether the frequency of the given fork will be higher or lower*

than that of other, one of the prongs of the given fork is loaded with a little wax, and the number of beats per second is again determined. The frequency of the fork is diminished by loading its prong. Hence if the number of beats per second obtained after loading the fork is *greater* than the number obtained before, the frequency of the given fork must be *less* than that of the known fork ; if the number be *less*, then the frequency of the unknown fork is *greater* than that of the known fork.

N.B.—The frequency of a fork may be increased by filing it.

The **uses of beats** are in (*a*) *finding frequency* ; (*b*) *tuning instruments*.

45. Beats and Interference compared :—Beats occur when the amplitudes of vibration of two waves travelling in the same direction are almost or exactly equal. In interference, the amplitudes of the two waves moving either in the same or in the opposite direction should be nearly equal. The displacement at any point due to the two wave-trains must be collinear in beats as well as in interference. In beats the frequencies of the two notes must not be equal but should be nearly so, whereas in interference the frequencies of the two notes must be identical. In interference the positions of maxima and minima are fixed so that the phase difference between two waves at any point remains constant. But in beats the phase difference between the waves at a point changes continually from 0 to π. In interference, as the pattern is fixed, the amplitude at a point remains constant but it varies from point to point. In beats the pattern of maxima and minima moves with the velocity of sound and due to its progressive nature the amplitude at a point varies from a maximum to a minimum with time.

46. Combinational Tone :—When two tones of frequencies n_1 and n_2 are produced simultaneously and preferably act on the same system, the resultant sound is found to consist of tones of various frequencies over and above the original ones (*i.e.* n_1 and n_2) and their overtones. These additional tones are found to possess frequencies which may be obtained as a result of combination of n_1 and n_2 in various ways, *e.g.* (n_1-n_2), (n_1+n_2), $(2n_1+n_2)$, (n_1+2n_2) etc. Some of them may have a high intensity as well. All these tones are known as *combinational tone*. The tone of frequency $(n_1 - n_2)$ was first observed by Sorge and later by Tartini and has been named by Helmholtz as the ***difference tone***. The tone of frequency $(n_1 + n_2)$ has been termed as the ***summation tone***. The tone which are overtones of the fundamentals, *i.e.* of frequencies $2n_1$ and $2n_2$, are called ***self-combination*** tones. All these different types have been included under the general name of ***combinational tone***.

Lagrange and Young suggested that the difference tones and the beats are identical. It is only when the beats are produced with sufficient rapidity that a new tone—called a *beat tone* is recognised. Accordingly it should only have a subjective existence. Only when number of beats are formed at a rate greater than 16 per sec. that the new tone (beat tone) will be perceived as a separate one. This ***beat tone theory*** was discarded by Helmholtz in view of his observation of tones of frequencies $(n_1 + n_2)$. The formation of such summation tones and other types of combinational tones were explained partly by Helmholtz in his ***intensity theory***. Here he assumes that the superposing tones are produced with a great force and as such have large amplitudes. The corresponding restoring force does not linearly vary with the displacement. It is rather of the form, $f = ax + bx^2 + cx^3 + \ldots$. Here the terms containing the even power of x introduce the asymmetry because with the change of sign of x they do not change.

Under the joint action of two forces the restoring force may be supposed to be linearly proportional to the sum of the displacements due to the two forces only when they are very small. But when these displacements are large their higher powers cannot be neglected which become rather prominent, thereby producing combinational tones. Considering the restoring force to be proportional only to the first and second powers of the displacement, Helmholtz has shown that due to double forcing (on account of two different sound sources), the frequencies n_1, n_2, $2n_1$, $2n_2$, $n_1 + n_2$ and $n_1 - n_2$ can be obtained theoretically.* Analysis with higher powers of displacement reveals the existence of a large number of these combinational tones. According to this theory the intensity of the difference tones will be of the same order as that of the summation tones. But actual experiment shows that the difference tones are generally more intense.

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* Both in the case of large amplitude vibrations as well as in asymmetric vibrations, the restoring force is of the form $ax + bx^2$, neglecting the terms containing higher powers of displacement. So if such a vibrating system be subjected to the joint action of two periodic forces, the corresponding equation of motion will be given by,

The experiments of Edser and Rucker using a double siren, and Michelson's interferometer conclusively established the *objective existence of the combinational tones.* The sounds from a double siren were concentrated by a horn on to a light plate attached to one prong of a mounted tuning fork. The other prong carried one of the mirrors of Michelson's interferometer, wherein a steady band system was secured by the usual arrangements. When the the natural frequency of the fork was so chosen as to correspond to one of the expected frequencies of the combinational tones, it began to vibrate and the

$$m\frac{d^2x}{dt^2}+ax+bx^2=F\cos n_1t+G\cos(n_2t-\delta);$$

or, $$\frac{dx^2}{dt^2}+\omega^2x+Kx^2=f\cos n_1t+g\cos(n_2t-\delta) \quad \dots \quad \dots \quad (1)$$

where $\omega^2=\frac{a}{m}$, $K=b/m$, $f=\frac{F}{m}$, $g=\frac{G}{m}$ and n_1, n_2 represent the circular frequencies of the impressed forces.

Ignoring the free vibrations for the time being and considering only the term involving the first power of the displacement (*i.e.* ω^2x), we get,

$$\frac{d^2x}{dt^2}+\omega^2x=f\cos n_1t+g\cos(n_2t-\delta) \quad \dots \quad \dots \quad \dots \quad (2)$$

whose solution will evidently be of the form,

$$x=A\cos n_1t+B\cos(n_2t-\delta).$$

Substituting these values in (2), we get, $A=\frac{f}{\omega^2-n_1^2}$ and $B=\frac{g}{\omega^2-n_2^2}$.

Putting this value in the Kx^2 term, the suggested solution will be of the form,

$$x=A\cos n_1t+B\cos(n_2t-\delta)+C+D\cos 2n_1t$$
$$+E\cos 2(n_2t-\delta)+F_1\cos[(n_1+n_2)t-\delta]+G_1\cos[(n_1-n_2)t+\delta] \quad \dots \quad (3)$$

Evaluating $\frac{d^2x}{dt^2}$, substituting the results in $\frac{d^2x}{dt^2}+\omega^2x$ and equating to the Kx^2 term, we get

$$C=-\frac{K}{2\omega^2}(A^2+B^2),\quad D=-\frac{KA^2}{2(\omega^2-4n_1^2)},\quad E=-\frac{KB^2}{2(\omega^2-4n_2^2)}.$$

$$F_1=-\frac{ABK}{[\omega^2-(n_1+n_2)^2]} \text{ and } G_1=\frac{-ABK}{[\omega^2-(n_1-n_2)^2]}.$$

So from (3), it is evident that as a result of double forcing the resultant vibration will have frequencies n_1, n_2, $2n_1$, $2n_2$, (n_1+n_2) and (n_1-n_2).

steadiness of the fringes was disturbed. In this way the existence of summation and difference tones was proved and the *objectivity* of the combinational tone established.

The main objection to Helmholtz's theory is that the large intensity of the primary, necessary for the production of difference tone (or combinational tone in general), is not actually corrcborated by experimental evidences.

Waetzmann introduced a *general theory of asymmetry* wherein he tried to reconcile the beat tone theory and the intensity theory of Helmholtz. He also established the objectivity of the combinational tone by the introduction of his theory. If an asymmetrically loaded membrane be subjected to double forcing, the resultant vibration can be photographed with the help of sand particles sprinkled on the membrane. This vibration can be shown by actual Fourier analysis to consist of tones of frequencies n_1, n_2, n_1-n_2 and n_1+n_2, etc. Thus Waetzmann supposed the combinational tone to be produced by an asymmetrically loaded membrane under the joint action of two forces. The drum-skin of the ear being loaded on one side by the bones, it may be considered as an asymmetric system. So under the action of two external forces (*i.e.* two sound waves) the combinational tones are ***subjectively produced in our ears.***

47. Diffraction of Sound : Acoustical shadow :—The bending of the wave at the edges of an obstacle is known as ***diffraction***. In light, we know that whenever an obstacle is placed in the path of the rays, a defined shadow of the obstacle is formed. On careful examination it is found that there is some amount of blurring at the bounding lines of the shadow. This is due to bending of the rays at the edges of the obstacle, *i.e.* due to diffraction of light. In the case of sound the effect is more pronounced. If in a closed room, one of the windows be slightly opened, the sound outside will readily spread out in all the nooks and corners of the room due to the considerable diffraction effect in sound. This is because of the larger wavelengths of sound waves. It is for this reason, the ***shadows of the acoustical obstacles*** are never as well defined as the optical shadows. Moreover, due to their larger wavelengths sound can easily bend round big obstacles without casting any shadow. But if the obstacle be very big compared to wavelength an ill-defined acoustical shadow may be obtained. To have an acoustical shadow of a moderately sized obstacle, the sound waves must be of smaller wavelengths.

Just as light waves can be made to form definite diffraction pattern when passed through a ruled grating, sound waves of shorter wavelengths can also be diffracted by a suitable grating of larger grating space (1 cm. or more). The wavelength of sound has been successfully determined by this method.

The action of *megaphones* provides us with an interesting example of acoustical diffraction. The sound coming out of the open end of a conical megaphone spreads out. The amount of this spreading is greater, the smaller the opening compared with the wavelength. This is due to the diffraction of the sound waves at the edge of the narrow opening. If, however, the opening be large compared with the wavelength, the diffraction is less and hence the sound spreads out to a lesser degree. So the directive property of the megaphone increases.

Examples

(1) *A set of 24 tuning forks is arranged in a series of increasing frequency. If each fork produces 4 beats with the preceding one and the last fork sounds the octave of the first, calculate the frequencies of the first and the last fork.*

(Bom. U. 1954)

Ans. Since the frequency difference between two consecutive forks is 4 and as there are 24 forks in all, the difference in frequency between the 1st and the last fork is $4\times23=92$. Again from the given condition of the problem the frequency of the last fork is double that of the first. So if n be the frequency of the first fork that of the last will be $2n$. Hence the difference is $2n-n=n$.

$\therefore$ $n=92=$the frequency of the first fork and $2n=184=$the frequency of the last fork.

(2) *Calculate the velocity of sound in a gas in which two waves of length 50 and 50·5 cm produce 6 beats per sec.*

Ans. We know $n\lambda=v$ where v is the velocity of the sound wave, λ its wave length and n its frequency. (Pat. U. 1959)

For the 1st wave $n_1\times50=V$ or $n_1=\frac{V}{50}$ and for the 2nd wave, $n_2\times50{\cdot}5=V$.

$$\text{or, } n_2=\frac{V}{50{\cdot}5}.$$

Here, $n_1>n_2$

$$\therefore\ n_1-n_2=6=V\left(\frac{1}{50}-\frac{1}{50{\cdot}5}\right)=V\,\frac{0{\cdot}5}{2525}$$

$$\therefore\ V=6\times2\times2525 \text{ cm./sec.}=30300 \text{ cm. per sec.}$$

$$=303 \text{ metres per sec.}$$

Questions (Chapter IV)

1. State the chief characteristics of stationary waves showing particularly how they differ from progressive waves. [C. U. 1936 ; *cf.* C. U. 1938]

2. Derive an analytical expression for stationary wave and deduce its characteristics. [E. P. U. 1953 ; *cf.* Raj. U. 1949]

3. Clearly distinguish between progressive and stationary wave. (Utk. U. 1965 ; *cf.* C. U. 1963]

A wave is propagating along a row of particles from a source having S. H. M. Find an expression for the displacement y of a particle at a distance x from the source at time t. Hence show that every particle successively attains all the phases that simultaneously exist in the row of particles. (C. U. 1937)

4. Write notes on stationary waves. (C. U. 1956)

5. Deduce equations of the progressive wave and of the stationary wave and from the latter derive the characteristic properties of stationary waves. (A. U. 1935, '44)

6. Distinguish between (*a*) transverse and longitudinal waves, (*b*) stationary and progressive waves. Which of these waves cannot exist in a gas and why ? (Bom. U. 1954)

7. Explain by means of a diagram the production of beats. (C. U. 1934)

8. Give mathematical investigation of the production of beats. (*cf.* Utk. U. 1966) (*cf.* C. U. 1963) (All. U. 1954)

9. Write a short note on combinational tone. (All. U. 1952 ; C. U. 1953)

10. Write short notes on the following :—(*a*) Interference of sound waves, (*b*) beats and (*c*) combinational tones. How can the existence of these phenomena be shown ? (*cf.* C. U. 1964)

11. Two tuning forks *A* and *B* are sounded together and 8 beats per second are heard. *A* is in resonance with a column of air, 32 cms. long in a pipe closed at one end and *B* is similarly in resonance when the length of the column is increased by one centimetre. Calculate the frequency of the forks. (C. U. 1950)

[*Ans.* 264 ; 256 vibs./sec.]

12. An organ pipe produces 8 beats per sec. when sounded with a fork of 256 vibrations per sec. the fork giving the lower note. How much must be the length of the pipe altered to bring it in unison with the fork ? (C. U. 1953)

[*Ans.* Increased by $l/32$]

13. Explain the difference observed when sunlight enters through an open window and when the audible sound waves enter through the same. What is an acoustical shadow ?

14. Sixty-four tuning forks are arranged in order of increasing frequency and any two successive forks give four beats per second when sounded together. If the last fork gives the octave of the first, calculate the frequency of the latter. (C. U. 1946)

[*Ans.* 252]

CHAPTER V

FREE AND FORCED VIBRATION : RESONANCE

48. Free Vibration :—Whenever a body emits sound, it vibrates. If it vibrates with a single period without any external interference, the vibration is said to be *free vibration*. Even a system of bodies connected together may possess a single vibration period which should also be termed as the *free or natural period of vibration*. A tuning fork when struck and left to itself will vibrate with its natural frequency.

Though in the above definition of natural period the vibration is supposed to be unrestricted in the true sense, no such vibration is ever possible. The medium offers a definite resisting or damping effect on the vibration. Even the elastic property of the vibrating body sets up a kind of ***internal friction*** so that all actual vibrations are always damped. But it can be shown that if the damping forces be not considerably large, its effect on the natural frequency is very small and as a consequence the frequency of vibration of the body even in the case of a resisted or damped motion may very well be taken to be its natural frequency of vibration.

Free or unresisted Vibration (Mathematical Analysis).—

Here evidently the motion is simple harmonic in nature. The equation of motion of a vibrating particle of mass m may be written as

$$m\frac{d^2x}{dt^2}+ax=0.$$

where $m\frac{d^2x}{dt^2}$ is the inertial force and ax is the restoring force proportional to displacement.

$$\therefore \quad \frac{d^2x}{dt^2}+\omega^2x=0, \text{ when } \omega=\sqrt{\frac{a}{m}}.$$

The solution of the equation will be of the form $x=A\sin(\omega t+\delta)$ where A and δ are two constants (*vide* General Physics, Chap. VI).

Here the *period of vibration* is evidently given by, $T=\frac{2\pi}{\omega}$ because after each interval of $\frac{2\pi}{\omega}$(*i.e.* for $t=\frac{2\pi}{\omega}$ or its multiples), the value of x will be repeated. This is the ***natural period*** of vibration of the body. The ***natural frequency*** of the undamped motion is consequently given by $n=\frac{1}{T}=\frac{\omega}{2\pi}$.

49. Damped Oscillations (Mathematical Analysis) :—When a body of mass m, vibrates in a medium which exerts a damping force on the body, the equation of motion will have to be modified. As the resisting force is found to be proportional to velocity, we can write it as $b\frac{dx}{dt}$ where b is a damping constant. This force acts against the inertial force $m\frac{d^2x}{dt^2}$ due to its motion. Over and above a restoring force, ax, proportional to the displacement and also directed opposite to the inertial force comes into play, where the constant a is the restoring force per unit displacement.

So the equation of motion may be written as,

$$m\frac{d^2x}{dt^2}=-b\frac{dx}{dt}-ax\ ;$$

$$\text{or,}\quad \frac{d^2x}{dt^2}+\frac{b}{m}\frac{dx}{dt}+\frac{a}{m}x=0\ ;$$

$$\text{or,}\quad \frac{d^2x}{dt^2}+2k\frac{dx}{dt}+\omega^2x=0 \qquad \ldots \qquad \ldots \qquad (1)$$

where $2k=\frac{b}{m}$ = the resisting force per unit mass per unit velocity

and $\omega^2=\frac{a}{m}$ = the restoring force per unit mass per unit displacement.

Let the trial solution of (1) be $x=Ae^{mt}$,

$$\text{then, } \frac{dx}{dt}=mAe^{mt} \text{ and } \frac{d^2x}{dt^2}=m^2Ae^{mt}.$$

Substituting in (1), $Ae^{mt}(m^2+2km+\omega^2)=0$.

Hence $m^2+2km+\omega^2=0$.

$$\therefore\ m=-k\pm\sqrt{k^2-\omega^2}.$$

So the general solution of equation (1) becomes,

$$x=A_1e^{(-k+\sqrt{k^2-\omega^2})t}+A_2e^{(-k-\sqrt{k^2-\omega^2})t}$$

$$=e^{-kt}\left[A_1e^{\sqrt{k^2-\omega^2}t}+A_2e^{-\sqrt{k^2-\omega^2}t}\right] \quad \ldots \qquad \ldots \quad (2)$$

Depending on the conditions of damping, this solution takes three distinct forms.

(*i*) If the damping be very large, *i.e.* if $k>\omega$. $\sqrt{k^2-\omega^2}$ is positive. Here the displacement x has two terms both diminishing exponentially and as such the displacement quickly falls to zero and no vibration is executed. This is called dead beat or over-damped motion as is found in a dead beat moving coil galvanometer.

(*ii*) When $k=\omega$,

the solution is of the form $x=(A_1+A_2)e^{-kt}$

$$=Ce^{-\frac{bt}{2m}}.$$

This represents a non-oscillatory motion which is critically damped.

(*iii*) When $k<\omega$, $\sqrt{k^2-\omega^2}$ is imaginary.

Let $\sqrt{k^2-\omega^2}$ be written as $i\beta$ where $i=\sqrt{-1}$ and $\beta^2=\omega^2-k^2$.

$$\therefore \quad x=A_1e^{-kt+i\beta t}+A_2e^{-kt-i\beta t}=e^{-kt}(A_1e^{i\beta t}+A_2e^{-i\beta t})$$
$$=C_1e^{-kt}\sin(\beta t+\delta),$$

where C_1 and δ are new constants.

The equation evidently represents a damped harmonic motion. The amplitude of the vibration is Ce^{-kt} which dies off exponentially with time. The natural frequency of the damped vibration is here $\beta/2\pi$ instead of $\omega/2\pi$, the latter being the natural frequency of an undamped free oscillation. Since $\beta=\sqrt{\omega^2-k^2}$ and k is generally very small, numerically β and ω are nearly identical for most of the cases.

Example

If the frequency of the damped vibration of a body be 200 c.p.s and the oscillation dies away to $\frac{1}{e}$ of its former amplitude in 1 second, calculate the frequency of its free vibration.

Ans. **At an instant, t, the amplitude $=C_1e^{-kt}$.**

After 1 second this will be $=C_1e^{-k(t+1)}$.

$\therefore$ From the given condition, $\frac{1}{e}\cdot C_1e^{-kt}=C_1e^{-k(t+1)}$

$$\text{or,}\quad e^{-1}=\frac{e^{-k(t+1)}}{e^{-kt}}=e^{-k},$$

$$\therefore \quad k=1.$$

Again $\beta = 2\pi \times \text{frequency} = \pi \times 400$. $\therefore$ $\beta^2 = \pi^2 \times 160000 = 1579200$ nearly.

$\because$ $\omega^2 = \beta^2 + k^2 = 1579201$. Or, $\omega = \sqrt{1579201}$.

$\therefore$ The frequency of free vibration $= \frac{\omega}{2\pi} = \sqrt{\frac{1579201}{4\pi^2}}$

$= 200$ c.p.s. very nearly.

50. Forced Vibration : Resonance (General Ideas) :—Forced Vibration.—Whenever a body or a system is made to vibrate, its amplitude gradually diminishes due to the presence of damping forces and the vibrations will finally stop. So in order to maintain this vibration an *external force* must be impressed on the body *periodically* so that the energy loss due to damping may be replenished.

This is why the motion of the pendulum in a clock is maintained by a wound spring attached to it and the vibration of the stringed instruments maintained by continually exciting them.

When an external periodic force is applied to the body capable of vibration, and if *the period of the force is not the same as the free period of the body*, the body at first tends to vibrate in its own way but will ultimately vibrate with a period equal to that of the applied force. Such vibrations of the body are called *forced vibrations.*

If a vibrating tuning fork is held by the stem in the hand, the sound will be almost inaudible even from a small distance, but, if the stem be lightly pressed on a table, the sound is much intensified. The reason is that vibrations of the fork are communicated to the table which is thus forced to vibrate at the same rate. Due to the vibrations of the table a large volume of the air in contact is made to vibrate, and the waves thus set up are added to those originating from the fork, and consequently, the sound becomes louder.

The diaphragm of a gramophone sound box is a common example of forced vibration, where the diaphragm vibrates with frequencies corresponding to the tones conveyed from the record. The vibration from the sounding boards of musical instruments like violin, piano, etc. are also forced vibrations. The sounding board of a violin is first set into forced vibration by the vibration of the strings, and then the large mass of air inside the board also vibrates and intensifies the sound.

Resonance.—When a body is forced to vibrate due to a periodic external force applied to it, it vibrates with a very small amplitude, if the period of the applied force is different from that of the natural period of the body ; but when these two periods are the *same*, the body vibrates with a much greater amplitude. The latter phenomenon is known as **resonance.** Thus *resonance is a particular case of forced vibration and is produced when one body forces vibration on a second body whose natural frequency of vibration is equal to that of the first.* The principles of forced vibration and resonance may be illustrated by the following experiment due to Barton.

An India-rubber cord *CC* (Fig. 30) three to four metres in length is horizontally stretched between two rigid supports. The cord should be rather a bit loosely stretched. A pendulum of heavy bob *A* is suspended from a point on this cord. The pendulum is known as the driver pendulum or simply the driver. A few other simple pendulums are also suspended from different positions of the cord. The bobs of these pendulum should be small paper-cones (because heavy bobs would bring in complexity in their vibrations which would no more be forced but rather coupled in nature). When the driver pendulum is set to vibration, a periodic force is applied to the elastic cord *OC* which begins to vibrate coperiodically with *A*. The pendulums P_1, P_2,...P_6 at first move rather irregularly but ultimately begin to vibrate with the same period as the driver and each with a constant amplitude depending on its length. So the periodicity of the vibration of *A* is *forced* on each of the pendulums. Among these pendulums if any one has its effective length identical with that of *A*, it is found that the forcing is almost spontaneous and it readily begins to vibrate with the frequency of the driver. This is because the lengths of both being identical, the natural

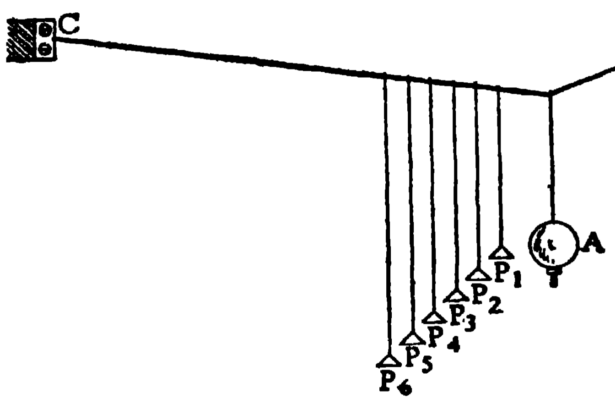

Fig. 30

frequency of vibration of the pendulum in question is same as the frequency of the driver and hence the former quickly responds to the driving force. This is an example of ***resonant vibration.*** In case the driven body or pendulum resonates with driver, the amplitude of the former may sometimes exceed that of the latter. It is due to such resonant vibrations that a suspension bridge was reported to have given way on account of a large swing set up therein when a battalion of soldiers were trying to cross the bridge with the regular marching order. So the soldiers are now-a-days ordered to break step while crossing a suspension bridge.

Again if two musical instruments are tuned to the same frequency and if one of them is sounded, the second is automatically excited when placed close by. Similarly, a ship at sea may be thrown into dangerous oscillation when the period of the striking waves tallies with the natural period of vibration of the ship.

51. Forced Vibration and Resonance (Mathematical Analysis) :—It has already been stated that the vibrations of a body or a system in a resisting medium gradually die out due to the damping effect of the medium and a periodic external force is to be impressed on it in order to maintain its oscillations. Let us now analytically consider the effect of this impressed periodic force on the motion of the body or the system.

The equation of motion in the resisting medium will be as before,

$$m\frac{d^2x}{dt^2}+b\frac{dx}{dt}+ax=F\sin pt$$

where $F\sin pt$ is the periodic force of period $T=\frac{2\pi}{p}$ and of frequency $n=p/2\pi$.

This equation can be rewritten as,

$$\frac{d^2x}{dt^2}+2k\frac{dx}{dt}+\omega^2x=f\sin pt \ldots \qquad \ldots \qquad \ldots \quad (1)$$

where $2k=\frac{b}{m}$=damping force per unit mass per unit velocity,

$\omega=\sqrt{a/m}$ and $\frac{\omega}{2\pi}$=natural frequency of free vibration of the body,

f evidently, being $\frac{F}{m}$, the amplitude of the impressed force per unit mass.

Let the particular solution of eqn. (1) be $x = A \sin (pt - \alpha)$.

Then $dx/dt = Ap \cos (pt - \alpha)$ and $\frac{d^2x}{dt^2} = -Ap^2 \sin (pt - \alpha)$.

Putting in (1), $-Ap^2 \sin (pt - \alpha) + 2kAp \cos (pt - \alpha) + \omega^2 A \sin (pt - \alpha) = f \sin [(pt - \alpha) + \alpha]$

$$= f \sin (pt - \alpha) \cos \alpha + f \cos (pt - \alpha) \sin \alpha.$$

Since this equation is valid for all values of t, when $(pt - \alpha) = 0$, we have, $2kAp = f \sin \alpha$ ... ... (2)
and when $(pt - \alpha) = \pi/2$, $-Ap^2 + A\omega^2 = f \cos \alpha$ ... ... (3)

Hence $\tan \alpha = \frac{2kp}{\omega^2 - p^2}$. This gives α in the solution.

Again squaring and adding (2) and (3),

$$f^2 = A^2\{(\omega^2 - p^2)^2 + 4k^2p^2\};$$

$$\text{or,} \quad A = \frac{f}{\sqrt{(\omega^2 - p^2)^2 + 4k^2p^2}}.$$

Thus A and α being known the *particular solution* is completely determined by the relation $= A \sin (pt - \alpha)$.

Again since $F \sin pt$ is periodic in nature, in between its maximum and minimum values ($+F$ and $-F$) it will pass through the transition state when it reduces to zero.

Hence the complete solution of (1) will contain a ***complementary solution***, which is nothing but the solution of the equation,

$$\frac{d^2x}{dt^2} + 2k\frac{dx}{dt} + \omega^2 x = 0.$$

This complementary solution, as has been found earlier, is of the form, $x = C_1 e^{-kt} \sin (\beta t + \delta)$, where $\beta^2 = \omega^2 - k^2$.

Hence complete solution of (1) is given by,

$$x = C_1 e^{-kt} \sin (\beta t + \delta) + A \sin (pt - \alpha) \quad \ldots \quad \ldots \quad (4)$$

In the beginning both these terms are prominent and the two oscillations of frequencies $\beta/2\pi$ and $p/2\pi$ may produce **beats**. But the first ***transient*** term gradually dies out with time because of the exponential factor and the system vibrates only with the forcing frequency in the steady state of its motion.

Note** :—As has been already stated, since the vibrating system vibrates in a resisting medium, the vibrations would gradually die out due to dissipation of energy. But when a periodic impressed force acts on the system, this external force would replenish this dissipated energy and the vibrations of the system would continue as usual. Or in other words, ***when a steady state of forced vibration has been attained, the average rate of energy loss due to the existence of the dissipative force is equal to the average rate of energy supply by the impressed force. This fact may be established by the simple mathematical analysis given below.

We know that the general equation of motion of the forced vibrating system in a resisting medium is given by

$$m\frac{d^2x}{dt^2}+b\frac{dx}{dt}+ax=F\sin pt.$$

or, $$\frac{d^2x}{dt^2}+2k\frac{dx}{dt}+\omega^2x=f\sin pt.$$

Here the dissipative force is $b\frac{dx}{dt}=b\dot{x}$.

Hence energy loss due to dissipation per sec.

$$=\text{rate of work done by the dissipative force}=\frac{dW_1}{dt}$$

$$=\frac{d}{dt}(b.\dot{x}\times\text{displacement})=\frac{d}{dt}(b\dot{x}\times x)$$

$$=b\dot{x}\frac{d}{dt}(x)=b\dot{x}^2, \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

because $b\dot{x}$ is constant so long as $\ddot{x}$ remains constant.

We know that the steady displacement,

$$x=A\sin(pt-\alpha)=\frac{f\sin(pt-\alpha)}{\sqrt{(\omega^2-p^2)^2+4k^2p^2}}$$

where ω, p, k and α have the same meaning as given above.

$$\therefore \quad \dot{x}=\frac{pf\cos(pt-\alpha)}{\sqrt{(\omega^2-p^2)^2+4k^2p^2}}=\frac{F\cos(pt-\alpha)}{z} \quad \ldots \quad (2)$$

where $F=fm$ and $z=\frac{m}{p}[(\omega^2-p^2)^2+4k^2p^2]^{\frac{1}{2}}$

$$=\left\{m^2\left(\frac{\omega^2}{p}-p\right)^2+4k^2m^2\right\}^{\frac{1}{2}}=\left\{\left(\frac{a}{p}-mp\right)^2+b^2\right\}$$

This z is generally known as the impedance of the system.

$\therefore$ From (1) and (2),

$$\frac{dW_1}{dt}=b\left(\frac{F}{z}\right)^2.\cos^2(pt-\alpha)$$

Taking the average over a period, $\frac{dW_1}{dt}=\frac{1}{2}b\frac{F^2}{z^2}$ ... (3)

Again the rate of energy supply by the forcing system (or the external impressed force) is, in the steady state, given by,

$$\frac{dW_2}{dt}=F\sin pt.\ \frac{d}{dt}(x)=F.\dot{x}.\ \sin pt.$$

substituting from eqn. (2),

$$\frac{dW_2}{dt}=\frac{F^2}{z}\sin pt.\cos(pt-\alpha)$$

$$=\frac{F^2}{z}[\sin pt\cos pt\cos\alpha+\sin^2 pt\sin\alpha].$$

Taking the average over a period, $\frac{dW_2}{dt}=\frac{1}{2}\frac{F^2}{z}\sin\alpha$

But $\sin\alpha=\frac{2kp}{\sqrt{(\omega^2-p^2)^2+4k^2p^2}}=\frac{b}{\frac{m}{p}[(\omega^2-p^2)^2+4k^2p^2]^{\frac{1}{2}}}$

$$=\frac{b}{z};$$

$\therefore$ $\frac{dW_2}{dt}=\frac{1}{2}b.\ \frac{F^2}{z^2}$ .... (4)

Thus from (3) and (4) we find that in the steady state of forced vibration the rate of energy dissipated by the dissipating force is equal to the rate of energy replenished by the forcing system.

Amplitude Resonance.—In the steady state of motion the amplitude of the forced vibration is $A=\frac{f}{\sqrt{(\omega^2-p^2)^2+4k^2p^2}}$.

Amplitude resonance occurs when this amplitude of the forced vibration has the maximum value.

A is maximum when the denominator is minimum, *i.e.*

when, $\frac{d}{dp}\,[(\omega^2-p^2)^2+4k^2p^2]=0$;

or, when $-4(\omega^2-p^2)p+8k^2p=0$;

i.e. when $(\omega^2-p^2)=2k^2$;

or, $p^2=\omega^2-2k^2$.

So the amplitude resonance occurs when the forcing frequency $\omega/2\pi$ is slightly lower than the natural frequency of free vibration of the body.

Velocity Resonance.—But by resonance we commonly mean that particular phenomenon wherein the energy taken in to maintain the forced oscillation is a maximum. Since the kinetic energy of a vibrating body is greatest when its velocity is maximum, the resonance in general indicates the velocity resonance (or **energy resonance**).

In the steady state of vibration,

$$x=A\sin(pt-\alpha).$$

$\therefore$ The velocity, $\frac{dr}{dt}=Ap\cos(pt-\alpha)$.

Hence the velocity amplitude $=AP=\frac{fp}{\sqrt{(\omega^2-p^2)^2+4k^2p^2}}$

$$=\frac{f}{\sqrt{\left(\frac{\omega^2-p}{p}\right)^2+4k^2}}.$$

This is maximum when $\omega^2=p^2$ or, $\omega=p$ and the maximum value of the velocity amplitude will be $f/2k$.

So when the forcing frequency is equal to the natural frequency of the free vibration of the body, the oscillation of the body is maintained by the impressed force and the *resonance* is said to occur.

At resonance the **phase** of the forced vibration lags behind the driving force by an angle $\alpha=\tan^{-1}\frac{2kp}{\omega^2-p^2}=\tan^{-1}\frac{2kp}{o}=\tan^{-1}\infty=\frac{\pi}{2}$.

When p is small, α is also small. As p increases, α also gradually increases till at resonance the above value $\pi/2$ is attained. Above the resonant frequency, as p tends to infinity, α tends to π. If the damping force be negligibly small (*i.e.* if $k\to o$) at resonance frequency, both the amplitude and the velocity tend to infinity. As k has always a definite value, however small it may be, the ideal condition of infinite amplitude is never realised in practice.

52. Sharpness of Resonance :—At the resonance frequency $\left(i.e.\ p/2\pi=\frac{\omega}{2\pi}\right)$ of the system, the energy intake being a maximum, the response is also greatest. *The response is generally defined as the ratio of the energy intake of the forced system at any arbitrary value of the frequency to the maximum energy intake at a resonance.* According to Barton response is the kinetic energy per unit force.

The energy intake is given by $E=\frac{1}{2}mv_a^2$, where v_a is the velocity amplitude.

Its maximum value at resonance is given by $E_{\max}=\frac{1}{2}mv_m^2$, where v_m= velocity amplitude at resonance.

∴ The response is given by, $R=\frac{E}{E_{\max}}=\frac{v_a^2}{v_m^2}$

$$=\frac{f^2p^2}{(\omega^2-p^2)^2+4k^2p^2}\div\frac{f^2}{4k^2}=\frac{4p^2k^2}{(\omega^2-p^2)^2+4k^2p^2},$$

When resonance occurs $\omega=p$, and we have,

$\frac{E}{E_{\max}}=1$, *i.e.* the response is cent per cent. The system is then said to be tuned. The difference (ω^2-p^2), generally known as the *mistuning* and is evidently proportional to the *mistuning ratio* p/ω.

Now if the energy intake be plotted against the mistuning or simply p/ω, we get curves of the nature shown in Fig. 31. When $\frac{p}{\omega}=1$ (*i.e.* $p=\omega$) the energy is maximum, hence also the response. When $\frac{p}{\omega}$ is greater than unity *i.e.* $p>\omega$, the curves slope down. Similar is the case when $\frac{p}{\omega}$ is less than unity, the sloping here being on the opposite direction.

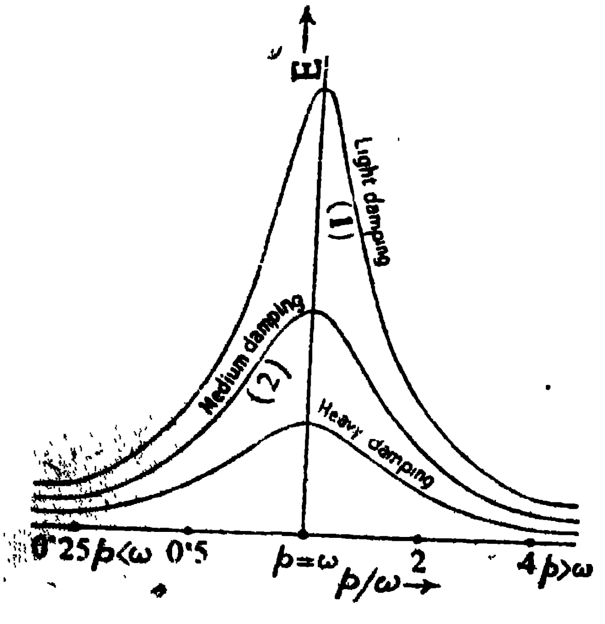

Fig. 31

It is found that the less is the damping of the system, the steeper will be the energy intake curve [as in curve (1)]. When the damping is zero, the energy intake curve cuts the E-axis at infinity at resonance. As the damping gradually increases the curves become more flat. Now the steepness of these resonance curves determines the *sharpness of resonance*. So the sharpness of resonance is determined by the rate of change of energy intake with mistuning. As is evident from the curves the smaller the damping the sharper will be the resonance. Mathematically, *the sharpness of resonance may be measured by the amount by which the ratio ω/p deviates from unity while the response falls to half its resonance value.* As at resonance response is unity, half its value is evidently $\frac{1}{2}$.

$$\therefore \quad \frac{4k^2p^2}{(\omega^2-p^2)^2+4k^2p^2}=\frac{1}{2};$$

$$\text{or,} \quad (\omega^2-p^2)^2=4k^2p^2 \; ; \text{ or, } \omega^2-p^2=\pm 2kp.$$

$$\therefore \quad p^2=\omega^2 \pm 2kp=\omega^2\left(1\pm\frac{2kp}{\omega^2}\right);$$

$$\text{or,} \quad p=\omega\left(1\pm\frac{kp}{\omega^2}\right)=\omega\pm\frac{kp}{\omega}.$$

$$\text{Hence } \omega=p\left(1\pm\frac{k}{\omega}\right) ; \text{ or, } \frac{\omega}{p}=1\pm\frac{k}{\omega}.$$

Thus the ratio $\frac{k}{\omega}$ is a measure of the sharpness of resonance. The smaller k is and the greater the natural frequency, the sharper is the resonance.

If we start from Barton's definition of response, exactly similar conclusions regarding the sharpness can be arrived at.

Again if we plot amplitude against the corresponding p/ω, almost a similar set of curves will be obtained. But here at $p=\omega$ (*i.e.* $p/\omega=1$) the maxima of the amplitudes would not lie, which will be shifted slightly to the $p>\omega$ side. So the curves will be slightly asymmetrical. Sharpness in case of amplitude resonance is also found to be controlled by the same factor, damping. The smaller the damping the sharper will be the amplitude resonance.

If a tuning fork be sounded on an air column it will respond but the sound will die out as soon as the fork is removed. This is due to a considerable damping effect. It is due to this damping that the resonance curve is rather flatter near the maximum region (see curves of medium and heavy damping). So the same column may respond to other frequencies near about the resonant frequency. But if two tuning forks of identical frequency be mounted on two resonance boxes (see next article) with their open ends facing each other and if one of them be excited, the other is also found to vibrate readily in response. Now if the original fork be touched so as to stop its vibration, the second one will still continue its vibration as the damping effect is negligibly small. Here the resonance is so sharp because of negligible damping that even if one of the forks be slightly loaded by wax, it would no more respond to the vibration of the other.

53. Applications of the Principle of Resonance :—The principle of resonance has many applications in various branches of Physics. A few of those applications—particularly those associated with acoustical phenomena are given below.

(1) Sounding (or Resonance) Boxes.—Tuning forks are often mounted on hollow wooden boxes (Fig. 32), called sounding or *resonance boxes.* The sizes of these boxes are so arranged that the enclosed mass of air has a free vibration whose natural period is the same as that of the fork. When the fork is struck, it sets the wood into forced vibration of the same period, and this agrees with the natural period of vibration of the enclosed mass of air; so the sound becomes louder due to resonance.

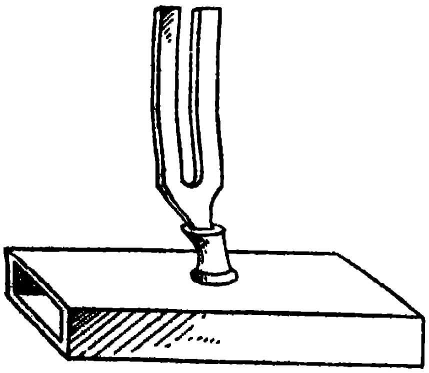

Fig. 32—Resonance Box.

Here the energy of the vibrating fork is quickly used up in setting the wood with the enclosed air into vibration, whereby loudness is gained at the cost of duration of sound. The action does not violate [illegible]tion of energy. Instruments like the [illegible] etc. are always provided with a large hollow wooden board, known as a **sounding board** whose principle of action is similar to what is explained above. When the handle of a tuning fork vibrating feebly is held on a table, the sound is intensified. Here the intensification is due to the vibration of a large volume of air which is made to vibrate by the forced vibration of the table.

(2) **Resonators.**—The great German scientist Helmholtz (1821-1894) constructed globes of brass, each having a large aperture *B* at the end of a small cylindrical extension called the neck for receiving sound waves and a small one *A*, called the *pip*, at the other side against which the ear is placed (Fig. 33). He utilised the principle of resonance in his investigations on the quality of notes emitted by various sources. These globes of various sizes are called **Helmholtz resonators.** In a given set of these resonators the size of each resonator is such that it can respond to a tone of a given fixed frequency and the tuning is so perfect that the particular tone, if present in a complex note, can be picked up with distinctness, by placing the ear at the small aperture *A*.

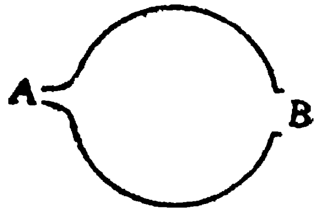

Fig 33

The natural frequency of each of such resonators depends on the volume of the air contained in the globe, the length and the cross-sectional area of the neck. So when the frequency of any component of a complex tone corresponds to this natural frequency resonance will be set up and the resonator would respond to it. The natural frequency of a resonator may be easily calculated in the following way.

When the air in the resonator is set to vibration, it is the air in the neck *B* that vibrates rather much more rapidly and that inside the globe may be supposed to be practically steady. But due to the vibration in the neck the gas inside the globe is also subjected to adiabatic compressions and rarefactions. Let ρ be the density of the air, v the volume of the resonator, l the length of the neck and s its cross-sectional area.

Then, mass of air in the neck $= \rho ls$.

Suppose due to a slight release of pressure, the air in the neck is displaced outwards through a distance x. If p be the pressure inside the neck and p_0 that outside then $(p-p_0)$ is the pressure change. So the corresponding force acting on the neck $=(p-p_0)s$.

$$\therefore\ \rho l s\frac{d^2x}{dt^2}=(p-p_0)s \qquad \ldots \qquad \ldots\ (1)$$

But as the adiabatic condition prevails during the rarefaction,

$p(v+sx)^{\gamma}=p_0v^{\gamma}$, where γ is the ratio of the two specific heats of air.

$$\therefore\ p\left(1+\frac{sx}{v}\right)^{\gamma}=p_0\ ;$$

$$\text{or, } p-p_0=-\frac{\gamma psx}{v}.$$

Hence equation (1) takes the form,

$$\rho l\frac{d^2x}{dt^2}=-\frac{\gamma ps}{v}x\ ;$$

$$\text{or, } \frac{d^2x}{dt^2}=-\frac{\gamma sp}{vl\rho}x \qquad \ldots \qquad \ldots \qquad \ldots\ (2)$$

This equation represents a S.H.M. whose time period is, $T=2\pi\sqrt{\frac{vl\rho}{\gamma sp}}$, *i.e.*, the frequency of vibration, $n=\frac{1}{2\pi}\sqrt{\frac{s}{vl}\times\frac{\gamma p}{\rho}}$.

But $\sqrt{\frac{\gamma p}{\rho}}=$ the velocity of sound $=c$ (say).

$$\therefore\ n=\frac{c}{2\pi}\sqrt{\frac{s}{vl}}.$$

Thus the natural frequency of vibration of such a resonator depends on the length and cross-sectional area of the neck and the total volume of the air enclosed in it. The ratio s/l is often called the ***acoustical conductivity of the neck.*** So different sizes of resonators can be constructed with different frequencies so that a complex sound may be analysed with their help.

Acoustic Filters :—These are acoustical arrangements for filtering out or sorting out tones of certain frequencies allowing others to pass through them. By proper arrangements the filters may be made to transmit high frequencies only or any desired band of frequency. A common type of filter is the Helmholtz's resonator which filters out all other frequencies excepting its own characteristic one. Stewart has devised acoustical filters which consist of uniform transmission pipes with side branches fitted to them at different convenient positions. The sound wave passing through the main pipe will divide into two parts at the junction of a side-tube. The portion going down the side-tube after reflection gets back to the junction. By proper adjustment of the length of this side-tube, this reflected component is made to superpose on the incident wave with a path difference of $\lambda/2$ and a minimum can be produced at the junction at all times. High frequency transmission is effected with a uniform straight pipe fitted with side-tube at regular intervals. Similarly low frequency or any other band frequency filters have been constructed by special designs of the transmission tube.

The filters are useful in purifying notes in scientific experiments and are successfully applied in filtering out the scratching sound of the needle in gramophones. The motor vehicles also employ such filters to muffle the sound from the exhaust.

(3) **Resonance of Air-Column.**—The air-column within a tube may also be made to vibrate by resonance, when a vibrating tuning fork is held close to the upper end of the tube.

Take a vibrating tuning fork *A* and hold it horizontally as shown in the figure, over a tall glass jar *B* (Fig. 34). Now gradually pour water into the jar and note that for a certain length *ED* of the air-column inside the jar a maximum sound is heard. Pour more water in and the sound disappears. This strengthening of the sound is called ***resonance***, which, in this case, takes place when the period of vibration of the tuning fork is equal to the natural period of vibration of the enclosed column of air.

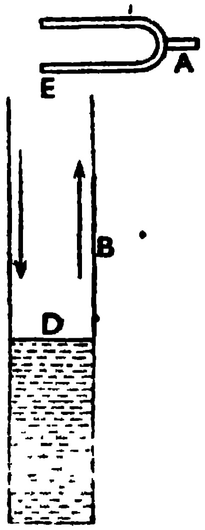

Fig. 34—Resonant Air-column.

It will be found that, for forks having different frequencies of vibration, the lengths of the air-column giving maximum resonance will be different. It will be greater or less as the frequency of vibration of the fork is lower or higher.

(4) **The Principle of Resonance in Wireless.**—In the reception

of wireless waves sent by a transmitting station, the oscillating electrical circuit in the receiving station must be set in unison with the oscillating circuit in the transmitting one. This is called the tuning of the receiving set. Here the sharpness of resonance is sufficiently pronounced so that a slight mistuning causes an indistinct reproduction whereas when the tuning is perfect the reproduction becomes very distinct and pronounced.

54. A Coupled System :—When two or more vibrating systems are so connected as to react on each other, the whole system, is said to be mechanically coupled. In a coupled system, the energy is transferred from one component to another whereupon a change in the natural frequency of the components occurs. At resonance, the energy-transfer from one component to the other is very great and almost complete so that the first component temporarily comes to rest while the second one acquires the maximum energy. Then begins the retransfer of energy and the process is repeated till the components finally come to rest due to the continuous, dissipation of energy caused by the resistance of the medium.

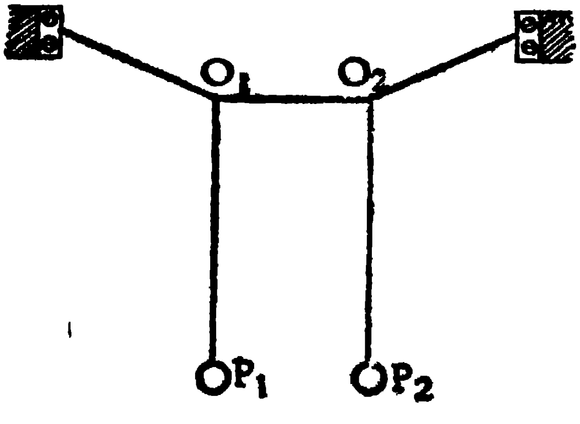

Fig. 35

The phenomenon of coupled vibration may be illustrated by two identical pendulum of heavy bobs hung from a thick flexible cord. If pendulum P_1 be given a swing, P_2 readily takes up the oscillation because of the equality of their natural frequencies (Fig. 35). The energy of the swing of P_2 is supplied by P_1 through the cord. According to Newton's third law P_2 exerts an opposing periodic force on P_1 and at a particular stage this opposing force will completely destory the gradually decreasing oscillations of P_1. Then the ocillations of P_2 will again freshly force now oscillations on P_1 and the state of affairs will be reversed, *i.e.* P_2 will again come to rest. This alternate energy transfer goes on till the vibrations of both P_1 and P_2 are stopped. The rapidity with which the oscillations of one pendulum increases at the expense of those of the other is determined by the *degree of coupling*. When the supporting cord is tight, the coupling is said to be *loose* or *small*, but when the cord is rather slack the coupling becomes more *tight* or predominant

Most of the musical instruments may be considered to be coupled systems, some elements of them being sharply tuned while the others are allowed to have a flat resonance curve.

Questions (Chapter V)

1. Write notes on forced and free vibrations. (C. U. 1955, '56)

2. Explain clearly what you mean by free and forced vibrations. Discuss the phenomena of resonance and give some of its practical applications. (Bom. U. 1943)

3. Write a short essay on resonance, pointing out how the subject may be experimentally illustrated. (C. U. 1933)

4. Discuss mathematically the phenomenon of forced vibration and explain the phenomenon of sharpness of resonance. (Nag. U. 1955)

5. A system executing a damped simple harmonic motion is subjected to an external periodic force having a frequency somewhat different from its own natural frequency without damping. Investigate the forced vibration and obtain the condition of resonance. Illustrate it by an example. (All. U. 1958)

6. Write notes on Helmholtz's resonator. (Nag. U. 1955)

7. Why the faint sound from a tuning fork becomes quite loud on placing the same upon a table top ? (C. U. 1941)

8. Write notes on coupled vibration.

9. Describe some of the applications of the principle of resonance in the practical field.

CHAPTER VI

VELOCITY OF SOUND : DOPPLER EFFECT

55. Velocity of a longitudinal wave through an elastic medium :—Since the sound waves are longitudinal in nature, the velocity of sound in an elastic medium will be identical with the velocity of propagation of a longitudinal wave in such a medium. Let us take the general case of propagation of a longitudinal wave in an elastic medium, fluid or solid, and find out an expression for the velocity of these waves in such a medium.

Consider a tube of the medium of *unit cross-section* and let A and B represent two sections of the tube perpendicular to its axis. Suppose initially in the undisturbed condition these planes are at distances x and $x+\delta x$ from some arbitrary zero position. Evidently the volume of the medium enclosed between these planes is $\delta x \times 1 = \delta x$ and the mass of the material therein is $\delta x \times \rho$ where ρ is the density of the material.

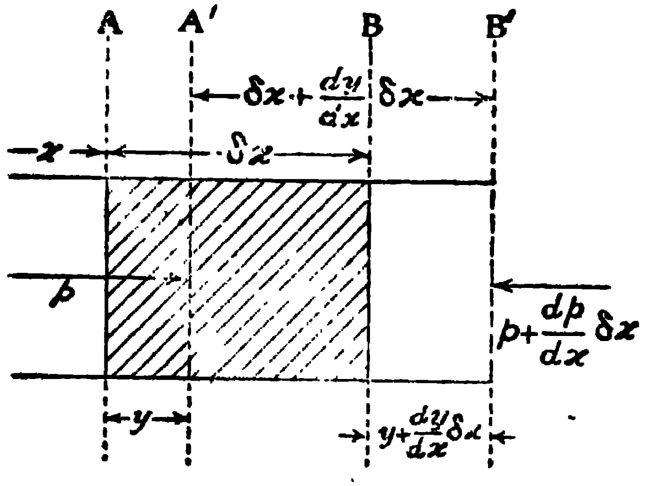
A
A′
B
B′
$\delta x + \frac{dy}{dx}\delta x$
x
δx
p
$p+\frac{dp}{dx}\delta x$
y
$y+\frac{dy}{dx}\delta x$

Fig. 36

Now, suppose, due to the application of an excess pressure p above the normal, acting along the positive direction of x, a longitudinal plane wave is set up in the medium. As a result of this excess pressure p, the section A is displaced to A'. The corresponding excess pressure on B is $p+\frac{dp}{dx}\delta x$ which displaces B to B' wherefrom the pressure is transferred on to the adjacent section of the medium and as such due to reaction an opposing pressure $\left(p+\frac{dp}{dx}\delta x\right)$ is set up in the direction from B' to A' acting in the negative direction of x. Let the displacement of A to A' be y (along the direction of x). Then the displacement of B to B' will be given by $y+\frac{dy}{dx}\delta x$. These displacements cause a change in the length of the medium enclosed between the planes.

The original length is δx and the final length is $\delta x+\frac{dy}{dx}\delta x$.

Hence the change in the length of the column of medium $=\frac{dy}{dr}\delta x$.

Since the cross-sectional area is unity, this also represents the change in volume, the original volume being δr.

Thus the expression $\frac{dy}{dx}\delta x/\delta x=\frac{dy}{dx}$ represents the change in volume per unit volume in the medium and also the change in length per unit length of the medium. Thus $\frac{dy}{dx}$ is the strain produced in the medium (volume strain in a fluid or a longitudinal strain in a solid bar) due to a stress p which is equal to the applied external excess pressure. But if p be positive the strain will be negative because a compression causes a diminution in volume or length.

Hence putting a negative sign to $\frac{dy}{dx}$ and taking p to be positive, the modulus of elasticity, E (volume or longitudinal as the case may be) can be written as,

$$E=\frac{\text{stress}}{\text{strain}}=\frac{p}{-\frac{dy}{dx}}\;; \qquad \text{or,} \quad p=-E\frac{dy}{dx} \qquad \cdots \quad (1)$$

$$\therefore \quad \left(p+\frac{dp}{dx}\delta r\right)=-E\frac{dy}{dx}+\frac{d}{dx}\left(-E\frac{dy}{dx}\right)\delta x$$

$$=-E\frac{dy}{dr}-E\frac{d^2y}{dx^2}\delta x \qquad \cdots \qquad \cdots \qquad \cdots \quad (2)$$

This pressure acts in the direction from B' to A' due to the reaction.

Hence the excess pressure acting on the medium is $E\frac{d^2y}{dx^2}\delta x$, from eqns. (1) and (2).

This force acting per unit area gives an acceleration $\frac{d^2y}{dt^2}$ to the molecules of the medium along the axis of the tube. Since the mass of the medium in the enclosed space $=\delta x.\rho$, the force acting on it is $\rho\,\delta x.\frac{d^2y}{dt^2}$.

Hence these two forces must be identical according to Newton's second law.

$$\therefore\quad \rho\delta x.\frac{d^2y}{dt^2}=E\frac{d^2y}{dx^2}\delta x\ ;$$

$$\text{or,}\quad \frac{d^2y}{dt^2}=\frac{E}{\rho}\frac{d^2y}{dx^2}\qquad\ldots\qquad\ldots\qquad(3)$$

This is the *D'* Alembert's form of the wave equation. Putting $\frac{E}{\rho}=V^2$,

eqn. (3) reduces to $\frac{d^2y}{dt^2}=V^2\frac{d^2y}{dx^2}$ ... ... (4)

This differential equation is satisfied if

$$y=f(Vt-x),\text{ where } f \text{ is a function.}$$

$$\text{Since } \frac{dy}{dx}=-f'(Vt-x),\ \frac{d^2y}{dx^2}=f''(Vt-x)$$

$$\text{and } \frac{dy}{dt}=Vf'(Vt-x),\ \frac{d^2y}{dt^2}=V^2f''(Vt-x).$$

Similarly, $y=f(Vt+x)$ also satisfies the above equation.

So the general solution for eqn. (4) should be

$$y=f_1(Vt-x)+f_2(Vt+x)\qquad\ldots\qquad\ldots\qquad(5)$$

where f_1 and f_2 may be identical or different functions. The first function, $f_1(Vt-x)$ of the solution shows that y at any place and time is exactly repeated if t increases by δt and x by $V\times\delta t$ *i.e.* it represents a disturbance travelling in the positive direction of x with a velocity V. Similarly, the second function represents a disturbance travelling in the negative direction of x also with an identical velocity V.

Thus the velocity of propagation of the wave is given by $V=\sqrt{E/\rho}$. In a fluid medium E = the volume elasticity K, and in a solid rod, E = Young's modulus Y.

[**An Alternative Method.**—An expression for the velocity of propagation of a plane longitudinal wave through a medium of fluid or of solid, may also be deduced in the following simple way.

Let us consider a long tube of the medium of cross-sectional area S and let AB represent a thin section of the medium at the left-hand extremity of the tube. Initially the medium is under normal conditions. But when a plane compressional wave is incident on the section AB, the latter is compressed. This compression is gradually transferred from layer to layer with a velocity, V equal to that of the incident longitudinal wave. So at the end of t seconds, the first compression received by AB moves to $A'B'$ through a distance Vt. During this time the medium of the first layer AB moves through ut to the position ab, where u is the velocity of the medium along the axis of the tube due to its vibrational motion.

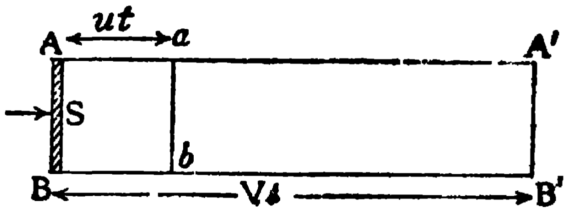

Fig. 37

So the medium which originally occupied a volume $Vt\times S$, now after t seconds occupies a volume $(V-u)t\times S$. As the mass of the medium is the same in both cases, we can write, $VtS\rho_0=(V-u)tS\rho$, where ρ_0 and ρ are the initial and the final densities of the medium.

Again supposing the medium in the initial state to be steady its momentum was zero. So the final momentum of the medium represents its momentum change in t seconds.

This final momentum $=(V-u)tS\rho\times u$.

$\therefore$ The extra force applied on the medium

$$=\text{the rate of change of momentum}=\frac{(V-u)tS\rho\times u}{t}$$

$$=(V-u)\rho Su \qquad \dots \quad (1)$$

If p be the excess pressure applied on the medium due to the incident wave, the above extra force is $p\times S$

$$\therefore \quad pS=(V-u)\rho uS\ ; \text{ or, } p=(V-u)\rho u \qquad \dots \quad (2)$$

In the case of propagation through a fluid medium, this pressure causes a change in the volume of the fluid and as such the volume elasticity comes into play. If the volume elasticity be written as k, then,

$$k = \frac{p}{\dfrac{\text{change in volume}}{\text{original volume}}} = \frac{p}{\dfrac{utS}{VtS}} = p \Big/ \frac{u}{V}.$$

$$\therefore\ p = \frac{uk}{V}.$$

Putting in (2), $(V-u)\rho u = \dfrac{uk}{V}$.

Since u is very small compared to V, we get, $V\rho = \dfrac{k}{V}$;

$$\text{or, } V = \sqrt{\frac{k}{\rho}} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3)$$

In the case of a medium in the form of a solid rod the pressure in eqn. (2) causes a change in the length of rod and the longitudinal elasticity comes into play. The modulus of this elasticity, Y

$$\text{(Young's modulus)} = \frac{p}{\dfrac{\text{change in length}}{\text{original length}}} = p \Big/ \frac{ut}{Vt}$$

$$= p \Big/ \frac{u}{V} \quad \therefore\ p = \frac{Yu}{V}.$$

Substituting in eqn. (2), we get, $\dfrac{Yu}{V} = (V-u)\rho u$.

Neglecting u in comparison to V, $\dfrac{Y}{V} = V\rho$;

$$\text{or, } V^2 = \frac{Y}{\rho}; \quad \text{or, } V = \sqrt{\frac{Y}{\rho}} \quad \ldots \quad \ldots \quad \ldots \quad (4)$$

Representing both the modulii k and Y by the general term E, we get $V = \sqrt{\dfrac{E}{\rho}}$.]

Pressure variation in a longitudinal wave.—

From relation (1) of Art. 55, we know that the pressure variation

p (or Δp) in a longitudinal wave propagated through a medium of co-efficient of elasticity E, is $p=-E\frac{dy}{dx}$.

In a longitudinal wave the displacement, y, is given by,

$$y=a\sin\frac{2\pi}{\lambda}(Vt-x) \qquad \ldots \qquad \ldots \qquad \ldots \quad (1)$$

$$\therefore \quad \frac{dy}{dx}=-\frac{2\pi a}{\lambda}\cos\frac{2\pi}{\lambda}(Vt-x).$$

Hence, $p=\frac{2\pi aE}{\lambda}\cos\frac{2\pi}{\lambda}(Vt-x)$

$$=\frac{2\pi aE}{\lambda}\sin\frac{2\pi}{\lambda}\left[\left(Vt-x\right)+\frac{\pi}{2}\right] \qquad \ldots \qquad \ldots \quad (2)$$

From relation (1) and (2), it is evident that the phase of the pressure variation (p) in a sound wave is ahead of that of the displacement y by $\frac{\pi}{2}$. So when $y=0$, p is $\frac{2\pi aE}{\lambda}$ and is maximum and when $y=a$ (*i.e.* max.), p falls to zero.

56. Newton's relation for the Velocity of Sound in Air :— The relation $V=\sqrt{E/\rho}$, was first deduced for gases by Newton. He assumed that during the propagation of sound in a gaseous medium, the compressions and rarefactions occur rather slowly. So any slight heat produced due to compressions is readily dissipated in the surroundings. Again, the cooling produced due to rarefaction is compensated for by the heat absorbed from the surroundings. The processes being supposed to be slow these dissipations and compensations are almost in step with the compressions and rarefactions. So the temperature remains unaffected in the processes, which may, therefore, be supposed to be ***isothermal*** according to Newton's assumption.

In the case of gases, ***stress*** is the change in ***pressure per unit area*** and ***strain*** is the corresponding change in ***volume*** produced ***per unit volume***.

Consider a gas of volume V c.c. under a pressure P dynes per unit area. Let the pressure be now increased by a very small amount p

per unit area, and let the volume be consequently diminished by a small amount v, the temperature remaining constant.

Then, the **isothermal** bulk elasticity,

$$E = \frac{\text{stress}}{\text{strain}} = \frac{\text{increase of force per unit area}}{\text{consequent decrease of volume per unit volume}}$$

$$= \frac{p}{v/V} = V\frac{p}{v}. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

Newton assumed that when sound travels through a gas, the change of pressure takes place under isothermal condition, i.e. it takes place so slowly that there is no change of temperature of the medium. So we have, according to Boyle's law,

$$PV = (P+p)(V-v) = PV + pV - vP - pv.$$

Since in the case of sound waves the changes in pressure and volume are very small, p and v are very small, and so the product pv is negligible.

$$\therefore \quad pV = vP \text{ ; or, } pV/v = P. \quad \therefore \quad E = P, \text{ from eqn. (1).}$$

Thus *the isothermal elasticity of a gas is equal to its pressure.*

[The same result may be arrived at by applying calculus as follows :

The bulk modulus $E = -dp/\frac{dv}{v} = -v\frac{dp}{dv}.$

For isothermal condition, pv=constant.

$\therefore$ Differentiating with respect to v, $v\frac{dp}{dv} + p = 0$, or, $-v\frac{dp}{dv} = p = E$]

Hence, according to Newton the velocity V of sound in gas is given by

$$V = \sqrt{\frac{P}{\rho}}.$$

Calculation of the Velocity of Sound in Air at N.T.P.—Normal pressure is the pressure exerted by a column of mercury 76 cms. in height at 0°C. at the sea-level at 45° latitude, *i.e.* $P = 76 \times 13{\cdot}596 \times 980{\cdot}6$ dynes/cm.^{2} $= 1{\cdot}013 \times 10^6$ dynes/cm.^{2}.

Again, density of air at $0°C = 0{\cdot}001293$ gms./c.c.

$\therefore$ Velocity of sound at N.T.P. $= \sqrt{\frac{P}{\rho}} = \sqrt{\frac{76 \times 13{\cdot}596 \times 980{\cdot}6}{0{\cdot}001293}} = 280$ metres/sec. (approximately). But this value of the velocity of sound at $0^\circ C.$ is not in agreement with the value obtained by actual experiment, which is 332 metres per second.

57. Laplace's Correction (isothermal and adiabatic Elasticities) :—The calculation of the elasticity of a gas, according to Newton, involved Boyle's law according to which changes of pressure and volume of a given mass of gas take place at a constant temperature. Newton assumed that the change in the air taking place in wave-motion had no effect on the temperature, *i.e.* the changes were *isothermal*. About 20 years later Laplace pointed out in 1817 that the changes of pressure, when sound waves travel through a gas, are *so rapid, and the radiating and conducting powers of a gas are so poor* that equalisation of temperature is improbable. So Newton's assumption that the temperature remains constant is not correct. According to him the changes that take place in a gas when sound waves travel through them are *adiabatic*, *i.e.* no heat enters the gas from outside, or leaves it from inside. That is, Laplace held that the alternate compressions and rarefactions take place so rapidly that the heat developed in the compressed layer remains fully confined to the compressed layer and has no time to be dissipated into the entire body of the gas, and similarly the cold caused in the rarefied layer cannot be compensated for by flow of heat into it from other layers. So Boyle's law does not apply to this case. Laplace's viewpoint was later on supported by Stokes from theoretical considerations also.

[When sound travels in air, or in any other gas, the-particles of the gas are suddenly compressed at the condensed part of the wave, and suddenly separated at the rarefied part of the wave. If a gas is compressed, or allowed to expand, suddenly, its temperature, rises or falls momentarily, and, with the rise or fall of temperature, the gas expands or contracts. Now consider the effects of changes of temperature on the elasticity of a gas. During compression the temperature of the gas rises owing to which the volume of it tends to increase and so a greater increase of pressure is necessary to produce a given diminution of volume than what is necessary if the temperature of the gas remained constant (*i.e.* if Boyle's law held good) during the compression. So the *elasticity* in the first case (when temperature increases) is *greater* than that in the second (when temperature is constant).

Similarly, during rarefaction the temperature of a gas falls owing to which the volume of it tends to diminish, and so a greater diminution of pressure is necessary to produce a given increase in volume than what is necessary if the temperature of the gas remained constant. So, here also the ***elasticity is greater*** than that in the isothermal case. Considering the above, Laplace said that the value for the elasticity E under ***adiabatic conditions should*** be used in the Newton's formula for the velocity of sound.]

It is known that the relation between the pressure P and volume V of a certain mass of gas under adiabatic conditions is given by $PV^\gamma = \text{constant}$,

$$\text{where } \gamma = \frac{C_p}{C_v} = \frac{\text{sp. ht. of the gas at constant pressure}}{\text{sp. ht. of the gas at constant volume}}.$$

The value of γ for a ***di-atomic*** gas like oxygen, nitrogen, or air is 1·41 (for a tri-atomic gas like CO_2 it is 1·33). Now suppose the pressure of any particular layer of air is increased adiabatically by a small amount p by which the volume is decreased by a small amount v; then, we have,

$$PV^\gamma = (P+p)(V-v)^\gamma = V^\gamma(P+p)\left(1-\frac{v}{V}\right)^\gamma$$

$$= V^\gamma(P+p)\left[1-\gamma\frac{v}{V}+\frac{\gamma(\gamma-1)}{2}\left(\frac{v}{V}\right)^2+\ldots\ldots\right]$$

(by the Binomial theorem).

But as (v/V) is very small, higher powers of it are still smaller and can be neglected. So we have,

$$P = (P+p)\left(1-\gamma\frac{v}{V}\right)$$

$$= P-\gamma\frac{Pv}{V}-\gamma\frac{pv}{V}+p, \text{ whence } p=\gamma\frac{Pv}{V}+\gamma\frac{pv}{V}.$$

But since p and v are each small, pv is very small and can be neglected.

$$\text{So, } p=\gamma\frac{Pv}{V}\text{ ; or, }\frac{pV}{v}=\gamma P.$$

$$\text{So the adiabatic elasticity} = \frac{pV}{v} = \gamma P.$$

This shows that the adiabatic elasticity of a gas is γ times the isothermal elasticity P.

[By applying calculus, we get the result in the following way :—

$$E = -v\frac{dp}{dv}.$$

For adiabatic condition, $pv^\gamma = \text{const.}$

Hence differentiating with respect to v, $v^\gamma . \frac{dp}{dv} + p\gamma v^{\gamma - 1} = 0$;

$$\text{or,} \quad -v\frac{dp}{dv} = \gamma p = E].$$

Therefore, the formula for the velocity of sound in air with Laplace's correction becomes, $V = \sqrt{\frac{\gamma \times P}{\rho}}$.

Hence the value of the velocity of sound in air

$$= \sqrt{\frac{1{\cdot}41 \times P}{\rho}} \text{ metres per sec.}$$

$$= 280 \times \sqrt{1{\cdot}41} = 332{\cdot}5 \text{ metres per sec.}$$

57(a). Relation between the velocity of Sound in a gas and the mean velocity of the molecules :—

We know that the velocity of sound in a gaseous medium is V, where $V^2 = \frac{\gamma P}{\rho}$.

Again, from kinetic theory we know that $P = \frac{1}{3}\rho C^2$ where C is the root mean square velocity of the gas molecules. Hence

$$V^2 = \gamma\frac{C^2}{3}.$$

But according to the kinetic theory, $C^2 = \frac{3\pi}{8}C_m^2$, where C_m is the mean velocity of the molecules.

$$\therefore \quad V = C_m\left(\frac{\gamma\pi}{8}\right)^{\frac{1}{2}}.$$

This relation shows that ***the velocity of sound in a gaseous medium is of the same order of magnitude as the mean velocity of the molecules.***

58. Effect of Pressure, Temperature, Humidity and Wind on the Velocity of Sound in a Gas :—

(1) Effect of Pressure :—If temperature remains constant, a change of pressure does not affect the velocity of sound through a gas. Let P_1 and P_2 be the pressures of a given mass of a gas, v_1 and v_2 the volumes, and ρ_1 and ρ_2 the corresponding densities. Temperature being constant, we have, by Boyle's law, $P_1v_1=P_2v_2$.

$$\text{or, } \frac{P_1}{P_2}=\frac{v_2}{v_1}.$$

But volume varies *inversely* as density, *i.e.* $\frac{v_2}{v_1}=\frac{\rho_1}{\rho_2}$.

because $v_2\rho_2=v_1\rho_1=\text{mass}=\text{a constant}$.

$$\text{Hence } \frac{P_1}{P_2}=\frac{\rho_1}{\rho_2}; \quad \text{or, } \frac{P_1}{\rho_1}=\frac{P_2}{\rho_2}=\text{a constant}.$$

Therefore, in the formula $V=\sqrt{\frac{1{\cdot}41P}{\rho}}$, the fraction $\frac{P}{\rho}$ remains unchanged. Hence, *the velocity of sound in a gas is independent of any change of pressure when temperature remains constant.*

(2) Effect of Temperature.—With change of temperature, there is a change of density and so the velocity of sound should be different. Let ρ_0 and ρ_t be the density of a gas at $0°C.$ and $t°C.$ respectively. Then by Charles' law,

$$\rho_0=\rho_t(1+\alpha t),$$

where α = coeff. of cubical expansion of the gas $=\frac{1}{273}$.

$$\text{That is, } \frac{\rho_0}{\rho_t}=1+\frac{t}{273}=\frac{273+t}{273} \qquad \dots \qquad \dots \qquad \dots \quad (1)$$

Let V_0 and V_t be the velocities of sound in the gas at $0°C.$ and $t°C.$ respectively and let the pressure of the gas have the same value P So we have,

$$V_0=\sqrt{\frac{1{\cdot}41P}{\rho_0}}, \text{ and } V_t=\sqrt{\frac{1{\cdot}41P}{\rho_t}}.$$

$$\therefore \frac{V_t}{V_0}=\sqrt{\frac{\rho_0}{\rho_t}}=\sqrt{\frac{273+t}{273}}=\sqrt{\frac{T}{T_0}}, \text{ from (1)} \qquad \dots \qquad \dots \quad (2)$$

when T and T_0 are the absolute temperatures corresponding to $t°C.$ and $0°C.$ respectively.

Therefore the velocity of sound in a gas is directly proportional to the square root of its absolute temperature. That is, velocity of sound in a gas increases with the rise of the temperature.

We have, from eqns. (1) and (2) above,

$$V_t = V_0 \left(1+\frac{1}{273}t\right)^{\frac{1}{2}} = V_0\left(1+\tfrac{1}{2}\times\frac{1}{273}\times t\right),$$ neglecting the terms containing t^2 and higher powers of t.

In the case of air, $V_0 = 332$ metres per second.

$$\therefore \quad V_t = 332\left(1+\frac{1}{546}t\right) \text{ metres per second.}$$

$$= (332+0{\cdot}61t) \text{ metres per second.}$$

Hence, *for each centigrade degree rise in temperature, the velocity of sound in air increases by about 0·61 metre or 61 cms. (or about 2 ft.) per second.*

(3) **Effect of Humidity.**—The density of water vapour is less than the density of dry air at ordinary temperatures in the ratio of 0·622 : 1. Therefore, the presence of water vapour in the air lowers the density of air and so increases the velocity of sound in it. Hence, for a given temperature, *the velocity of sound in damp air is greater than that in dry air.*

Correction for the Presence of Moisture in the observed Value of the Velocity of Sound in Air.

If V_m = velocity in moist air at pressure P mm. and temperature t°$C.$,
V_d = velocity in dry air at pressure 760 mm. and temperature t°$C.$,
ρ_m = density of moist air at pressure P mm. and temperature t°$C.$,
ρ_d = density of dry air at pressure 760 mm. and temperature t°$C.$,

then, $$V_m = \sqrt{\frac{\gamma \times P}{\rho_m}}\ ; \quad V_d = \sqrt{\frac{\gamma\times 760}{\rho_d}}$$

Now, if f = saturation pressure of water vapour at t°$C.$, we have ρ_m = mass of 1 c.c. of moist air at pressure P mm. and temperature t°$C.$ = mass of 1 c.c. of dry air at pressure $(P-f)$ mm. and temperature t°$C.$ + mass of 1 c.c. of moisture at pressure f mm. and temperature t°$C.$ (Dalton's law).

We know that the mass of 1 c c. of water vapour = 0·622 × mass of 1 c.c. of dry air.

Now because the density of a gas at a constant temperature varies directly as its pressure, we have

$$\rho_m = \frac{P-f}{760} \times \rho_d + 0{\cdot}622 \times \frac{f}{760} \times \rho_d = \frac{\rho_d}{760}\,(P - f + 0{\cdot}622 \times f)$$

$$= \frac{\rho_d}{760}\,(P - 0{\cdot}378f) \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

$$\therefore \quad \frac{V_d}{V_m} = \sqrt{\frac{760 \times \rho_m}{P \times \rho_d}} = \sqrt{\frac{P - 0{\cdot}378f}{P}} = \sqrt{1 - 0{\cdot}378\frac{f}{P}}.$$

$$\therefore \quad V_d = V_m\sqrt{1 - 0{\cdot}378\frac{f}{P}}.$$

(4) **Effect of Wind.**—The velocity of sound is affected by velocity of the medium, *i.e.* if wind blows it affects the velocity of sound in air. If the wind blows in the direction of sound propagation, the velocity of the latter will be observed as $V + w$ where w is the wind velocity. If w be directed oppositely to V, the observed velocity will be $V - w$. If the wind blows in an inclined direction, w shall be replaced by the proper component.

***59. Acoustical Resistance and Impedance** :—The propagation of sound through a mass of gas contained in a tube is somewhat analogous to the flow of electricity through a conducting wire. As an electric current flows only when a potential difference is established between the ends of the wire, so does an acoustical disturbance pass from one end of the tube to the other when a pressure difference exists between the ends. Again, as in a direct current circuit, the ratio of the potential difference E to the current strength $\left(i = \frac{dQ}{dt}\right)$ is called the electrical resistance, so is the ratio of the steady pressure excess (p) to the particle velocity $\left(\frac{dy}{dt}\right)$ of the medium known as the *acoustical resistance*. This resistance arises due to the viscosity of the medium, and is comparatively small in gases. But in a liquid or a solid medium acoustical resistance is rather more pronounced.

If the disturbance in a tube is supposed to be propagated in the [illegible]n of the X-axis, which is also the direction of the axis of the and $0^\circ C$, is at a distance x from the origin the longitudinal displacement of the particle be y (also along X-axis), the excess pressure in tube will be given by $p = -E\frac{dy}{dx}$, [*Vide* eqn. (1), Art. 55].

But the particle velocity is

$$\frac{dy}{dt} = -\frac{dy}{dx} \cdot \frac{dx}{dt} = -c \cdot \frac{dy}{dx} \ (c = \text{vel. of sound}).$$

$\therefore$ The acoustical resistance $R_a = \frac{p}{dy/dt} = \frac{E}{c} = \frac{E}{\rho} \cdot \frac{\rho}{c} = c^2 \frac{\rho}{c} = c\rho$,

$(\because \ c = \sqrt{E/\rho})$.

For air this acoustical resistance at *N.T.P.* is about 41 whereas for water it is three thousand times greater. The acoustical resistance is also found to obey Ohm's law just like the electrical resistance.

When the applied pressure is alternating in nature, over and above the viscosity, the frequency also plays an important part in opposing the movement of the gas. If the viscous resistance in air be neglected and if the displacement, y, along the X-axis be considered along the axis of a tube of cross-sectional area A, then the volume displacement $X = Ay$.

Hence the particle velocity $= \frac{dy}{dt} = \frac{1}{A}\frac{dX}{dt}$.

The acoustical impedance Z in such a case is defined by the relation $p \sin wt \Big/ \frac{dX}{dt}$.

Neglecting the viscous forces, if we consider a cylindrical volume of a gas of cross-sectional area A and of mass m to be subjected to an alternating force $F \sin wt$, the differential equation of the forced vibration can be written as,

$$m\frac{d^2y}{dt^2} + ky = F \sin wt \qquad \ldots \qquad \ldots \quad (1)$$

where k = the elastic restoring force per unit displacement.

Since volume displacement, $X = Ay$, $y = \frac{X}{A}$ and $\frac{d^2y}{dt^2} = \frac{1}{A}\frac{d^2X}{dt^2}$.

Moreover, the alternating pressure excess, $p \sin wt = \frac{F \sin wt}{A}$.

Substituting in eqn. (1), we get,

$$\frac{m}{A^2}\frac{d^2X}{dt^2}+\frac{kX}{A^2}=p \sin wt \qquad \ldots \qquad \ldots \quad (2)$$

Case (I). When the inertia is predominant and the restoring force negligible, $(k=0)$, eqn. (2) takes the form, $\frac{m}{A^2}\frac{d^2X}{dt^2}=p \sin wt.$

Integrating, $\frac{m}{A^2}\frac{dX}{dt}=-\frac{p \cos wt}{w}.$

$\therefore$ Acoustical impedance, $Z=\frac{-p \cos wt}{dX/dt}=\frac{mw}{A^2}=Lw,$

where $L=\frac{m}{A^2}$ is the *acoustical inductance* or *inertance.*

Case (II). When the elastic force is predominant and $m=0$,

$$\frac{k}{A^2}X=p \sin wt.$$

Differentiating with respect to t, $\frac{kdX}{A^2dt}=wp \cos wt,$

$$\therefore Z=\frac{p \cos wt}{dX/dt}=\frac{k}{A^2w}=\frac{1}{Cw},$$

where $C=\frac{A^2}{k}$ is the *acoustical capacitance.*

Case (III). When both m and k are finite,

$$\frac{dX}{dt}=\frac{p \cos wt}{\frac{1}{Cw}-Lw}. \qquad \ldots \qquad \ldots \qquad \ldots \quad (3)$$

where the *acoustical impedance*, $Z=\frac{1}{Cw}-Lw,$ the inertance and the capacitance acting in opposition to each other.

For resonance, the denominator of eqn. (3) is minimum.

$$\therefore \frac{1}{Cw}=Lw\,; \qquad \text{or,} \quad w^2=\frac{1}{LC}=\frac{A^2}{m}\times\frac{k}{A}=\frac{k}{m}=w_0^2,$$

where w_0 represents the *natural* circular frequency (pulsatance) of the medium.

***59. (a) Alternative Deduction of the expression for Intensity**

of plane sound wave :—We know that the intensity of sound is defined as the amount of energy that is transmitted across unit area per second. Since pressure indicates the force per unit area, the intensity will evidently be given by the product of the excess pressure p and the particle velocity $\frac{dy}{dt}$.

$\therefore$ the rate of energy flow per unit area is,

$$\frac{dW}{dt}=p.\frac{dy}{dt} \text{ at an instant } t.$$

From the expression of acoustical resistance we know,

$$p=R_a\frac{dy}{dt}=\rho c\frac{dy}{dt}, \text{ and } \frac{dy}{dt}=\frac{p}{\rho c}$$

$$\therefore \quad \frac{dW}{dt}=R_a\left(\frac{dy}{dt}\right)^2=\rho c\left(\frac{dy}{dt}\right)^2,$$

$$\text{and} \quad \frac{dW}{dt}=\frac{p^2}{\rho c}.$$

When the average of $\frac{dW}{dt}$ is taken over a period, we get intensity, I.

$$\therefore \quad I=R_a\overline{\left(\frac{dy}{dt}\right)^2} \text{ and } I=\frac{\overline{p^2}}{\rho c},$$

where $\overline{\left(\frac{dy}{dt}\right)^2}$ and $\overline{p^2}$ represent the *mean square* values of the particle velocity and the excess pressure.

Since $y=a \sin \omega t$ and $\frac{dy}{dt}=\dot{y}=a\omega \cos \omega t=\dot{y}_m \cos \omega t$,

$$\overline{\left(\frac{dy}{dt}\right)^2}=\frac{1}{T}\int_0^T \dot{y}_m^2 \cos^2 \omega t \, dt=\tfrac{1}{2}\dot{y}_m^2.$$

Hence $I=\frac{1}{2}R_a \dot{y}_m^2$ (where y_m = the max. particle velocity).

Similarly, $\overline{p^2}=\frac{1}{2}p_m^2$ and $I=\frac{1}{2}p_m^2/\rho c=\frac{1}{2}\frac{p_m^2}{R_a}$,

where p_m is the maximum excess pressure.

Since $R_a=\rho c$ and $\dot{y}_m=a\omega$, we have

$$I=\tfrac{1}{2}\rho c a^2\omega^2.$$

Here c is the velocity of sound which is the same as V used in Art. 22.

60. (a) The Velocity of Sound in Different Gases :—We know that the velocity of sound in air, $V_a=\sqrt{\frac{\gamma P}{\rho_a}}$, where ρ_a is the density of air and V_a the velocity of sound in it. Under similar condition of pressure and temperature, the velocity of sound in another di-atomic gas (for which the value of γ is the same), say hydrogen, is,

$$V_h=\sqrt{\frac{\gamma P}{\rho_h}} \therefore \frac{V_a}{V_h}=\sqrt{\frac{\rho_h}{\rho_a}}.$$

So the velocity of sound in a gas is inversely proportional to the square root of its density. Thus if V_0, V_h be the velocities, and ρ_o, ρ_h, the densities of oxygen and hydrogen respectively under the same conditions of temperature and pressure,

$$\frac{V_0}{V_h}=\sqrt{\frac{\rho_h}{\rho_0}}=\sqrt{\frac{1}{16}}=\frac{1}{4}.$$

(b) Velocity of Sound in a Mixture of Gases.—

If P and γ be the pressure and the ratio of the two specific heats of the mixture, the velocity of sound in the mixture of gases will as usual be given by $V=\sqrt{\gamma P/\rho}$ where ρ, the density of the mixture is given by

$$\rho=\frac{p_1\rho_1+p_2\rho_2+p_3\rho_3+\dots}{P}$$

and the ratio of specific heats γ by,

$$P/(\gamma-1)=p_1/(\gamma_1-1)+p_2/(\gamma_2-1)+p_3/(\gamma_3-1)+\dots$$

where p_1, p_2...etc., ρ_1, ρ_2...etc., and γ_1, γ_2...etc. refer to the constituent gases.

61. Experimental Determination of the Velocity of Sound in Air :—

(a) Open-air Method.—Some members of the Paris Academy first determined the velocity of sound in open air in 1738. Their findings show that the velocity of sound (*i*) does not depend upon any changes of atmospheric pressure ; (*ii*) increases with temperature and humidity ; (*iii*) increases in the direction of the wind and decreases against it. According to the Dutch physicists Moll, Van Beck and others, the velocity of sound at 0°C. is 332·26 metres per sec. Bravais and Martins determined the velocity of sound along a slope, the difference of altitude between Faulhorn, the upper station and the Lake of Briez, the lower station, being 2079 metres while their distance was 9860 metres. They found velocity corrected to 0°C. to be 332·37 metres per sec. During the Arctic expeditions of Parry and Greely, the experiments were done at very low temperatures and almost the same result was found.

Arago did the following experiment in 1829 : Two observers were stationed on the tops of two hills several miles apart. One of them was provided with a gun while the other had accurate stop-watch. The first man fired his gun, the second man started his watch on seeing the flash and kept a continuous record of the time until the sound of the firing was heard. A large number of observations under similar atmospheric conditions were taken and the mean value (t secs.) of the recorded times was taken. If x ft. is the distance between the two stations, the velocity of sound (v) is given by,

$$v = x/t \text{ ft. per sec.}$$

Such determinations are liable to two principal errors, *viz.* (1) *the error due to the wind velocity*, and (2) *the personal equation of the observer.*

The first error is that the velocity of sound is affected, though slightly, by the velocity by the wind, it being greater in the direction of the wind and less against it. It is corrected by the *method of reciprocal observations* in which both the observers are provided with a gun and a stop-watch. When one fires, the other records the time and *vice versa*. Suppose t_1 and t_2 are the mean values of the time recorded by the first and the second observer respectively. If the wind is blowing in the direction of the second station from the first at the rate of c ft./sec.

$$v + c = x/t_1, \text{ and } v - c = x/t_2.$$

$$\therefore \quad v = \tfrac{1}{2}\left(\frac{x}{t_1} + \frac{x}{t_2}\right) \text{ ft./sec.}$$

Thus the effect of the wind is eliminated.

The second error is that every man is apt to delay some fraction of second to start the watch after he actually sees the flash of the firing and this delay period various from person to person and is a personal factor of the person making the experiment. This error can be avoided by making electrical arrangements for the recording of the exact moment of the gun-fire at one station and for the reception of the sound at the other.

(b) Regnault's Method.—

Regnault took precautionary measures against both the errors mentioned above and in 1864 determined the velocity of sound at Versailles. To avoid the personal equation of the observer electrical devices were adopted.

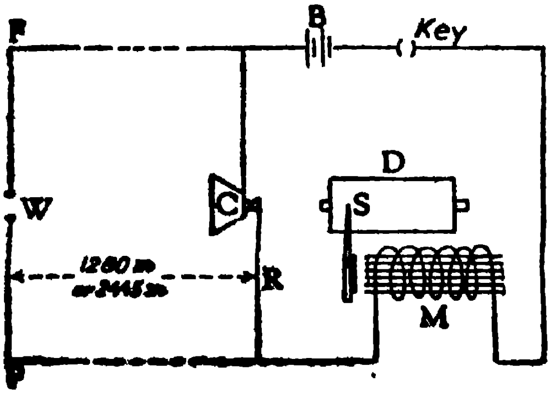

Fig. 38

The apparatus used by Regnault is shown in Fig. 38. Two such arrangements were used at two stations. In the figure *D* represents the drum of a chronograph rotated at a constant known speed. As it rotates it advances along its axis. *S* is a pen or stylus which strikes against *D* and makes a mark there when attracted by an electromagnet *M*, which can be actuated by the current from the battery *B* when the circuit is closed. The circuit is normally closed through the line *FWF* extending up to the firing station. The portions *BMC* along with the drum attachments are installed at the receiving station. *C* is a conical wooden receiver at the narrow end of which there is a vibrating thin metallic diaphragm. This is connected in parallel to the main circuit and normally the connecting rod *R* is slightly out of contact. But as the diaphragm vibrates, the contact is established with *R*.

To start with the experiment the circuit *BFWMB* is closed by plugging in the key and connecting the gap at *W* by means of a piece of metallic wire. The stylus records its continuous mark on *D*. Now at the sending station the muzzle of the gun is directed towards *W* and the gun fired. The circuit is thereby disconnected and the moment of firing is recorded on *D* by the discontinuity in the stylus mark.

Again as soon as report of the firing reaches the receiving station, it is received by the wooden cone C wherein the membrane is set to vibration. This temporarily completes the $BMCB$ circuit and the stylus is again pressed against the drum and makes a mark there. Now from a knowledge of the speed of rotation and axial movement of the drum, the time interval between the firing and the reception of the corresponding sound can be computed from the positions of discontinuity of the stylus mark and its reappearance on the drum at the instant of reception. The distance between W and R being accurately measured the velocity of sound can be found. In actual experiment a reciprocal observation is taken with the second set of a similar apparatus and the wind effect is thus eliminated. Regnault carried out his experiment for two different distances 1280 metres and 2445 metres respectively and found that the velocity of sound slightly varies with the loudness of a report. In his actual calculations Regnault even accounted for the slight error which may arise due to the inertial lag of the component parts of the apparatus.

(c) Hebb's Telephone Method.--

At the suggestion of Michelson, Hebb in 1905 and later in 1919 determined the velocity of sound by the arrangements described below. The principle of the measurement lies in determining the wavelength λ of sound emitted from a source whose frequency n is known. The velocity V is then given by $n\lambda$.

In the actual arrangements M_1 and M_2 are two paraboloidal mirrors of plaster of Paris arranged coaxially. F_1 and F_2 are the foci of the two mirrors. A whistle whose fundamental frequency was previously determined by comparing it with a standard fork, was blown steadily at F_1 so that its frequency remains constant. T_1 and T_2 are two telephone transmitters placed at F_1 and F_2. These are connected to the primaries P_1 and P_2 of a special type of induction coil through two batteries. The secondary of the coil is connected to telephone receiver T_3. An adjustable rheostat (r) is connected in the circuit of T_1 in order to make the sounds received in T_3 from T_1 and T_2 of equal intensities. When the whistle is sounded, T_1 directly receives the sound and T_2 receives it after two reflections as indicated in the figure (Fig. 39). Due to the extra path $F_1M_1M_2F_2$ traversed by the reflected sound there will be a phase difference between the two sounds and their resultant effect will be heard at T_3. Now

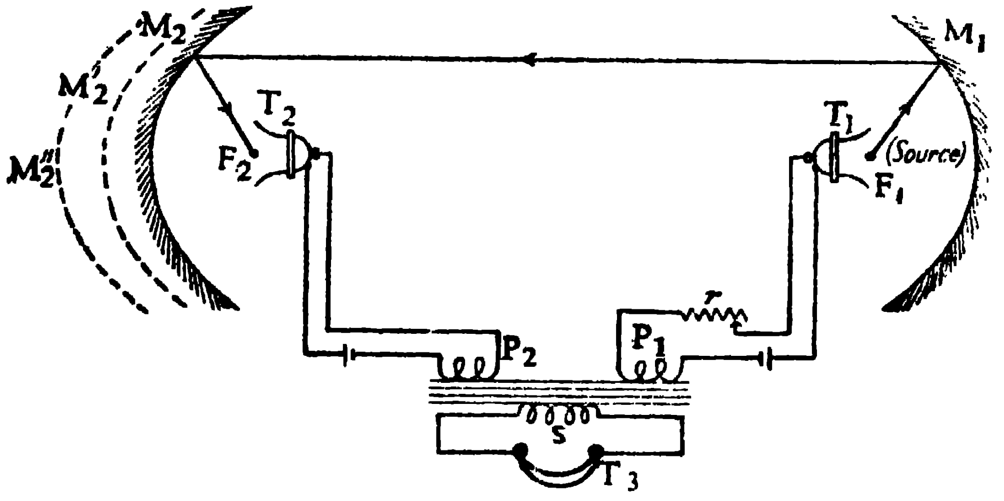

Fig. 39

since the amplitudes or intensities have been adjusted to the almost the same in both the components, it is possible to adjust the phase by shifting M_2 to different positions like M'_2 M''_2, etc. so that alternate destruction and reinforcement of sound in T_3 is obtained. The position of destruction of sound *i.e.* of minimum intensity can be more accurately detected by T_3 than that of reinforcement. Now plotting these maxima M and minima (m) as indicated by T_3 against the corresponding shifts of the mirror M_2, a graph as shown in Fig. 40 may be obtained. The wavelength, λ, of sound is evidently the distance between two successive minima or maxima. Thus knowing λ, the distance between two successive minima in the curve and from a previous determination of n of the sound emitted by the whistle, the velocity of sound can be obtained from the relation $V=n\lambda$.

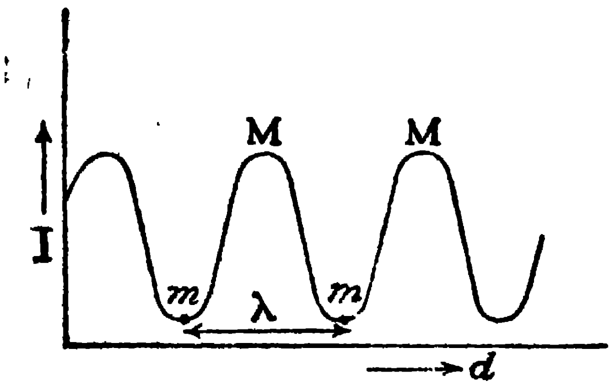

Fig. 40

The velocity obtained by Hebb corrected to 0°*C.* comes to 331·29 metres per sec. Later experiments of Hebb gave the velocity at 0°*C.* as 331·44 metres per sec.

(d) Laboratory Methods.—

(*i*) By resonance of air column (*vide* Chap. VIII).

(*ii*) By Kundt's tube method (*vide* Chap. VIII).

(e) Recent Determinations. -

Esclangon in 1917 modified Regnault's method for an accurate determination of the velocity of sound. The sound of the gun-fire was received by a tuned hot wire microphone whose hot wire formed an arm of balanced Wheatstone's bridge, the detecting galvanometer in the circuit being an Einthoven string galvanometer. When the sound was received by the microphone the bridge was out of balance and a kick was recorded by the galvanometer. The value of V at $0°C$. obtained by this method is 330·9 metres per sec.

Angerer and Ladenburg in 1922 redetermined V by a similar method. In 1925 Pierce made a precise determination of V in air by using high frequency sound and found that V changes with the frequency of the source. The mean value of the velocity of sound in dry air at S. T. P. may be taken to be 331 metres per sec. for all practical purposes.

62. (a) Dependence of the Velocity of Propagation on the amplitude and frequency.—

In Regnault's experiment with a distance of 1280 metres the velocity of sound in air obtained was 331·37 metres/sec. But with a distance of 2445 metres he found that the velocity of sound in air was 330·7 metres per sec. This discrepancy was accounted for by supposing that at a near distance the report of the gun was more intense while at the greater distance it was of weaker intensity. So it may be said that the velocity is a function of the intensity. It seems to be rather reasonable because a high intensity sound produces a greater compression in the air which produces a greater heating effect and hence a higher temperature. So the velocity of sound is also proportionately increased.

Since V depends only on the properties of the medium namely γ, P and ρ, the frequency of the source of sound has nothing to do with the velocity of propagation. Since $V = n\lambda$, if n is varied λ will be accordingly adjusted so as to retain the constancy of V. Pierce's observation that V changes with n in the high frequency regions probably suggests that the adiabatic condition, as suggested by Laplace, does not exactly represent the thermal state of the medium during the propagation of a high frequency sound wave.

(b) Anomalous Propagation of Explosive Sounds :—

In the propagation of sounds produced by large explosions, the following anomalies have been noticed :—

In the neighbourhood of the source abnormally large velocities are produced which are generally attended with a considerably large air movement. Next appears a zone of normal velocities and then a *zone of complete silence*. Lastly, in the fourth zone the sound reappears with the usual intensity but after an unusually long interval.

Various theories have been put forward to account for these anomalies. But the latest theory known as *the temperature theory* seems to be in a position to explain most of the above facts. According to this theory, after the initial fall in temperature with altitude, at about 17 kms. above the earth's surface the temperature of the atmosphere begins to rise. It reaches the surface value (nearly 15°C.) again at about 35 kms. At 60 kms. the temperature becomes as high as 70°C. On the assumption of this reversal of the temperature gradient, the above anomalies may be mostly explained because due to the variation of densities of the different layers the sound rays will be bent down from the upper atmosphere and reach the fourth zone after a long interval and thereby create an intervening zone of silence (See Art. 33). The validity of the above assumptions has also been partly established by certain observations on meteors.

63. The Velocity of Sound in Water :—(*a*) The velocity of sound in water was determined by Colladon and Strum in 1825 in the lake of Geneva, where a large bell, hung below the surface of water from the side of a boat was struck by a hammer. The sound was received through a sort of ear-trumpet fixed in the water to another boat, which was placed at a distance of 2 miles. There was an arrangement in the first boat such that, at the instant the hammer was struck, a charge of gunpowder was ignited giving a flash in the air which could be seen by the observer in the second boat. The interval between the flash and the report was noted by a stop-watch and the velocity was calculated in the usual way. The value of V at $8^\circ C$. was found to be 1435 metres per sec.

(*b*) Radio-acoustic method was adopted for determining the velocity of sound in water by A. B. Wood and Browne in 1922 in collaboration with the Admiralty. A similar method was also used by Stephenson in 1923 in America. Stephenson denoted a bomb at a depth of 10 metres of water and simultaneously sent a radio-signal. This signal as well as the sound signal were recorded on a photographic plate with the help of microphone and string galvanometer arrangements at 5 different stations equidistant from the source. The velocity thus determined was 1453·5 metres per sec. which agrees well with the theoretical value.

(*c*) Kundt's Tube method has also been used for determining the velocity of sound in liquids.

Theoretical Calculation.—

Velocity of sound in water, $V_w=\sqrt{\dfrac{\text{adiabatic elasticity}^{*}}{\text{density}}}$. For water, density$=1$ gm. per c.c. and adiabatic elasticity$=2{\cdot}1\times10^{10}$ dynes per sq. cm.

$$\therefore\ V_w=\sqrt{\frac{2{\cdot}1\times10^{10}}{1}}\ \text{cms. per sec.}=1449\ \text{metres per second.}$$

This agrees fairly well with the experimental result. Note that this is nearly 4 times the velocity of sound in air.

64. The Velocity of Sound in Solids :—Sound travels much faster in solids than in air. The velocity of sound in cast iron was determined by Biot by striking with a hammer one end of a long series of cast iron pipes of total length L metres joined end to end. The sound travels through the walls of the pipes and through the air inside them with unequal speeds. An observer at the other end

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[*] In all ordinary measurements on the elasticity of liquids isothermal elasticity is obtained. Adiabatic elasticity is γ times this isothermal value E_T. But γ, the ratio of specific heats of a liquid cannot be obtained by ordinary methods. It is obtained from the thermodynamic relation $C_p-C_v=\alpha^2VTE_T$ and a knowledge of C_p from experiments. Here α is the volume coeff. and T the absolute temperature.

noted the interval between the sounds transmitted by the metal and that by the air. The interval between the sounds was t seconds.

Therefore, if V=velocity of sound in cast iron, and V_1 that in air, the time taken by sound to travel L metres through cast iron $=L/V$, and that through air$=\frac{L}{V_1}$. $\therefore \frac{L}{V_1}-\frac{L}{V}=t$.

Assuming the value of the velocity of sound in air at the particular temperature, the velocity in cast iron was determined, but the result was not quite accurate. The velocity obtained was only 3500 metres/sec.

Theoretical Calculation.—When a compression wave is transmitted along a solid rod, its velocity is given by $V=\sqrt{Y/\rho}$, where Y =Young's modulus of elasticity for that material. For annealed steel, $Y=21{\cdot}4\times10^{11}$ dynes per sq. cm. and $\rho=7{\cdot}63$ gms./c.c.

$$\therefore \quad V=\sqrt{\frac{21{\cdot}4\times10^{11}}{7{\cdot}63}}\text{ cms. per sec.}=5291\text{ metres per second.}$$

The present accepted value of the velocity of sound in iron is 5130 metres per second.

(a) The Velocity of Sound in Other Forms of Solids.—The velocity of transverse waves in solids, when in the form of a string, can be experimentally determined in the laboratory by a sonometer. When the solid is in the form of a rod, the velocity of sound is conveniently determined by Kundt's method (*vide* Chapter III) which is based on the principle of resonance.

From the table of velocities of sound it will be seen that sound travels faster in solids and liquids than in air. If the ear is applied to one end of a long wooden or metal board while somebody lightly scratches the other end, the sound of the scratching will be clearly heard, but it may not be audible when the ear is removed from contact with the board, *i.e.* when sound travels through air.

Similarly, any sound made under water may be easily heard at a considerable distance by means of a submerged **hydrophone** (*vide* Art. 65) which is an under-water microphone receiver with a sensitive metal diaphragm for recording sound waves. *But sounds do not readily pass from one medium to another when the media differ greatly in* *density*. For this reason, when your ears are under water you will not be able to hear the shouts of people around you made in air.

The sounds made by a running horse's hoofs will be heard from a very long distance if the ear is applied to the ground though they may be inaudible when the listener is standing up, and similarly the ear in contact with a railway line will catch the sounds of an approaching train long before they can be heard by others. This principle is applied by the water works inspector in detecting leaks in the water mains under the street. This is done by applying rod to the ground above the pipe and pressing the ear to the rod, that is, by making a continuous solid connection from the pipe to the ear, the sound of water running in the pipe will be readily audible.

The principle may be applied for preventing sound from passing from one room to another of a building by making cavity walls, that is, walls with an air space between them.

65. The Hydrophone :—It is a microphone receiver used for reception of sound under water and for finding the direction of a sound. It is largely used in echo depth-sounding, location of submerged objects and ice-bergs by methods of echo-sounding, location of submarines by the method of sound ranging in the sea-water and similar acts of sound-reception under water.

Ordinarily, it is a carbon-granule type of transmitter adapted for use under water. It consists of a heavy annular metallic ring *R* provided with a central thin diaphragm *D* made also of metal. One end of a stylus *S* is fixed to the centre of the diaphragm and the other end to a carbon-granule box *C*. The diaphragm *D* and the back-end *E* of the box are separately joined to two wires from a cable by which the receiver is dipped into the sea. The ends of the

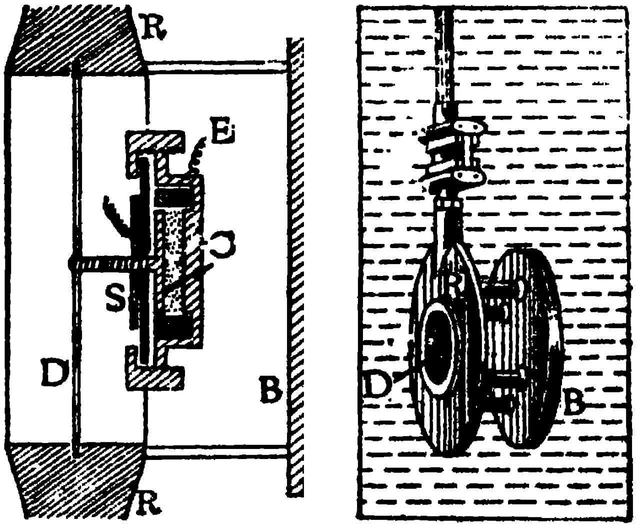


Fig. 41—The Hydrophone.

wires at the other end of the cable are connected in series to a headphone and a battery of cells. The back side of the ring *R* is provided with a screen *B*, called the baffle (or the deaf side), since it cuts off the reception of sound at that end. The movement of the diaphragm, due to the incidence of any vibratory disturbance on it, causes a fluctuation of resistance in the carbon-granule box and so of the current in the headphone circuit. For correct reception of sound, the receiver is rotated in all possible directions until the maximum sound is heard in the headphone. The direction of the sound is normal to the plane of the diaphragm at this position.

66. Sound Ranging :—In war engagements the position of an enemy gun can be located by noting the times taken by the report of the gun to reach several sound-detecting stations.

These stations are usually selected on a common base line at a distance of some miles from the enemy front line, separated from each other by intervals of few hundred yards. Each station is provided with a **hot-wire microphone** which is a sensitive electrical apparatus for detecting sounds. These microphones are electrically connected to a central station where the instant of reception of sound by each microphone is automatically recorded.

Suppose there are three different stations *A*, *B* and *C* (Fig. 42). If the report of the gun reaches *B*, t seconds later than it reaches *A*, then taking V for the velocity of sound, the gun at *G* must be $V \times t$ farther from *B* than from *A*, or $(GB-GA) = V.t$. If, again, the report reaches *C*, t_1 second later than at *A*, then $(GC-GA)=V.t_1$. If now circles with radii $V.t$ and $V.t_1$ respectively are drawn with centres *B* and *C*, the gun *G* will be at the centre of a circle passing through *A* and touching each of these two circles. This new circle is usually drawn by trial and its centre represents the position of the gun.

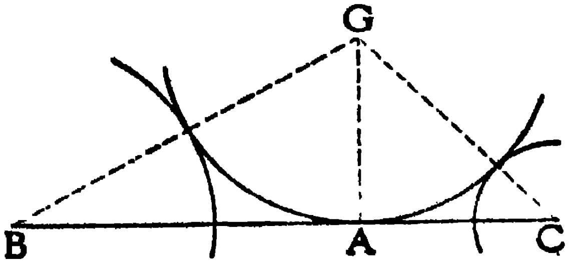


Fig. 42—Location of a Gun.

The nature of the above two equations suggests that the source of sound *G* must lie on the point of intersection of two hyperbolas—

one having its foci at A and B, the difference of the focal distances being Vt and the other having its foci at A and C, the difference of the focal distances here being Vt_1. The actual procedure is much simplified by the fact that the source distance is large compared with the distance between the microphones so that in the neighbourhood of the source the hyperbolas may be identified with their asymptotes. So the geometrical method as described above, which was adopted in earlier sound ranging devices, is justified. Now-a-days, however, quicker mechanical devices have been devised wherein six microphones are stationed at different positions. Each of them gives an electrical record of the reception in its own galvanometer unit. These records (six spots) are registered on a photofilm on which hundredths of a second are accurately impressed photographically. From these records the source can be quickly located by a direct method with reference to a map.

67. Determination of Ship's Position :—In foggy weather when a ship finds it difficult to get its bearing, it sends out simultaneously two signals,—a wireless signal and another under water sound signal —to the stations on the coast, which are suitably equipped for their reception and which in turn inform the ship by wireless the interval between receptions of the two signals. Thus, if the interval is t secs. at the station A, then the ship is $V' \times t$ ft. from A, while an interval of t_1 secs. at B would indicate that the ship is at a distance of $V' \times t_1$ ft. from B, where the velocity of sound in sea-water is V ft./sec. Therefore, the ship's position will be obtained by intersecting arcs drawn on the chart with centres A and B and of radii $(V' \times t)$ ft. and $(V' \times t_1)$ ft. respectively.

Examples

(1) Calculate the velocity of sound in air at 10° C. when the pressure of the atmosphere is 76 cms.

A sound is emitted by a source at one end of an iron tube 950 metres long and two sounds are heard at the other end at an interval of 2·5 secs. Find the velocity of sound in iron.

Ans. : We know that, $V = \sqrt{\frac{1 \cdot 41 P}{\rho}}$.

∴ The velocity of sound in air at 0°C. and 76 cms. pressure.

$$\sqrt{\frac{1·41 \times 76 \times 13·6 \times 981}{·001293}} \text{ cms. per sec.} = 332·5 \text{ metres per sec.}$$

∴ The velocity of sound at $10°C = 332·5 + 0·61 \times 10 = 338·6$ metres per sec.

If V be the velocity of sound in iron expressed in metres per sec., the time taken by the sound to travel 950 metres along the iron tube is $950/V$ sec. The time taken by the sound to travel through the same distance in air at 10°C. is $950/338·6$ sec.

The velocity of sound in solids is greater than that in air; hence the time taken by the sound to travel through the iron of the tube is smaller than the time taken to travel through the air inside the tube.

$$\therefore \quad 2·5 = \frac{950}{338·6} - \frac{950}{V}, \text{ whence } V = 3107·92 \text{ metres per sec.}$$

(2) *Assuming that the velocity of sound in air at N.T.P. is 1090 ft./sec., find the velocity at 50°C and 70 cm. pressure.* (C. U. 1958)

Ans. : As the velocity of sound is independent of pressure, the temp. effect is only to be considered here.

$$\therefore \quad V_t = V_0(1+\alpha t)^{\frac{1}{2}} = V_0(1+\tfrac{1}{2}\alpha t) = 1090\,(1+\tfrac{1}{2}.0·00367 \times 50) = 1185 \text{ ft./sec.}$$

(3) *The wavelength of the note emitted by a tuning fork of frequency 512 is 66·5 cm. in air at 17°C. If the density of air at N.T.P. is 1·293 gm. per litre, calculate the ratio of the two specific heats of air.*

Ans. : Since $V = n\lambda$, the vel. of sound at $17°C = V = 512 \times 66·5$ cm./sec.

But, $\frac{V_0}{V} = \sqrt{T_0/T} = \sqrt{\frac{273}{273+17}} = \sqrt{\frac{273}{290}}$. $\therefore \; V_0 = \sqrt{\frac{273}{290}} \times 512 \times 66·5$.

Again, $V_0 = \sqrt{\frac{\gamma P}{\rho}}$. Here $P = 76 \times 13·6 \times 980$ dynes/cm.^{2} and $\rho = 1·293/1000$ $= ·001293$ gm. per c.c.

$$\gamma = \frac{V_0^2 \times \rho}{P} = \frac{273 \times 512^2 \times 66·5^2 \times ·001293}{290 \times 76 \times 13·6 \times 980} = 1·39.$$

(4) *Calculate the velocity of sound in air at 100°C. if the density of air at N.T.P. is 0·001293 gm./c.c. and the specific heat of air at constant pressure is 0·2417 and that at constant volume is 0·1715.*

Ans. : The velocity of sound, $V = \sqrt{\frac{\gamma P}{\rho}}$.

Here $\gamma = \frac{0·2417}{0·1715}$; $P = 76 \times 13·6 \times 980$ dynes/cm.^{2} and $\rho = 0·001293$ gm./c.c.

$$\therefore \quad V_0 = \sqrt{\frac{2417 \times 76 \times 13·6 \times 980}{1715 \times 0·001293}} \text{ at } 0°C. \quad \text{But } \frac{V_{100}}{V_0} = \sqrt{\frac{273+100}{273}} = \sqrt{\frac{373}{273}}$$

$$\therefore \quad V_{100} = V_0\sqrt{\frac{373}{273}} = \sqrt{\frac{373 \times 2417 \times 76 \times 13·6 \times 980}{273 \times 1715 \times 0·001293}} = 38880 \text{ cm./sec.}$$

68. Doppler Effect :—If there is a change in the relative motion of a source of sound and the observer, the apparent frequency of the note as heard by the observer seem to differ from that emitted by the source. It is found that the pitch of the whistle of a train appears to rise when the train approaches the hearer and to fall as the engine recedes from him. Similar effect is noticed when the observer in a running motor car passes a warning bell on a side road. Such apparent change in the pitch of a note as perceived by the observer due to the relative motion of the source, the observer or the medium is called the *Doppler Effect* after the name of Christain Doppler (1803-52). Doppler first applied this principle to explain the colour of the stars. It was later found to apply equally to all wave motions and was successfully applied to acoustics. It is with the help of this phenomenon that the traffic police can put a check on high speed driving in big cities because from a knowledge of the frequency change of horns the velocities of the vehicles can be ascertained.

A general explanation of the phenomenon can be put forward from Huygen's principle. If the source be fixed, the positions of the wave-fronts drawn at intervals of one period will be so many concentric spheres, the successive ones being one wavelength apart each as shown in Fig. 43. Now if the source *S* moves towards the observer at *O* due to its motion towards the observer this equidistant wave-fronts will be crowded in the region between *S* and *O* as shown in Fig. 44, where the gradual shift of *S* is indicated by 1, 2, 3, 4. The circles 1, 2, 3, 4

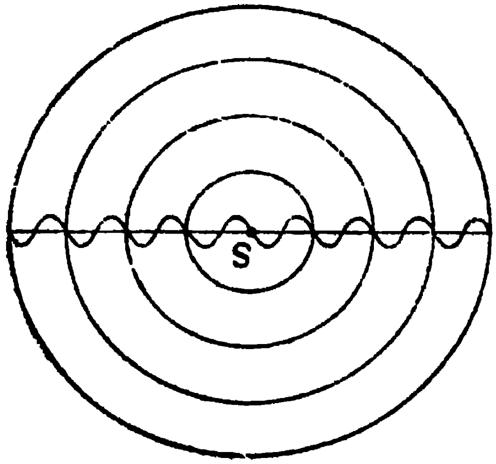


Fig. 43

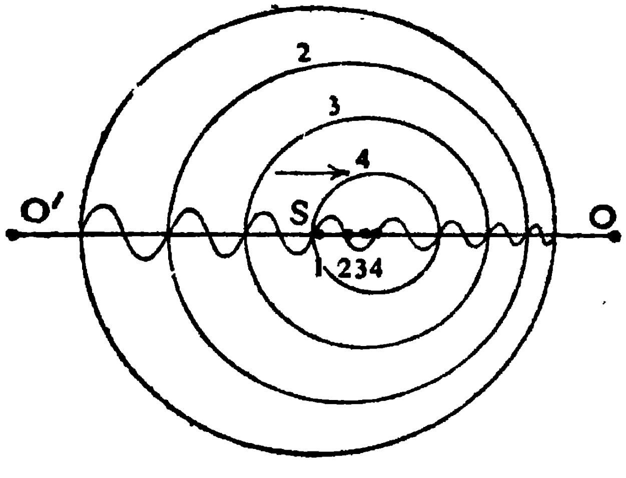


Fig. 44

indicate the wave-fronts emitted from the source at those shifted positions. As a result of this crowding, the wavelengths will be shorter and the frequency will be higher. Similarly, if the source move away from the observer at O' the wavelengths between the S and O' increase and the frequency falls. Similar will be the explanation for the motion of the observer, or the medium when the source itself remains stationary. The actual calculation of this change in frequency under different conditions may be done as follows :—

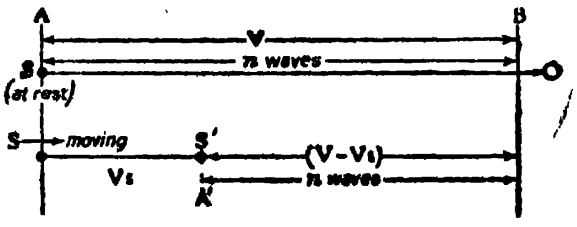


Fig. 45

(a) Source moving, observer at rest.—Here we suppose that the observer is at rest at O and the medium (*i.e. wind*) *has neither any velocity.* If n be the frequency of the source, then supposing it to be at rest the number of waves sent by it *per second* will be n which will occupy a space equal to the velocity V of sound in length. This length is represented by AB in Fig. 45. Here $n=\frac{V}{\lambda}$. But if the source S moves towards the observer O with a velocity V_s, by the time the first impulse of emitted sound reaches B at the end of the second, S moves to S' at A' as shown in the lower portion of the same figure. So all the n waves emitted by the source in one second will be confined within the region $A'B=(V-V_s)$. So the original wavelength λ will be altered to λ' due to the clustering of the waves. Here $\lambda'=\frac{V-V_s}{n}$. Now since the velocity of sound remains unaltered, the frequency of the note will be correspondingly altered to n', where

$$n'=\frac{V}{\lambda'}=n.\frac{V}{V-V_s} \qquad \cdots \qquad \cdots \quad (1)$$

If the source moves away from the observer, V_s will be negative and hence eqn. (1) will take the form,

$$n = n\frac{}{V+V_s} \qquad (2)$$

Thus while the source approaches the observer, the frequency increases and while it recedes, the frequency falls.

If the medium moves (*i.e.* if wind blows) in the direction of the sound with a velocity V_m, then due to the motion of the medium the effective velocity will be increased from V to $V+V_m$.

∴ Eqn. (1) will take the form,

$$n'=n.\ \frac{V+V_m}{V+V_m-V_s}.$$

If the wind blows in the opposite direction, V_m is negative with respect to V and hence,

$$n'=n.\ \frac{V-V_m}{V-V_m-V_s}.$$

(b) Source Stationary, observer moving.—Let the source emitting the sound with a velocity V at a frequency n be fixed at S and let the observer O move with a velocity V_0 in the same direction as V. If the observer were fixed at B, then in 1 second all the n waves starting from S would have reached him in the plane BO. But as in this interval of 1 sec. the observer moves from O to O' through a distance V_0, all the waves emitted cannot reach him during this interval. The waves limited within the shaded space of length V_0 cannot reach the observer within 1 sec.

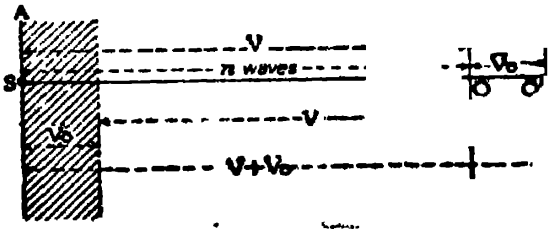


Fig. 46

Now since the source and the medium are stationary there cannot be any change in λ in the space. Hence the frequency or number of waves emitted is $n=\frac{V}{\lambda}$. Out of this number, the moving observer does not receive the waves contained in the shaded region whose number is $n_1=\frac{n}{V}\times V_0$. So the total number of waves reaching the observer per second which is identical with the apparent frequency of the note received by him will be given by

$$n'=n-n_1=n-\frac{nV_0}{V}=n.\frac{V-V_0}{V} \qquad (1)$$

If the motion of the observer be towards the source, V_0 will be negative and hence,

$$n' = n\frac{V+V_0}{V} \qquad \ldots \qquad \ldots \qquad (4)$$

In case the *wind blows with the sound*, eqn. (3) will be modified as

$$n' = n\frac{V+V_m - V_0}{V+V_m},$$

where V_m = velocity of the medium or wind.

When the *wind blows against the sound*,

$$n' = n\frac{V - V_m - V_0}{V - V_m}.$$

(c) **Both source and observer moving.**—Suppose that the velocity of the source, V_s, and that of the observer V_0 be in the direction of the velocity V of the sound reaching the observer and that the medium is stationary.

If the observer be supposed stationary for the time being, the changed wavelength of the sound waves caused by the motion of the source is given by,

$$\lambda' = \frac{V-V_s}{n} \text{ [\textit{vide} case } (a)]$$

and the corresponding changed frequency is,

$$n' = n\frac{V}{V-V_s}.$$

Now when we consider the motion of the observer together with that of the source, we can suppose that the source is now a stationary one emitting a sound of this changed frequency n'. So due to the motion of the observer the new altered frequency n'' will be given by,

$$n'' = n'\frac{V-V_0}{V} \text{ [\textit{vide} case } (b)].$$

Substituting the value of n' we get,

$$n'' = n\frac{V}{V-V_s} \times \frac{V-V_0}{V} = n\,\frac{V-V_0}{V-V_s} \qquad \ldots \qquad (5)$$

If the motion of the observer or the source be negative there will be only a corresponding change in the sign of the quantity or quantities in expression (5).

Now, if the velocity of wind V_m be considered, the effective velocity of sound should be $V+V_m$, when V_m is in favour of V.

The altered frequency due to the simultaneous motions of wind, observer and source, all in the same direction, is therefore given by,

$$n'' = n\frac{V + V_m - V_0}{V + V_m - V_s} \quad \ldots \quad \ldots \quad \ldots \quad (6)$$

If V_m be oppositely directed to V, its sign will be negative in expression (6).

(d) Source moving at an angle to the line joining the source and the observer—

Let O [Fig. 46(a)] be the stationary position of an observer and let the source S be moving with a velocity V_s in a direction SQ, making an angle α with $\overrightarrow{SO}$. The velocity component along SO is $V_s \cos \alpha$. The observer will therefore hear a note of higher frequency n' which is given by,

$$n' = n\frac{V}{V - V_s \cos \alpha}.$$

where V is the velocity of sound and n is the frequency of the sound emitted.

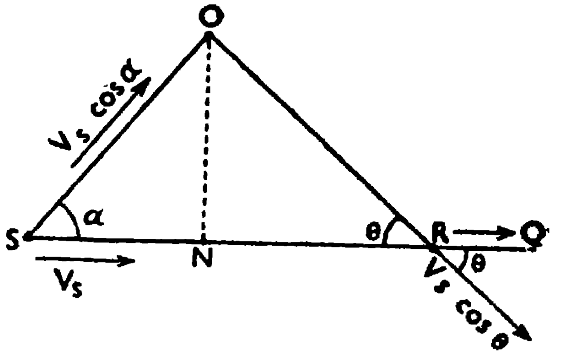


Fig. 46(a)

As the source gradually approaches N, the perpendicular drawn from O on SQ, α gradually increases and n' also increases till at N where $\cos \alpha$ being 0, n' becomes same as n. When the source reaches R on the other side of N, the velocity component $V_s \cos \theta$ will act along $\overrightarrow{OR}$ and hence the frequency of the sound apparently heard by the observer will be,

$$n' = n\frac{V}{V + V_s \cos \theta}.$$

69. Experimental Verification and Application of Doppler's Principle :—Doppler's principle was experimentally tested by Ballot and Russel in 1845 and later in 1861 by Mach. The easiest method adopted is, however, that due to König. He took two tuning forks of nearly equal frequencies mounted on two resonance boxes. When both of them were stationary, they produced only four beats when set to vibration. Then on keeping one of the forks fixed in position the second one was moved either towards or away from the stationary observer. This produced a change in the number of beats heard which was evidently due to the fork being moved. From a knowledge of the change in the number of beats, the frequency change could be calculated and hence the Doppler's principle was verified.

Doppler applied this principle to find out the velocities of moving heavenly bodies from a knowledge of the frequency shift of the light (*i.e.* colour change) emitted by them. The principle also successfully explains the finite breadth of a spectral line. With the help of this principle the speed of rotation of the sun can be calculated from the spectrum of light coming from the periphery of the sun. The differential rotations of the inner and outer edges of the Saturn's rings have also been ascertained with the help of this principle. [For applications in Optics, refer to Art. 99, Physical Optics, College Physics, Vol. III.]

Examples

(1) *What is the apparent frequency of the sound of a whistle of frequency 600 from an engine which is approaching a stationary observer at 10 metres per sec.? (Velocity of sound = 332 metres per sec.)*

Ans.: Here $V_0=0$, $V_m=0$ and $V_s=10$ metres per sec.

Apparent frequency, $n_1=n\times\dfrac{V+V_m-V_0}{V+V_m-V_s}$.

$$=600\times\frac{332+0-0}{332+0-10}=600\times\frac{332}{322}=619 \text{ (approximately) per sec.}$$

(2) *Calculate the apparent frequency if the note of a whistle of frequency 1000 per sec. is heard from a train which is approaching a station at 44 ft./sec. where the whistle is blown (velocity of sound = 1100 ft./sec.)*

Ans.: Here $V_s=0$, $V_m=0$, and $V_0=-44$ ft./sec.

$\therefore$ Apparent frequency, $n_1=n\times\dfrac{V+V_m-V_0}{V+V_m-V_s}$

$$=1000\times\frac{1100+0-(-44)}{1100+0-0}=1000\times\frac{1144}{1100}=1040 \text{ per sec.}$$

(3) *An engine approaches a bridge at 5 ft. per sec. while sounding a whistle of frequency 500 c. p. s. The sound is reflected from the bridge Calculate the frequency of the beats heard by a stationary observer behind the engine* ($V=1100$ *ft./sec.*).

Ans. : Since the engine is receding from the observer, its apparent frequency as heard by him is,

$$n'=\frac{1100}{1100+5}\times 500.$$

As the engine approaches the bridge, the *image* of the engine may also be supposed to approach towards the bridge and hence to the observer. So the pitch of the reflected sound will be,

$$n_1=\frac{1100}{1100-5}\times 500.$$

∴ The beat frequency $=n_1-n'=500\times 1100\left(\frac{1}{1095}-\frac{1}{1105}\right)=4\cdot55$ per sec.

(4) *A train passing an observer at 80 miles per hour is sounding its whistle. Determine the interval (ratio of the frequencies) between the notes heard as the engine approaches and recedes* ($V=1120$ *ft./sec.*).

Ans. : 80 miles per hour $=117\cdot33$ ft./sec.

When the engine approaches the observer, the apparent frequency is,

$$n_1=n\frac{V}{V-V_s}=n\frac{1120}{1120-117\cdot33}=n.\frac{1120}{1002\cdot67}.$$

When the engine recedes, the corresponding apparent frequency is

$$n_2=n\frac{V}{V+V_s}=n\frac{1120}{1120+117\cdot33}=n\frac{1120}{1237\cdot33}.$$

Hence the required interval is $\frac{n_1}{n_2}=\frac{1237\cdot33}{1002\cdot67}=1\cdot23$ (approx.).

(5) *An engine in a cutting between two ridges, is whistling when its velocity is 1/20th of the velocity of sound in air. Compare the frequencies of the echoes from the ridges to an observer between them.*

Ans. : Here $V_s=\frac{V}{20}$, where $V=$vel. of sound in air.

Let A and B be the two ridges

When the engine recedes from the observer and approaches the ridge A, its image seems to approach the observer due to reflection from A. So the frequency of the echo from A is

$$n_1=n.\frac{V}{V-V_s}=n.\frac{V}{V-\frac{V}{20}}=n\times\frac{20}{19}.$$

Again, since the engine recedes from B, the corresponding image will also recede from B and hence from the observer. So the frequency of the echo from B will be,

$$n_2 = n.\frac{V}{V+V_s} = n\frac{V}{V+\frac{V}{20}} = n\frac{20}{21}.$$

$$\therefore\ \frac{n_1}{n_2} = \frac{21}{19} = 1{\cdot}1058 \text{ (approx.)}.$$

(6) *Calculate the velocity at which a source of frequency 10 thousand per second should approach an observer at rest in order to produce a Doppler's effect of 200 per second.* (*C U. 1954*)

Ans. **: We know when the observer is at rest and the source approaches him, the altered frequency is given by,**

$$n' = n\frac{V}{V-V_s},$$

Here $n = 10000$ and $n' = 10000+200$. Taking the velocity of sound in air as 1120 ft./sec., we have,

$$10200 = \frac{1120}{1120-V_s} \times 10000$$

or
$$1120 - V_s = \frac{112}{102} \times 1000\ ;\ \text{or, } V^s = 1120\left(1-\frac{100}{102}\right).$$

$$\therefore \qquad V_s = 21{\cdot}9 \text{ ft./sec.}$$

(7) ***A person standing on one side of a road blows a whistle which has a frequency of 600. A cyclist passes at 10 miles per hour. What does the frequency of the whistle appear to be to the cyclist before and after passing? Velocity of sound in air = 1100 ft/sec.*** (***Bihar, Suppl. 1953***)

Ans. **: Velocity of the cyclist $= V_0 = \frac{10 \times 1760 \times 3}{60 \times 60} = \frac{44}{3}$ ft./sec.**

When the cyclist approaches the whistle, the apparent frequency,

$$n_1 = n\frac{V+V_0}{V} = 600 \times \frac{1100+\frac{44}{3}}{1100} = 608 \text{ vib./sec.}$$

When the cyclist recedes from the whistle, the apparent frequency,

$$n_2 = n.\frac{V-V_0}{V} = 600 \times \frac{1100-\frac{44}{3}}{1100} = 592 \text{ vib./sec.}$$

(8) ***A locomotive whistle emitting 2000 waves per sec. is moving towards you at the rate of 60 miles an hour on a day when the thermometer stands at 24°C. Calculate the apparent pitch of the whistle.***

(***Velocity of sound at 0°C=1093 ft./sec.***) **(C. U. 1960)**

Ans. **: Here $V_{24} = (V_0 + 2 \times 24) = (1093+48) = 1141$ ft./sec.**

As the source is approaching the observer, the apparent pitch is given by,

$$n_1 = n\frac{V}{V-V_s} = 2000 \times \frac{1141}{1141-88} = 2167{\cdot}2 \text{ vib./sec.}$$

(9) Two aeroplanes are approaching each other at the speed of 100 and 150 m.p.h. The frequency of note emitted by the first as heard by passengers in the other is 1000. Calculate the true frequency of the note as heard by its own passengers. Take the velocity of sound as 750 m.p.h. (*R. U. 1948*)

Ans. : Let n be the true frequency.

Here $V=750$ m.p.h. ; $V_o=150$ m.p.h. and $V_s=100$ m.p.h.

Apparent frequency due to motion of the source (1st aeroplane) is,

$$n_1=n\frac{V}{V+Vs}.$$

Due to the motion of the observer (2nd aeroplane) towards the source, the modified apparent frequency is $n_2=1000$, where

$$n_2=n_1\frac{V+V_o}{V}=n\frac{V+V_o}{V-Vs}.$$

$$\therefore\quad n=n_2\frac{V-Vs}{V+V_o}=1000\times\frac{750-100}{750+150}=722{\cdot}2 \text{ vib. per sec.}$$

(10) A spectroscopic examination of a star shows that the apparent wavelength of a certain spectral line is 5001 A.U., whereas the observed wavelength of the same line produced by terrestrial source is 5000 A.U. In what direction and at what speed do these figures suggest that the star is moving relative to the earth.

(*C. U. 1947*)

Ans. : Since the observed wavelength is greater, the star is evidently receding from the earth.

$\therefore$ The apparent wavelength $\lambda'=\frac{V+Vs}{n}=\frac{V+Vs}{V}\lambda$

where λ is the actual wavelength, V=vel. of light, Vs=vel. of the star.

Hence, $\lambda'-\lambda=\frac{V_s}{V}\lambda$; or, $V_s=\frac{V(\lambda'-\lambda)}{\lambda}$

$$\therefore\quad Vs=\frac{3\times10^{10}(5001-5000)\times10^{-8}}{5000\times10^{-8}}=6\times10^{6}\text{ cm./sec.}$$

Questions (Chapter VI)

1. Show that the velocity of propagation of sound waves in a gas is given by the expression $\sqrt{E/\rho}$, where E is the elasticity and ρ is the density of the gas. What expression will you use for the value of E? Give your answers for choosing this particular value. (C. U. 1930)

2. Derive an expression for the velocity of sound in a gas. (Raj. U. 1949)

3. Derive a general expression for the velocity of sound in a gas, and discuss the formulæ due to Newton and Laplace. (*cf.* C. U. 1965 ; *cf.* Utk. U. '70S)

Calculate the velocity of sound in air at N.T.P. (the ratio of specific heats for air $=1{\cdot}4$: density of air at N.T.P. $=\cdot001293$). (All. U. 1943, '52 ; C. U. 1938)

[*Ans.* 332·5 metres/sec.]

4. Derive an expression for the velocity of sound in gases. Describe any method of determining the velocity of sound in air. (C. U. 1950, '58)

Find the velocity of sound in air at 50°C and 70 cm. pressure. The velocity at N.T.P. is 1090 ft. per sec. (C. U. 1958)

[*Ans.* 1190 ft./sec. (approx.)]

5. Find the expression for velocity of propagation of longitudinal vibrations in a solid rod.

Discuss how from an acoustical experiment Young's modulus can be determined. (*cf.* C. U. 1944, '54, '62)

6. Young's modulus for steel is $2{\cdot}14\times10^{12}$ dynes/cm.^{2} and its density is 7·8 gms./c.c. Find the velocity of propagation of sound through a steel bar. How would the temperature of the bar affect the result ?

[*Ans.* 5237 metres/sec.]

7. Explain why sound travels faster and is heard at greater distances in liquids and in solids than in gases. (C. U. 1931)

8. Find an expression for longitudinal waves in a metal bar, and describe a method of determining the velocity of the sound in it experimentally. (Nag. U. 1955)

9. Deduce a theoretical formula for the velocity of sound in gases. A sound is emitted by a source placed at one end of an iron tube, one kilometre long, and two sounds are heard at the other end at an interval of 2·8 secs. If the velocity of sound in air in the condition of the experiment is 330 metres per sec., find that in iron. (C. U. 1942)

[*Ans.* 4312 metres/sec.]

10. Give an account of the classic determination of the velocity of sound in free air and state the results obtained. (C. U. 1949)

11. Mention the different factors that influence the velocity of sound as determined by open-air observations and explain how they may be corrected in order to obtain an absolute value. (*cf.* C. U. 1963)

12. Find an expression for the velocity of sound in air from the theory of dimensions assuming that it depends only on pressure and density. Is the formula so obtained verified experimentally ? Give reasons for your answer.

[HINTS :—Since velocity V depends on pressure and density, $V=k.\,P^x\rho^y$. As the dimension of V is LT^{-1}, that of P, $ML^{-1}T^{-2}$ and that of ρ, ML^{-3} equating the dimensions we get,

$$LT^{-1}=M^xL^{-x}T^{-2x}.M^yL^{-3y}=M^{x+y}L^{-x-3y}T^{-2x}.$$

$$\therefore\ x+y=0; \qquad \text{or,}\ x=-y.$$

$$\text{and}\quad 2x=1; \qquad \text{or,}\ x=\tfrac{1}{2}.$$

$$\therefore\ y=-\tfrac{1}{2}.$$

$$\therefore\ V=KP^{\frac{1}{2}}\rho^{-\frac{1}{2}}=K\sqrt{P/\rho}.$$]

13. Give Newton's formula for the velocity of sound in air. In what respect was it defective and what correction was found necessary to make it yield the accurate result ?

14. State the law connecting the velocity of sound through a gas with its temperature and pressure. (C. U. 1938)

15. A weight of 5 Kgs. suspended from the lower end of a uniform wire of length 1 metre and diameter 0·36 mm. produces an extension of 2·5 mm. Find the frequency of the note emitted when the stretched wire is stroked by a resined cloth. The density of the material of the wire is 7·8 gms. per c.c.

[*Ans.* 2490 c.p.s.]

16. Write notes on : (*a*) Hydrophone, (*b*) sound ranging (Utk.U.), (*c*) echo depth sounding (C U.), (*d*) determination of the position of a ship in foggy weather, (*e*) effect of humidity on the velocity of sound.

17. Explain Doppler's principle and explain how it can be easily demonstrated.

18. It was reported that a musical policeman could estimate the speed of a motor car by the drop in pitch of the horn sounded as the car passed him. Give a theoretical explanation to substantiate this report (Bom. U. 1932)

19. What is Doppler's Principle ? (C. U. 1964)

A train approaches a stationary observer at a speed of 75 kilometres per hour sounding a whistle of frequency 1000. What will be the apparent frequency of the whistle to the observer ? (C. U. 1948)

[*Ans.* 1067 c.p.s. (nearly)].

20. A line of wavelength $4·8\times10^{-5}$ cm. in the spectrum of the light from a star is found to be displaced from its normal position nearer to the red end of the spectrum by an amount corresponding to a change in wavelength of $1·2\times10^{-8}$ cm. What velocity of the star in the line of sight would account for this shift ? (*cf.* All. U. 1954)

[*Ans.* $0·25\times10^{-3}$ c. (c=velocity of light)].

21. Explain clearly Doppler effect in sound. Show that the Doppler effect is greater when the source approaches the observer than when the observer approaches the source with the same speed. (*cf.* Utk. U. '70*S* ; Bom. U. 1954 ; C. U. 1954)

22. What is Doppler effect in sound ? If a sounding tuning fork of high frequency is rapidly moved to-and-fro between the ear and the surface of a flat wall, beats will be heard. Explain how two notes of different frequencies are produced. (Nag. U. 1954)

23. What is Doppler effect ? Derive and explain the formulæ for the apparent frequency due to it.

A train is passing a railway station with a speed of 40 m.p.h. and blowing continuously a whistle of frequency 256 per sec. What will be the frequencies apparent to a person waiting on the platform when the train is (*a*) approaching, (*b*) departing ? What is the interval between these two notes ? (vel. of sound = 1120 ft /sec.) (C. U. 1945)

[*Ans.* (*a*) 270·15 c.p.s.; (*b*) 243·26 c.p.s.; Interval = 1·147 secs.

24. Show that if the source moves away with the velocity of sound from an observer who is at rest, the frequency of vibration heard is halved. (C.U. 1958)

25. A locomotive whistle emitting 2000 waves per sec. is moving towards you at the rate of 60 m.p.h. on a day when the thermometer stands at 24°*C*. Calculate the apparent pitch of the whistle (vel. of sound at 0°*C*=1093 ft./sec.) (C.U. 1940)

[*Ans.* 2167 8 c.p.s.]

26. Calculate the percentage difference between the frequency of a note emitted by a whistle of a train approaching an observer with a velocity of 100 ft./sec. and that heard by the observer, the velocity of sound in air being 1100 ft./sec. (C. U. 1957)

[*Ans.* 10 per cent.]

27. An observer on a railway station platform observed that as the train passed the station at 60 m.p.h., the frequency of the whistle appeared to drop by 400 vib./sec. Find the frequency of the whistle. (Bihar U. 1958)

(Velocity of sound=1100 ft./sec.)

[*Ans.* 2484 vib./sec.]

28. An engine sounding a whistle of frequency 500 vib./sec. is approaching an observer at a velocity of 60 m.p.h. What is the frequency of the note heard by the observer ? Velocity of sound=1120 ft./sec. (C. U. 1959)

[*Ans.* 542·6 vib./sec.]

CHAPTER VII

VIBRATION OF STRINGS : FORK : PLATE : MEMBRANE : BELL

70. Vibration of Strings :—In sound a *string* usually means a wire or a cord of any material, which is flexible and uniform in cross-section. These conditions are found to be satisfactorily fulfilled by thin metallic wires or catgut. Strings may vibrate in two ways : *transversely* and *longitudinally*. A string can be vibrated longitudinally by rubbing it along the length with a piece of chamois leather covered with resin, or by a piece of wet flannel. It can be vibrated transversely by plucking it to a side, by bowing it with a violin bow, etc. When a stretched string is plucked to one side, it tends to return to its original (straight) position of rest. But owing to inertia that it possesses, it overshoots the mark like the motion of a pendulum and goes over to the other side and goes on swinging to and fro with gradually decreasing amplitudes and after sometime it stops. The vibration in this case is mainly due to the tension in the string which when the string is deflected, tends to bring it back to initial straight position. *In stringed musical instruments, only the transverse vibrations of strings* are employed.

71. Reflection of Waves in Transverse Vibration.—(*a*) *Reflection of waves in a string.*—Let a wave travelling along a wire, say from left to right, meet a fixed support and let the wave meet the support in the form of crest. The end of the wire will exert a force on the support tending to move it in the direction of the force. Then according to Newton's *Third Law of Motion*, the support will react and exert an equal and opposite force on the wire which causes a rebound, so that the pulse is thrown over towards the other side of the string and a reversed pulse starts travelling back along the string from right to left. Thus, **in this case reflection takes place at the fixed ends**

with change of type, *i.e. a crest is reflected back as a trough and a trough is reflected back as a crest.* It should be noted, however, that in the case of water-waves, which are *transverse waves*, a crest meeting a rigid wall is reflected back as a crest and a trough is reflected back as a trough, like longitudinal sound waves. Also, the important difference between the reflections of sound waves at the closed and open ends of a pipe should be noted.

(*b*) *Reflection of Water-waves.*—When a water-wave travels along it has both potential and kinetic energy. Part of the energy is *potential*, because a force must have been applied to and work done upon the water to raise it above its normal level, and a part is *kinetic*, because the molecules are in motion. When the waves strike a rigid wall or a denser medium, the motion of the molecules towards the wall is arrested, their kinetic energy is reduced, which is then converted into potential energy, thus increasing the amount of potential energy. So the average elevation of the water in the crest is increased and the water is piled up against the obstruction, which then runs down and away from the wall producing a crest like the original wave and travelling in the opposite direction. Thus, in the case of a water-wave meeting a rigid wall a *crest is reflected as a crest and similarly a trough is reflected as a trough.*

72. Stationary Waves in a String :—When a stretched string is plucked aside, a wave will travel along its length with a definite velocity. The transverse wave will be propagated to both ends and will be reflected at these points. If a complete wave consisting of a crest and a trough is sent along a string crest first, it will return as a trough first after reflection at the fixed end. These reflected waves will return to the centre of the string when they pass each other, and go on to the ends, to be once again reflected, and so on. These incident and reflected waves, travelling to-and-fro along the string in opposite directions with equal velocities combine to form *transverse stationary waves whose position of nodes and antinodes are fixed.*

73. The Velocity of Transverse Waves along a String :—When a string stretched under tension is displaced laterally, transverse waves are set up in it. The waves travel along the string with a velocity dependent on the ***tension*** and the ***linear density*** of the string.

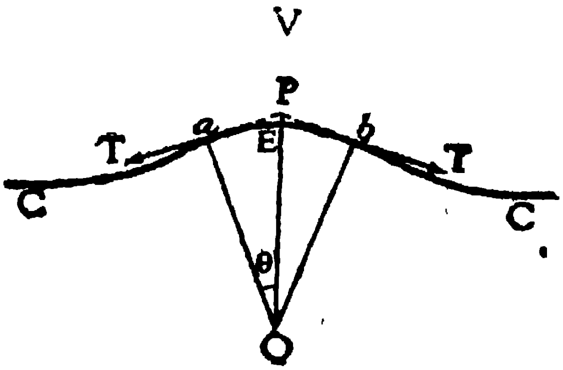


Fig. 47

Suppose the string CC stretched under tension T [Fig. 47] is displaced perpendicularly to its length so as to make transverse vibrations (by plucking, bowing or striking), due to which, suppose, the summit aEb of the displaced position is bent into an arc of a circle. The transverse wave travelling along the string from left to right with velocity V may be imagined to be due to the hump also travelling with the same velocity. For the circular motion of an element near the summit E of the hump, the necessary centripetal force is supplied by the tension at a and b.

Suppose $aE=Eb$ and O is the centre of the curvature aEb. Let the angle aOE be θ. Join Oa and Ob. Draw tangents T at a and b, representing the tension of the string which produced backwards meet at P on OE. Suppose the length aEb is L, mass per unit length of the string is m, and the radius of the curvature is R.

The components of the tension T at a and b in the direction PO (each equal to $T\sin\theta$) constitute the centripetal force $\frac{m.\,L.\,V^2}{R}$ on the hump, while the components of T at right angles to PO cancel each other. Therefore,

$$\frac{m\,L.\,V^2}{R} = 2T\sin\theta = 2T.\theta \text{ (approximately)}$$

$$= 2T\times\frac{L/2}{R} = \frac{TL}{R}, \qquad \therefore \quad V=\sqrt{\frac{T}{m}}.$$

***Alternative Method.**—Let a string WW' be stretched along the X-axis. If the wire be under a tension T and then displaced along Y-axis to the position $WABW'$, the tension would act uniformly over the whole loop in absence of any stiffness. Suppose AB is a very small element of the wire of length ds, the point A having co-ordinates x, y with reference to any arbitrary origin. The tension, T, at A acts downwards along the tangent there and the tension at B acts tangentially upwards as shown in Fig. 48.

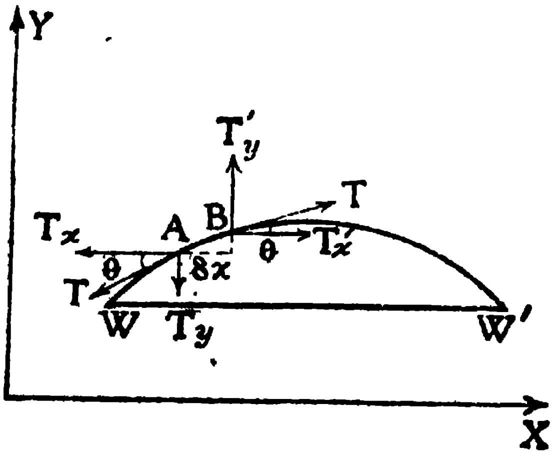


Fig. 48

At A let the inclination of the tangent with the X-axis be θ. Then the horizontal component of T there is $T_x = T \cos \theta = T$, since θ is generally very small. This T_x is directed in the negative direction of the X-axis. Again the vertical component of T at A is given by

$$T_y = T \sin \theta = T \tan \theta = T\frac{dy}{dx}. \qquad (\because \theta \text{ is very small}).$$

This component acts along the negative direction of the Y-axis. The first component $T_x = T$ being independent of x can be shown to be identical with the corresponding component of T'_x of the tension at B, T and T'_x being oppositely directed and equal in magnitude cancel each other as points A and B are almost coincident.

Now since the y-component T_y of the tension at A is $T\frac{dy}{dx}$, the corresponding y-component of the tension at B will be

$T'_y = \left[T\frac{dy}{dx} + T\frac{d}{dx}\left(\frac{dy}{dx}\right)\delta x \right]$, where $\frac{d}{dx}\left(T\frac{dy}{dx}\right)$ is the rate of variation of y-component along X-axis. This component is directed along the positive direction of the X-axis.

$\therefore$ The resultant transverse force acting at right angles to the original length of the wire is given by

$$T'_y - T_y = T\frac{d^2y}{dx^2}\delta x = T\frac{d^2y}{dx^2}ds \qquad (1)$$

ds is practically equal to δx for small values of θ.

It is due to this effective component of the tension that the wire or the string is forced to vibrate transversely. Now if m be the mass per unit length of the wire, the mass of length ds is $m.\ ds$.

This element ds of the above mass is, under the above force, made to vibrate with an acceleration $\frac{d^2y}{dt^2}$.

$\therefore$ The force acting on the element is $mds\,\frac{d^2y}{dt^2}$.

This is equal to the force due to tension as derived in expression (1).

$$\therefore\quad mds\frac{d^2y}{at^2}=T\,\frac{d^2y}{dx^2}ds\ ;$$

$$\text{or,}\quad \frac{d^2y}{dt^2}=\frac{T}{m}\frac{d^2y}{dx^2}\qquad \ldots\qquad \ldots\ (2)$$

Eqn. (2) represents the differential equation of a wave. Since the displacement in a simple harmonic wave is given by $y=a\sin\frac{2\pi}{\lambda}(Vt-x)$, where V is the velocity of the wave, we evaluate $\frac{d^2y}{dt^2}$ and $\frac{a^2y}{dx^2}$ from this displacement expression and find that

$$\frac{d^2y}{dt^2}=V^2\frac{d^2y}{dx^2}\qquad \ldots\qquad \ldots\ (3)$$

Relation (3) also represents the equation of a simple harmonic wave moving with a velocity V. Hence comparing eqns. (2) and (3),

$$V=\sqrt{T/m}.$$

[**Note :** From above we get the velocity of a transverse wave in a string as $V_T=\sqrt{\frac{T}{m}}=\sqrt{\frac{T}{S\times\rho}}$ where S and ρ are the cross-sectional area of the wire and the density of the material respectively.

Again the velocity of a longitudinal wave along the same wire is given by $V_L=\sqrt{Y/\rho}=\sqrt{\frac{YS}{S\rho}}$.

$\therefore$ If $V_L=V_T$, we have $T=YS$; or $Y=T/S$.

i.e. Young's modulus = Tension applied per unit cross-sectional area = Longitudinal stress.

But by definition, $Y = \dfrac{\text{Longitudinal stress}}{\text{Longitudinal strain}}$.

$\therefore$ If $V_L = V_T$, longitudinal strain $= 1$; or, in other words, the extension under the tension must be identical with the original length. No wire is known to possess so abnormally high an elastic limit. So V_L and V_T cannot be equal. As strain is always less than unity, $Y > T/S$; or $YS > T$. $\therefore$ $V_L > V_T$.

So under all available tensions, the velocity of a longitudinal wave through a wire will be greater than that of a transverse wave through it.]

74. Frequency of Transverse Vibration of Strings :—The velocity of a transverse wave along a stretched string is given by

$$V = \sqrt{\frac{T}{m}} = \sqrt{\frac{Mg}{m}} \quad \dots \quad \dots \quad \dots \quad (1)$$

where T = tension of the string expressed in dynes; m = mass in grams per unit length of the string; M = mass of load on the string.

When the string gives out its *fundamental i.e.* the note of the lowest pitch the length of the string, l cm. = distance between two consecutive notes $= \lambda/2$.

$$\therefore \quad V = n\lambda = 2nl.$$

Substituting the value of V in eqn. (1) we get,

$$2nl = \sqrt{\frac{T}{m}}\,; \text{ or, } n = \frac{1}{2l}\sqrt{\frac{T}{m}} \quad \dots \quad \dots \quad (2)$$

Again, if ρ be the density of the material of the wire and r be its radius, then $m = \pi r^2 \rho$, and so we have from eqn. (2),

$$n = \frac{1}{2l}\sqrt{\frac{T}{\pi r^2 \rho}} = \frac{1}{2rl}\sqrt{\frac{T}{\pi\rho}} = \frac{1}{ld}\sqrt{\frac{T}{\pi\rho}} \quad \dots \quad \dots \quad (3)$$

where d is the diameter of the wire.

The frequency, $n = \dfrac{1}{2l}\sqrt{\dfrac{T}{m}}$, of the string corresponds to the fundamental note emitted by the string when it is vibrating in the simplest possible way with two nodes at the two ends and an antinode in the middle. If this fundamental frequency by designated by n_1, we can write, $n_1 = \dfrac{1}{2l}\sqrt{\dfrac{T}{m}}$. When the string vibrates in two segments with a third node in the middle, the *second harmonic* or the *first overtone* is produced. Here $l=\lambda$.

$\therefore$ The frequency of the second harmonic is $n_2=\frac{1}{l}\sqrt{\frac{T}{m}}=2n_1$.

In the next higher mode of vibration, the string vibrates in three segments and the *third harmonic* emitted has the frequency, $n_3=3.\frac{1}{2l}\sqrt{\frac{T}{m}}=3n_1$. Similarly, the frequency of the *sth harmonic* is

$$n_s=\frac{s}{2l}\sqrt{\frac{T}{m}} \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (4)$$

Relation (4) gives us the general expression of the frequency emitted by a transversely vibrating string.

75. Laws of Transverse Vibration of Strings :—From relation (2) of Art. 74, we get the following laws of the transverse vibration of strings, which are generally known as **Mersenne's laws.**

(1). *Law of Length*—The frequency of a note emitted by a string varies inversely as the length, the tension remaining constant ; that is, $n \propto 1/l$, when T and m are constant.

(2). *Law of Tension.*—The frequency of a note emitted by a string varies directly as the square root of the tension, the length being kept constant ; that is $n \propto \sqrt{T}$, when l and m are constant.

(3). *Law of Mass.*—The frequency of a note emitted by a string varies inversely as the square root of the mass per unit length of the string, the length and tension remaining constant ; that is $n \propto 1/\sqrt{m}$, when l and T are constant.

Again, from relation (3) of Art. 74, the law of mass may be put into two additional laws for strings of circular section as given below :

3(*a*). *Law of Diameter.*—The frequency of the note produced by a string varies inversely as the diameter of the string—the length, the density of the material of the string and the tension remaining constant ; that is, $n \propto 1/d$, when l, ρ and T are constant.

3(*b*). *Law of Density.*—The frequency of the note emitted by a string varies inversely as the square root of the density of the material of the string, length and diameter of the string and tension remaining constant, that is, $n \propto 1/\sqrt{\rho}$, when l, d and T are constant.

76. Experimental Verifications of the Laws of Transverse Vibration of Strings (By Sonometer)—The laws of transverse vibration of strings can be verified by means of an instrument, called a **sonometer.** It consists of a hollow wooden box *AA*, on which one (in case of a **monochord**) or more wires can be stretched (Fig. 49). Each wire is attached to a peg at one end and passes over two wedge-shaped hard wood b_1, *B*, called the ***bridges*** and finally over a pulley at the other end. The string is kept taut by weights *E* attached at this end. A third bridge *C* can be placed in any position between the other two in order to set any desired length of the string into vibration.

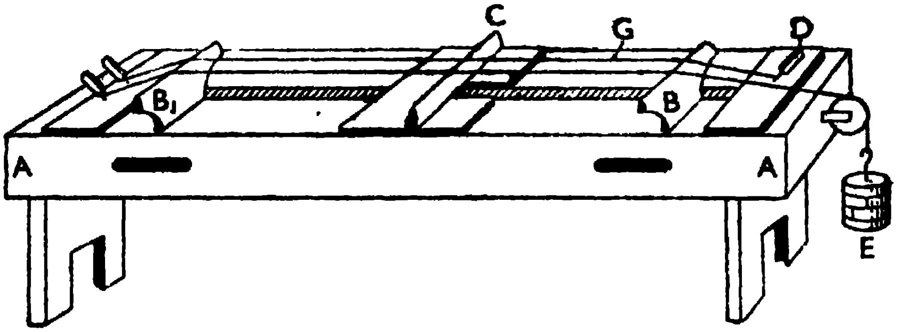


Fig. 49—The Sonometer.

Law 1. *To verify* $n \propto 1/l$.—To verify the law of length, two tuning forks of known frequencies n_1 and n_2 are taken. One of the forks is made to vibrate, and altering the position of the movable bridge *C*, the length B_1C of the sonometer wire (under a given tension) is so adjusted that the note emitted by that length of the wire, when plucked in the middle, is in unison with the note yielded by the fork (*vide* Art. 77). Then the frequency n_1 of the fork is equal to the frequency of the wire of length l_1. Repeating the experiment with the other tuning fork, another length of the wire is similarly determined. Let n_2 be the frequency of this fork, and l_2, the corresponding length of the wire. It will be found by experiment that $\frac{n_1}{n_2} = \frac{l_2}{l_1}$; or, $n_1l_1 = n_2l_2$. Repeating the experiment with other forks, it will be found that $n_1l_1 = n_2l_2 = n_3l_3$, etc., *i.e.* $nl =$ a constant. This verifies the law.

Note that the same wire is used and the tension is kept the same while adjusting the length of the wire for unison with different forks.

Law 2. *To verify $n \propto \sqrt{T}$.*—Stretch another wire, called the comparison wire, by the side of the first wire. Let T_1 be the tension on the first wire. A length of the comparison wire is then adjusted which is in unison with the note yielded by the first wire.

Let the length be l_1. Now increase the tension on the first wire to T_2; so the frequency of the note emitted increases. Again another length l_2 of the comparison wire is found which is in unison with the note of the first wire. Let n_2 and n_1, be the frequencies of the notes of the comparison wire of lengths l_1 and l_2, and so of the first wire corresponding to tensions T_1 and T_2 respectively. We have, by the law of length, $\frac{n_1}{n_2}=\frac{l_2}{l_1}$. Again, it is found by experiment that $\frac{l_2}{l_1}$ $=\sqrt{\frac{T_1}{T_2}}$. So, $\frac{n_1}{n_2}=\sqrt{\frac{T}{T_2}}$.

By applying different tensions T_1, T_2, T_3, etc. to the first wire and determining corresponding attuned lengths l_1, l_2, l_3, etc. for which the respective frequencies are n_1, n_2, n_3, etc. it may be shown that $n/\sqrt{T}$ is constant. This verifies the law of tension.

Law 3. *To verify $n \propto 1/\sqrt{m}$*—To verify the law of mass, two wires of different mass per unit length are taken. The wires may be of the same material or of different materials. One of them is stretched by the side of the comparison wire with a suitable load. Taking any length of the wire whose mass per unit length is m_1, a length l_1 of the comparison wire is determined, which is in unison with the note of the first wire. Replacing the first wire by the second wire of mass m_2 per unit length, and keeping ***tension*** the ***same***, the above experiment is repeated, taking the ***length*** of the second experimental wire ***same*** as that of the first. A length l_2 of the comparison wire is found which is in unison with the note of the second wire. Then a measured length of each of the two wires is taken and each of them is weighed. From these weights, the masses of unit length (m_1 and m_2) for the two wires are found.

Let n_1 and n_2 be the frequencies corresponding to lengths l_1 and l_2 of the comparison wire. We have, by the law of length, $\frac{n_1}{n_2}=\frac{l_2}{l_1}$.

and it will be found by the experiment that $\frac{l_2}{l_1} = \sqrt{\frac{m_2}{m_1}}$.

Hence, $\frac{n_1}{n_2} = \sqrt{\frac{m_2}{m_1}}$; or, $n_1 \sqrt{m_1} = n_2 \sqrt{m_2}$. Repeating the experiment with other wires of different mass per unit length, it will be found that $n \sqrt{m} =$ a constant. This proves the law of mass.

Law 3(a). *To verify* $n \propto 1 \sqrt{d}$—Take two wires of different diameters but of the same material and proceed just as in the above experiment (Law 3). Let l_1 and l_2 be the lengths of the comparison wire which are found to be in unison with notes produced by equal length of the two wires having diameters d_1 and d_2 respectively. Now, measure d_1 and d_2 with a screw-gauge. From Law 1, we have $n_1/n_2 = l_2/l_1$, and it will be found by experiment that, $l_2/l_1 = d_2/d_1$.

Hence $\frac{n_1}{n_2} = \frac{d_2}{d_1}$ which verifies the law.

Law 3(b). *To verify* $n \propto 1 \sqrt{\rho}$, take two wires of different materials but of the same diameter, and repeat the experiment exactly as in the verification of Law 3(a). It will be found that $\frac{n_1}{n_2} = \sqrt{\frac{\rho_2}{\rho_1}}$. This verifies the law.

N.B.—It should be noted that this experiment gives a method of determining acoustically whether two wires are made of the same material or not.

77. Notes on Tuning :—In tuning two strings, or a tuning fork and a string, or any two notes, the following two methods may generally be adopted.

(i) **"By Resonance".**—Tune as nearly as possible by ear. Then place an inverted *V*-shaped paper rider, or a thin wire rider on the middle of the string, and place the stem of the vibrating tuning fork on the sonometer box. It will set the string into vibration by resonance and the rider will be thrown off, if the tuning be accurate. If, however, this does not occur, adjust the length of the string by moving the movable bridge until the rider is thrown off.

(ii) **"By Beats".**—By adjusting the length of the string by the movable bridge until the two notes (of the string and of the fork) are very nearly of the same frequency, beats will be heard, *i.e.* the resultant sound will appear to give alternate maxima and minima of loudness. On adjusting the length still further, beats will become slower, and will cease entirely when tuning is exact, *i.e.* when the frequencies of the two notes are exactly equal.

78. **Determination of the Pitch by Sonometer** :—(a) The frequency of a note can be determined either by keeping the length of the sonometer wire constant and adjusting the tension, or by adjusting the length of the wire keeping the tension constant, until the string is in unison with the note, the pitch of which is to be determined. The later method is, however, convenient. If the frequency of a tuning fork is to be determined, its stem is lightly pressed against the sonometer box after it is made to vibrate. The resonant length of the wire is then measured and the mass of the string per unit length is determined. The stretching weight is noted. The tension is calculated by multiplying it by the acceleration due to gravity. The frequency n is then calculated by the formula,

$$n=\frac{1}{2l}\sqrt{\frac{T}{m}}.$$

N.B.—By knowing n, the *density* of the material of the wire can be determined from formula (3) of Art. 74.

(b) The pitch of a tuning fork can also be determined by taking another standard fork of known frequency and then determining as above a length of the same wire stretched by the same weight until this fork and the wire are in unison again. If n be the frequency of the standard fork, n' the unknown frequency, and l and l' be the corresponding lengths of the wire, then, we have, $\frac{n}{n'}=\frac{l'}{l}$, whence n' can be determined.

79. **Determination of frequency of an a.c. source by a Sonometer** :—If the sonometer steel wire under a tension T be connected to an alternating current source and the bridges so adjusted that the length between them has a natural frequency equal to that of the a.c. source, the segment will vigorously vibrate when placed between

the poles of a strong magnet. When the resonance condition is adjusted by trial, the frequency of the a.c. source is $n=\frac{1}{2l}\sqrt{\frac{T}{m}}$.

80. Certain Terms :-

Note, Tone.—A *note* is a general term denoting any type of musical sound. A musical sound is a complex sound made up of two or more simple component sounds of different pitches. Each of the simple component sounds is called *tone.* A tone cannot further be divided into simpler components and therefore, has a single frequency. In other words, a tone is a sound of single frequency, while a note consists of some pure tones.

Fundamental, Overtone, Harmonic and Octave.—When a body vibrates, generally there are present in the note several tones of frequencies which are multiples of the frequency of a *fundamental* which is the tone of the lowest pitch. The other tones, except the fundamental, are called *overtones.* When the frequencies of the overtones are exact multiples of the frequency of the fundamental, they are, in particular, called *harmonics.*

The tone whose frequency is twice that of the fundamental is said to be an *octave higher*, or called the first harmonic of the fundamental. All tones of frequencies between any number n and $2n$ are said to be in the *same octave.*

81. The Harmonics of a Stretched String :—(*i*) A string can be made to vibrate in different modes. When it vibrates as a whole it is the simplest mode of its vibration. Such vibration is produced where the string is plucked at its centre. It has been pointed out in Art. 37 that when a string vibrates, the waves generated are reflected from the fixed ends, and the incident and the reflected waves give rise to transverse stationary waves having definite nodes and antinodes. In the present case there will be produced *two nodes* (N, N) *at the two fixed ends and one antinode* A in the middle as shown in Fig. 50, (I) ; in this case the length of the string, $l=\lambda/2$.

$\therefore n=\frac{V}{\lambda}=\frac{V}{2l}$. But the string may vibrate in other ways also.

(*ii*) If the string be plucked at a point *one-fourth* the length of the wire from one end at the same time the middle point of the wire is lightly touched, it will vibrate in two segments. In this mode of vibration there are *three nodes* and *two antinodes* as shown in Fig. 50, (II). In this case, $l = \lambda$.

$\therefore \quad n_1 = V/l$; or, $n_1 = 2n$.

This tone is an *octave higher* and is called the *first harmonic of the fundamental tone.*

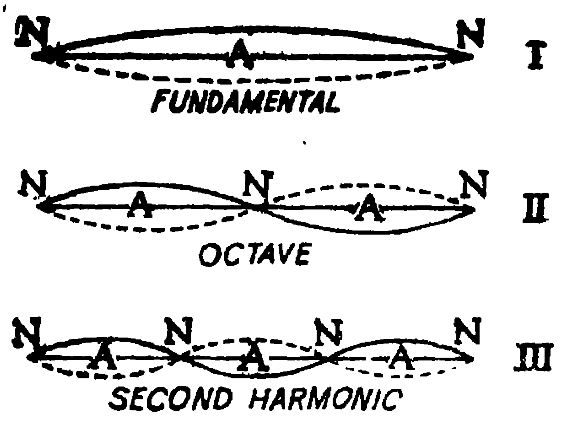


Fig. 50

(*iii*) In the next mode of vibration, if the string is held at one-third of its length and if the middle of the shorter segment is bowed, the wire will vibrate in three segments, and in that case there will be *four nodes* and *three antinodes* as shown in Fig. 50, (III). In this case $\lambda = \frac{2}{3}l$.

$$\therefore \quad n_2 = \frac{3V}{2l} \text{ ; or, } n_2 = 3n.$$

This tone is called the *second harmonic* of the fundamental tone. In this way it will have all the *odd* and *even* harmonics. But if the bowing is at random such that the segments into which the string vibrates are not regular, tones will also be produced which are not exact multiples of the fundamental. These higher tones are called **Upper tones or Overtones or Upper partials.**

82. Melde's Experiments :—Melde 1859-60 devised a beautiful experiment for demonstrating vibrations of a stretched string and also for verifying the laws of transverse vibration of a string. Here the string under test is stretched by a load at one end, the other end being tied to the prong of an electrically maintained tuning fork. The experiment may be conducted in two ways :—

(*i*) When the fork is so mounted that the motion of its prongs is at right angles to the length of the string, the arrangement is known as the *transverse arrangement* ; (*ii*) when the prongs vibrate in the direction of the string the arrangement is known as the *longitudinal arrangement.*

(i) **Transverse Method.**—The transverse arrangement is shown in Fig. 51. Here the string is attached at one end to a rigidly clamped tuning fork which is free to vibrate at right angle to the length of the string while the other end passes over a smooth pulley and carries a small scale pan for putting suitable weights on it. As the fork is set to vibration by switching on the circuit of the electromagnet operating it, the string will also be forced to vibrate. The transverse waves set up in the string will move in the direction of the string with a velocity depending on the linear density and tension of the string. These waves will then suffer reflection at the pulley and travel back towards the fork. Thus two identical waves travelling in the opposite senses along the string will give rise to stationary waves having definite nodal and antinodal positions on the wire. The definite positions of these nodes and antinodes are ensured only when the condition of resonance is satisfied, *i.e.* when the natural frequency of the string under tension exactly coincides with that of the fork. Before this condition is satisfied the nodes and antinodes may appear due to forcing but in that case the vibration of the string would be rather small. But adjusting the length of the segment of the string and its tension, the resonance condition may be attained so that the vibration amplitude becomes maximum. Now the length of the string from the fork to the pulley is measured and the number of fixed loops formed is also counted. If s be this number, then the frequency of the vibration of the string is given by $n=\frac{s}{2l}\sqrt{T/m}$.

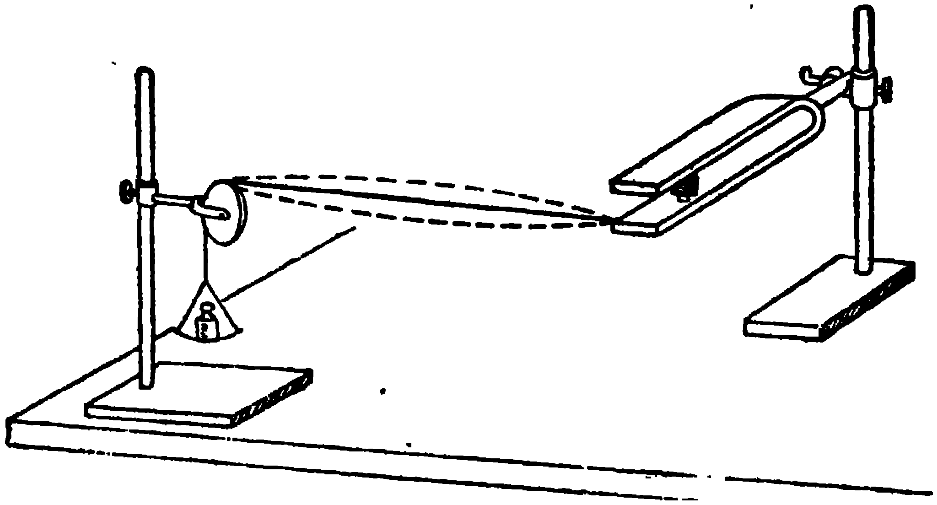


Fig. 51

Since for resonance this n is the same as the frequency of the fork, the calculated frequency as arrived at from the above formula should agree with the frequency of the fork. This is actually found to be so thereby verifying the laws of transverse vibration of string.

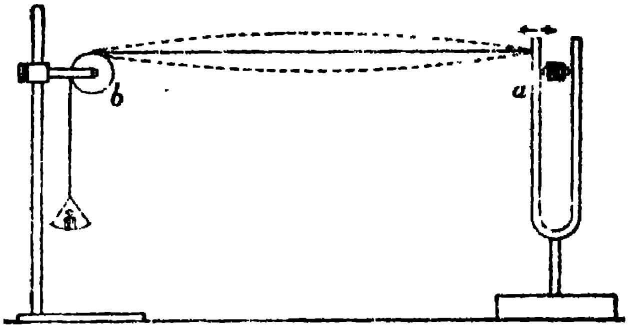


Fig. 52

(ii) Longitudinal method.—In the longitudinal arrangement also one end of the string passes over a smooth pulley and carries a scale pan for applying the requisite tension. The other end is attached to an electrically driven tuning fork so adjusted that its prongs may vibrate along the direction of the stretched string.

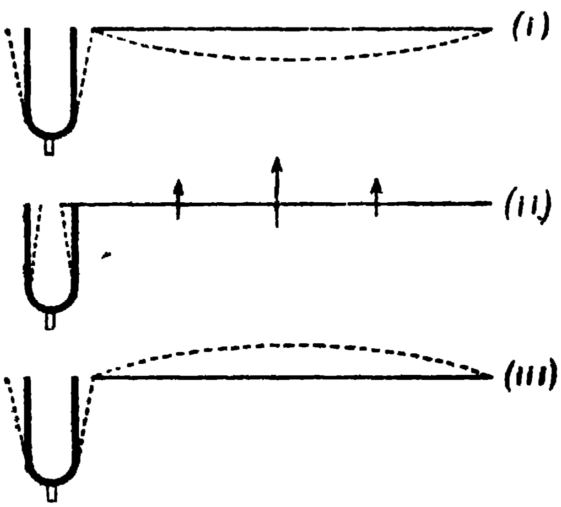


Fig. 53

Due to the superposition of identical incident and reflected waves standing waves are formed. The string is found to have definite nodes and antinodes when resonance is established. The natural frequency of this vibrating string is also given by $n = \frac{s}{2l}\sqrt{T/m}$, where l is the resonant length of the string, s is the number of loops formed. On calculation it is found that the frequency, n, as obtained from the above formula is only half the frequency of the driving fork and not equal as was the case in the transverse arrangement.

The explanation of this peculiarity may be found out if the function of the driving fork is considered. In course of its vibration, when the displacement of the prong, *a* attached to the string is maximum towards the other end *b* of string let us assume that the string also has its maximum sag as shown by the dotted curve in Fig. 53(*i*). As the prong *a* gradually recovers its position this sag gradually diminishes and finally becomes zero when the prong undergoes maximum displacement on other side as shown in Fig. 53(*ii*). Again when the prong, *a*, returns to its maximum displaced position next time towards *b*, the string is not sagged but is rather displaced upwards [Fig. 53(*iii*)] due to inertia of motion. So during the time in which the fork makes one complete vibration, the string forms a single loop, *i e*. it undergoes only half a complete vibration. So the frequency *n* as derived from the above formula should be only half that of the fork. Experimentally also this is found to be the case thereby establishing validity of the laws of transverse vibration of a string.

It is to be noted that under the same tension, a particular string vibrates in a transverse arrangement with a number of loops which is double of that in a longitudinal arrangement.

83. **Transverse Vibration of bars** :—In the considerations of the vibration of strings it has been assumed that the strings are perfectly flexible and the restoring force is purely supplied by the tension and that the rigidity (or stiffness) plays no part in the phenomenon. In the case of vibrating rigid bars or rods the state of affairs is just the reverse. Here tension has nothing to do in the actual vibration of the bar, restitution being completely brought about by the rigidity (or stiffness) of the bar. The transverse vibration of bars may be divided into three distinct categories according to their mode of support. When a bar is clamped at one end, it is called a *fixed-free bar*. When it rests on two supports near the two ends it is known as a *free-free bar* and when both ends are rigidly clamped it is a *fixed-fixed bar*. Let these three types be separately considered one by one.

(a) **Fixed-free bar.**—Here the bar is rigidly clamped at one end and the free-end is excited either by bowing or striking. The attached figures (*a*), (*b*), (*c*) (Fig. 54) show the first three simplest modes of vibration: (a) indicates the fundamental mode of vibration where at the free-end an antinode is formed the fixed end being the position of the adjacent node, (b) represents the mode of vibration, producing the first overtone. Here in between the antinode at the free-end and the node at the clamped end, there are one more node and a second antinode. Similarly, in the next higher mode of vibration as in (c), there are in all three antinodes and three nodes. So the modes of vibration are seen to be similar to the vibration of air-columns in a closed organ pipe (*vide* Chapter VIII). In a closed organ pipe we know that while emitting the first overtone the position of the new node will be at a distance $l/3$ from the free-end and the frequency of this overtone will be thrice the frequency of the fundamental. Again the case of production of the second overtone the additional nodes in the organ pipe will be at distances $\frac{l}{5}$ and $\frac{3l}{5}$ from the free-end and the corresponding frequency will be five times the fundamental frequency. But in rods the positions of the nodes are not so regular and the *overtones are neither the harmonics of the fundamental tone.* In the case of production of the first overtone in a fixed free rod the first node is at $0{\cdot}2261l$ from the free-end and its frequency is 6·267 times the fundamental. While emitting the second overtone, the first and the second nodes are respectively at $0{\cdot}1321l$ and $0{\cdot}4999l$ from the free-end and the corresponding frequency is 17·55 times the fundamental.

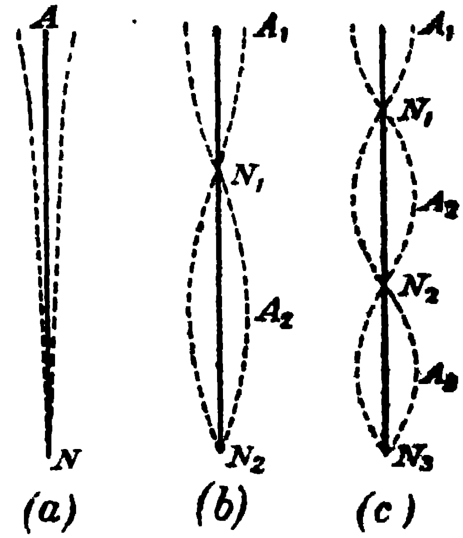


Fig. 54

In case of transverse vibration of bars it must be noted that if the bar is of a rectangular cross-section it can readily vibrate only in two mutually perpendicular planes while a bar of a circular cross-section is capable of vibration in any plane.

(b) Free-free bar.—Here the bar is to be so supported on rubber pads near the two ends that the positions of these pads form the nodes. For the fundamental mode of vibration [Fig. 55 (*a*)] these nodal positions are at a distance of $0\cdot2242l$ from either of the free-ends. For getting the first overtone the pads should be at $0\cdot1321l$ from the ends, the third node being midway between the two pads [Fig. 55(*b*)]. The frequency of the first overtone will be 2·756 times that of the fundamental and as such it is not a harmonic of the fundamental tone.

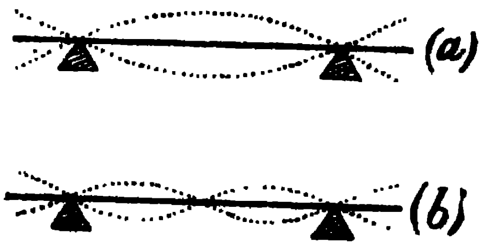


Fig. 55

(c) Fixed-fixed bar.—Here both the ends of the bar are rigidly clamped. When excited by a bow it begins to vibrate emitting exactly the same series of tones as is found to be produced in a free-free bar.

In all types of transverse vibration of bar the frequency is found to vary,

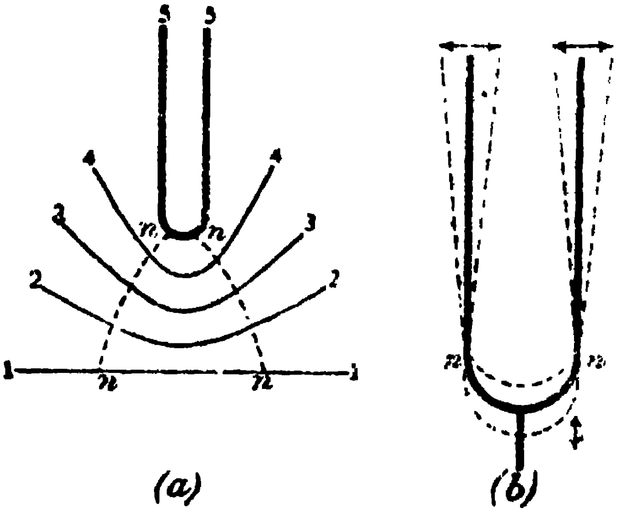
(*i*) inversely as the square of its length,

(*ii*) directly as its thickness, and

(*iii*) directly as the velocity of propagation of the longitudinal waves through it.

84. Tuning Fork :—The tuning fork is one of the most commonly used acoustical instrument. It was first constructed by König. It is so very useful in sound because of its constancy of frequency which may be maintained almost indefinitely. This frequency varies only slightly with amplitude of vibration and temperature. A tuning fork may be considered to be a dynamically balanced system wherein the prongs simultaneously move inwards and exactly after half a period again simultaneously outwards.

According to Chladni a tuning fork may be considered to be a free-free bar bent in the form of the letter U. Here the nodal points approach each other due to gradual bending as shown in Fig. 56(*a*). This



(a) (b)

Fig. 56

increases the length of the vibrating free-ends (or arms) of the system and hence the frequency is reduced but at the same time amplitude of vibration at the central loop diminishes. When an additional metal piece, called the stem, is attached to the middle of the central loop, the stiffness of the antinode is still increased and the amplitude decreases considerably. But the transverse vibration of this loop being along the length of the stem, the latter is set into a longitudinal vibration, that of the arms or prongs being still transverse in character [Fig. 56(*b*)]. This longitudinal vibration of the stem is in a position to set up vibrations in a resonance box against which the stem is pressed. This method of representation readily supplies us with the interpretation of the fact that the prongs alternately approach each other and recede from each other during a vibration—because the ends of a free-free bar either move upwards or downwards simultaneously. A tuning fork has been considered by Lord Rayleigh to be equivalent to two fixed free bars of identical size joined at the fixed end and embedded in a metal block. This view point also explains the longitudinal vibration of the stem of the fork inspite of the transverse vibration of its prongs. From Rayleigh's standpoint an expression of the frequency of a tuning fork can be obtained as

$$n=\frac{m^2 k}{2\pi l^2}\sqrt{\frac{Y}{\rho}} \qquad \dots \qquad \dots \qquad (1)$$

where $k=a\sqrt{12}$, a being the thickness of the prong, l=the length of the prong, $m=1{\cdot}875$, Y=Young's modulus and ρ=density of the material of the fork.

Putting Y and ρ for steel, n becomes $84590\,\frac{a}{l^2}$.

Thus from (1) it is evident that the frequency of a fork is proportional (*a*) directly to the thickness of the prong, (*b*) inversely as the square of the length of the prong and (*c*) directly as the velocity of propagation of the longitudinal wave through the material of the prong. Evidently n does not depend on the breadth of the prong.

As has been mentioned above the frequency of a fork is found to vary slightly with the temperature and the relation can be mathematically written as, $n_t = n(1 - \cdot 000114t)$,

where n_t = the frequency when the temperature is raised through t°C. from the original lower temperature at which the frequency was n. König experimentally found that the coefficient of decrease is rather ·000112 instead of 0·000114 as derived theoretically.

Tuning forks are used as standard of frequency in all acoustical measurements. When they are maintained by electrical circuits, they also serve as the standards of electrical frequency. Since the periods of vibration of tuning forks are very constant they are also used as accurate standards of time.

The reason behind the use of a tuning fork as a standard is that when properly designed and constructed, the note emitted by a fork is constant in pitch and remains so over a considerable time. Moreover the note is free from overtones provided it is not vigorously excited. It is indeed very difficult to excite overtone in a good tuning fork and even if they are excited they are very feeble and readily die out within a few seconds. The forks mounted on resonance boxes emit still purer tones because these boxes speak only under the exact equality of the frequency of the fork and the fundamental frequency of the box. So the intense sound produced will have the exact frequency of the fork because the slightest inequality (or mistuning) makes the response almost negligible.

85. Transverse Vibration of Plates :—The theoretical study of the vibration of plates is rather difficult but the experimental methods devised by Chladni for studying these vibrations are quite interesting. A square metal plate is clamped at the centre and sand particles are uniformly sprinkled on the plate. Now a resined bow is

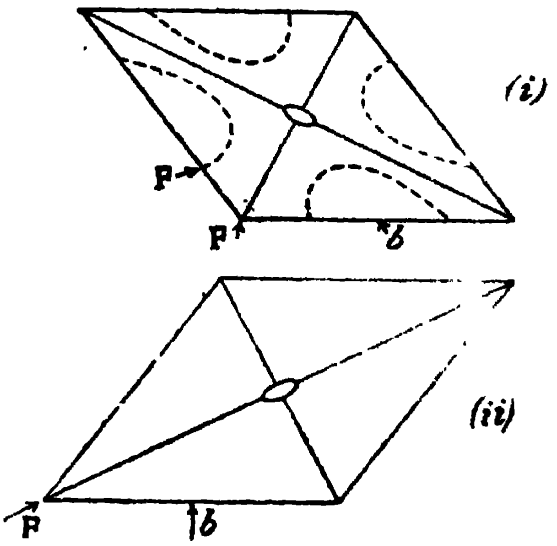


Fig. 57

pulled vertically and slowly across the edge. By placing the bow at different positions and touching the different points of the edge by fingers, *i.e.* damping the vibrations there, different patterns can be obtained wherein the sand particles are found to arrange along the nodal lines. The knowledge of the positions of these nodal lines makes the study of such vibrations comparatively easier. The point where the bow is applied indicates the antinodal position and the point or the points where fingers touch the plate modify the forms of the nodal lines. In Fig. 57(*i*) two fingers are applied at *FF* and the bow at *b* and the nodal lines are indicated by the discontinuous curves and diagonal lines in the figure along which the sand particles accumulate. In Fig. 57(*ii*), the finger is placed at one corner and the bow applied in the middle of an edge and the pattern forms two intersecting diagonals.

Chladni actually got a large number of such figures (known as *Chladni's figures*) by taking different shaped plates—circular, square, etc. In circular or elliptical plates, the nodal lines were either radial or concentric with the plates. If instead of sand, lycopodium or cork powder be sprinkled on the plates, the antinodal region will be covered up by them and none of these light particles would lie along nodal lines. But if the figures are produced in vacuum, even these lighter particles would collect along the nodal lines. Hence it seems that the apparently anomalous behaviour of these lighter particles are probably due to the vibrations set up in air.

86. Transverse Vibration of Membranes :—In plates the stiffness of the material is one of the major factors which controls its vibration. But in the case of a theoretical membrane we suppose it to be perfectly flexible. A membrane is a very thin lamina of negligible rigidity which is stretched outwards in all directions by a constant force. In a rectangular membrane, the frequency of the series of partials emitted by it when properly excited is given by $n=\sqrt{\frac{T}{4m}\left(\frac{p^2}{a^2}+\frac{q^2}{b^2}\right)}$ where a and b are lengths of the sides, T is the tension, m the mass per unit area and p and q are integers. Some of the overtones are harmonic but other lower tones, which lie very close together, produce a noise along with the prominent note.

In circular membranes the fundamental frequency is given by $n=\frac{0{\cdot}765}{2a}\sqrt{\frac{T}{m}}$, a being the radius of this circle. Here the overtones are not harmonics. The nodal lines are a series of concentric circles. The overtones may however be made harmonic by gradually decreasing the mass per unit area from the centre to the marginal parts as in an *Indian tabla*. Though the theoretical conditions of a membrane cannot be ordinarily fulfilled, membrane of paper, skin, etc. may be successfully applied to demonstrate the formation of nodal lines. Paper is often chosen as a membrane for the construction of loud speakers. A *kettle-drum* is an application of a skin as a membrane. Here the resonant air cavity gives it a definite fundamental tone. When it is struck about half way along a radius by a soft hammer fewest anharmonic overtones will be produced and the fundamental will be predominent with the least amount of noise attending the sound.

Indian Tabla.—In an Indian Tabla, stretched membrane is mounted on a hollow wooden cylinder closed at one end. The tension on the membrane can be varied with the help of the stretching strings and a number of small cylindrical wooden blocks. The central region of the membrane is loaded with a paste of gum and iron filings whose thickness gradually decreases from centre to

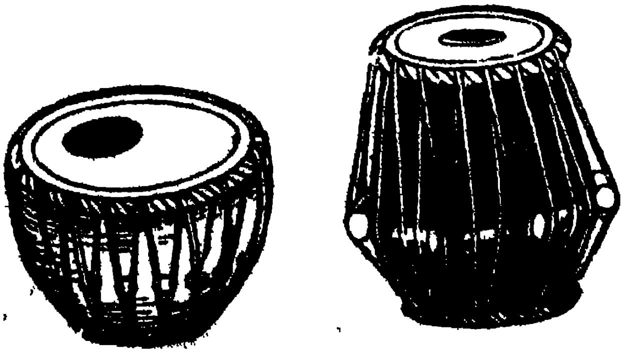


Fig. 57(a)

edge. This changes the otherwise anharmonic vibration of the membrane into a series of harmonic vibrations. This harmonic nature of the vibrations enables it to be used as an important musical instrument. The quality of tone here is superior to other types of drums. The mode of vibrations of the membrane of a tabla and the positions of the nodal lines during excitation can be demonstrated by sprinkling sand particles on the membrane. The exact character of the sound produced by a tabla and also its nature of vibration, however, depend to a large extent on the boundary conditions which are not rigidly fixed.

87. Vibration of Bells :— The bells may be of different shapes and forms. But all varieties are supported at their centre of symmetry and are excited by striking them near the edge. A bell may be regarded either as a development of a hollow cylinder closed at one end or as a bent circular plate. When excited the vibrations will to some extent resemble those of a circular plate. Here the nodal lines may be either (*a*) radial lines or nodal meridians extending from the points of support to points on the periphery thereby dividing the bell into even number of equal sections, or (*b*) nodal circle at various distances from the point of support. Sometimes both nodal meridians and circles are simultaneously present.

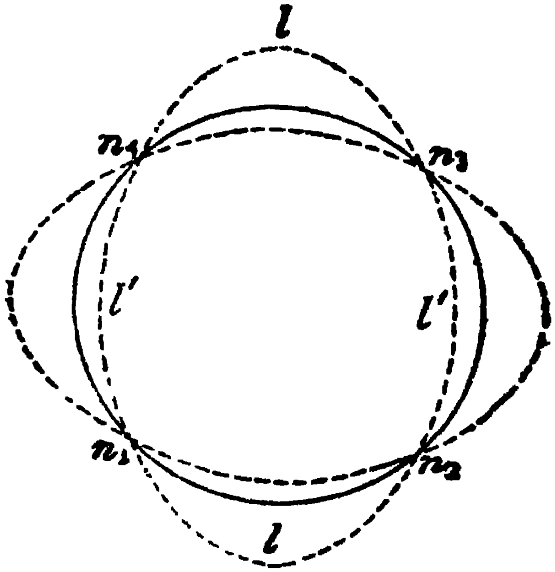


Fig. 58

The lowest tone is produced when there are only 4 nodal meridians and no nodal circle (Fig. 58). During vibrations when one segment between two adjacent nodal meridians n_1, n_2 move outwards to l, the adjacent segments move inwards to l' l' and the opposite segment also moves outwards as shown in Fig. 58. The adjacent segments are therefore in opposite phases during their vibrations. The second tone is emitted with the formation of 4 nodal meridians and 1 nodal circle and higher tones are still more complicated. The overtones are not harmonic in nature. The characteristic alternate waxing and waning of the ringing of a bell is due to the air-turbulence which causes a slow rotation of the nodal lines as the bell vibrates. So the nodes and antinodes are alternately presented to the observer which accounts for the above waxing and waning.

Examples

(1) *Two similar wires vibrate transversely in unison. When the tension in one is increased by 2·01 per cent. and the two wires vibrate simultaneously, three beats are produced per second. Find the original frequency of vibration of the two strings.* (C. U. 1957)

Ans. **If T be the original tension of the strings and m, the mass per unit length, the original frequency of the strings is**

$$n_0=\frac{1}{2l}\sqrt{\frac{T}{m}}.$$

When the tension of one string is increased by 2·01 per cent. the new tension becomes $\left(1+\frac{2{\cdot}01}{100}\right)T=T_1.$

Hence the new frequency of that string is $n=\frac{1}{2l}\sqrt{\frac{T_1}{m}}$

From the given condition $n=n_0+3$.

$$\therefore\quad \frac{n}{n_0}=\frac{n_0+3}{n_0}=\sqrt{\frac{T_1}{T}}=\sqrt{1+\frac{2{\cdot}01}{100}}\simeq 1+\frac{2{\cdot}01}{200}\ \text{(approx.)}.$$

$$\therefore\quad 1+\frac{3}{n_0}=1+\frac{2{\cdot}01}{200};$$

$$\text{or,}\quad n_0=\frac{600}{2{\cdot}01}=298{\cdot}5 \text{ vibrations per sec.}$$

(2) *When the vibrating length of a wire is adjusted to 80 cms. the note it emits when plucked is in tune with a standard fork. On adding a further weight of 100 gms. the vibrating length has to be altered by 1 cm. in order to restore the tuning. What is the initial weight on the wire?*

Ans. We know, $n=\frac{1}{2l}\sqrt{\frac{T}{m}}=\frac{1}{2l}\sqrt{\frac{Mg}{m}},$

where Mg is the original weight.

$$\therefore\quad n=\frac{1}{2\times80}\sqrt{\frac{Mg}{m}}.$$

Again adding the weight of 100 gms. and altering the length by 1 cm.

$$n=\frac{1}{2\times 81}\sqrt{\frac{(M+100)g}{m}}.$$

$$\therefore\quad \frac{1}{80}\sqrt{\frac{Mg}{m}}=\frac{1}{81}\sqrt{\frac{(M+100)g}{m}}.$$

$$\text{or,}\quad \frac{81}{80}=\left(\frac{M+100}{M}\right)^{\frac{1}{2}};$$

$$\text{or,}\quad \frac{81^2}{80^2}=1+\frac{100}{M};\quad \text{or,}\quad M=\frac{80^2\times 100}{81^2-80^2}=3975 \text{ gms. (approx.)}$$

$\therefore$ Initial weight on the wire was 3975 gms.-wt.

(3) *In a transverse arrangement (Melde's Expt.) the string vibrates in 3 loops when the tension is 200 gms. Calculate the tension required to make the string vibrate in two loops in the longitudinal arrangement.* *(C. U. 1956)*

Ans. When the string vibrates in 3 loops, $\frac{\lambda}{2}=\frac{l}{3}$.

$\therefore\quad \lambda=\frac{2}{3}l$. Now $n=\frac{1}{\lambda}\sqrt{\frac{T}{m}}$; or, $n=\frac{3}{2l}\sqrt{\frac{200g}{m}}$.

The frequency of vibration in two loops $=\frac{1}{l}\sqrt{\frac{T}{m}}$, ($\because$ Here $\lambda=l$).

As this vibration is longitudinal, this frequency is $\frac{n}{2}$.

$\therefore\quad \frac{1}{2}\cdot\frac{3}{2l}\sqrt{\frac{200g}{m}}=\frac{1}{l}\sqrt{\frac{T}{m}}$, or, squaring, $\frac{9}{16}\times 200g=T$. $\therefore$ $T=112\cdot5$ gms.-wt.

(4) *Three strings A, B, C of the same length are stretched on a sonometer. Their relative masses per unit length are 2 : 8 : 18 and their tensions are in the ratio 12 : 12 : 27. Find the ratio of the frequencies of notes emitted by A, B and C.*

Ans. As l is same for all the strings their $n\propto\sqrt{\frac{T}{m}}$

$$\therefore\quad n_a : n_b : n_c=\sqrt{\frac{T_1}{m_1}}:\sqrt{\frac{T_2}{m_2}}:\sqrt{\frac{T_3}{m_3}}=\sqrt{\frac{12}{2}}:\sqrt{\frac{12}{8}}:\sqrt{\frac{27}{18}}$$

$$=\sqrt{6}:\sqrt{\frac{3}{2}}:\sqrt{\frac{3}{2}}=2:1:1.$$

Questions (Chapter VII)

1. Prove that the frequency of vibration of a stretched string is equal to $\frac{1}{2l}\sqrt{T/m}$, where T is the tension, l, the length and m, the mass per unit length of

the string. Explain what harmonics will be present and what absent when the string is struck at the middle point. (C. U. 1932)

[Hints :—For the last part, Art. 114(*b*)]

2. Explain with diagrams the nature of the vibration of a tuning fork. For what special feature, it is a valuable instrument for study of sound. (C. U. 1917)

3. Explain the production of nodes and antinodes in the case of a transversely vibrating string.

4. Write a short note on Chladni's figures. (Luck. U. 1946)

5. State the laws which govern the vibration of string. One end of a stretched string is fastened to one of the prongs of a vibrating fork : compare the vibrations excited in the string when the prong is moving parallel to the string with those when the prong moves transversely. (C. U. 1943)

6. Deduce an expression for the velocity of transverse waves in a sonometer wire. What are their frequencies for the case when the wire is plucked at its middle point ? (*cf.* Utk. U. 1966 ; All. U. 1953 ; C. U. 1952, '67)

7. Prove that the velocity of a pulse along a stretched string varies directly as the square-root of the tension and inversely as the square-root of the linear density. (Del. U. 1946)

8. Enumerate the laws of vibrations of a stretched string. (C. U. 1965)

Wires of equal length of brass and steel are stretched on a sonometer and adjusted to emit some fundamental note. If the tensions in the two cases are 5 and 8 kgms.-wt. respectively and the diameter of the steel wire is 0·8 mm. find that of the brass wire ; the densities of brass and steel being 8·4 and 7·8 respectively.

[*Ans.* 0·887 mm.] (C. U. 1948)

9. Find an expression for the velocity of transverse waves along a stretched string and determine the frequency of its fundamental note.

A sonometer is arranged to emit a note of frequency n. How much the tension is to be varied to increase the frequency of the note to $\frac{5n}{8}$? If the tension of the wire is maintained constant in what other way could the same change in frequency be made. (Bom. U. 1954)

10. Describe Melde's experiment for determining the frequency of a tuning fork. Compare the vibrations excited in the string in the longitudinal arrangement with those in the transverse arrangement. (C. U. 1956, *cf.* '65)

11. Write notes on : (*a*) Vibration of tuning fork ; (*b*) Vibration of membranes and its application : (*c*) Vibration of bells.

12. Two wires vibrate transversely in unison. The tension of one of them is increased by 1 per cent and now, when they vibrate simultaneously, three beats are heard in 2 seconds. What was the original frequency of the two wires ?

[*Ans.* 300 c.p.s.] (Burd. U. 1963)

13. Experimentally it is observed that the velocity of a transverse wave along a stretched string depends on the tension of the string, its total mass and length.

Derive, from these observations the formula for the velocity of transverse waves by the dimensional method.

[HINTS : –

Let $V = KT^x M^y l^z$.

where T=tension, M=total mass of string and l=its length.

Dimensions of V is $[LT^{-1}]$, that of T is $[MLT^{-2}]$, that of M is $[M]$ and that of l is $[L]$.

$$\therefore \quad LT^{-1} = K.M^x L^x T^{-2x}.M^y L^z$$
$$= K.M^{x+y} L^{x+z} T^{-2x}.$$

$$\therefore \quad 2x=1 \; ; \text{ or, } x=\tfrac{1}{2} \; ; \quad x+z=1 \; ; \text{ or, } z=\tfrac{1}{2},$$
$$\text{and } x+y=0 \; ; \text{ or, } y=-\tfrac{1}{2}.$$

$$\therefore \quad V = K\sqrt{\frac{Tl}{M}} = K\sqrt{\frac{T}{M/l}} = K\sqrt{\frac{T}{m}}.$$

Experimentally, $K=1$.

$$\therefore \quad V = \sqrt{\frac{T}{m}}. \;]$$

14. Find the fundamental frequency of transverse vibration of a steel wire. 1 metre long and 1 mm. in diameter hanging vertically from rigid support with a load of 2 kgms. attached to its lower end. (Density of steel = 7·9 gms./c.c.).

[*Ans.* 88·9 per sec. nearly.] (Utkal U.)

15. Two parts of a sonometer wire divided by a movable knife-edge differ in length by 2 mms. and produce 1 beat per sec. when sounded together. Find their frequencies if the length of the whole wire is 1 metre.

[*Ans.* 250·5 and 249·5]

16. A piano wire 50 cms. long and of mass 6·5 gms. is stretched so that the frequency of its fundamental mode of vibration is 80 cycles/sec. Find the stretching force in gms.-wt. (C.U. 1964)

[*Ans.* 8·49 × 10^3 gms.-wt.]

CHAPTER VIII

VIBRATIONS OF AIR-COLUMNS : LONGITUDINAL VIBRATIONS OF RODS (DUST-TUBE EXPERIMENT) : MAINTAINED VIBRATIONS

88. Stationary Vibration of the Air-Column within an Organ Pipe :—The column of air enclosed in a pipe can be set into momentary vibration when any sudden disturbance is communicated to it, or the pressure at the mouth of the pipe is suddenly altered. For example, a sound is produced by suddenly withdrawing a cork from a tightly-corked cylindrical bottle, because the sudden withdrawal of the cork disturbs the air-pressure at the mouth of the bottle which is the cause of the vibrations of air in the bottle. The whistling sound produced by blowing across the open end of the barrel of a key is also another example of vibration of air-column. In various musical instruments such as the flute, clarionet, etc. the musical sound is produced and maintained by vibrating the air-column enclosed within the pipe. ***Air-column in a pipe, closed or open, vibrates longitudinally when disturbed at the mouth.***

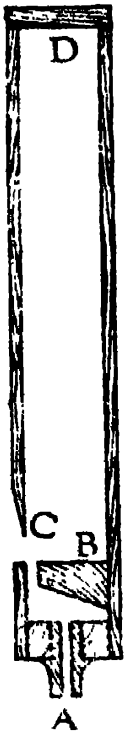


Fig. 59—A Closed Organ Pipe

An organ pipe is the simplest form of a wind instrument. Fig. 59 shows a longitudinal section of an organ pipe. It consists of a hollow tube BD in which air can be blown through a pipe A. The air issues through a narrow slit B, and strikes against the sharp edge C, called the *lip*, of the mouthpiece. This sets up vibration in the air-column enclosed in the pipe. When the blast is directed into the pipe, it produces compression, and, when directed outwards it can, by suction produce rarefaction at the lower end of the air-column. Sondhaus in 1853 showed that whenever an wedge of small angle (*i.e.* edge) is placed in front of a slit and set parallel to it, then on blowing air through the slit certain strong tones of stable frequencies will be produced. They are known as *edge tones*. These stable tones are responsible for the musical applications of an organ pipe. **An organ pipe is called closed or open according as it is closed at one end or open at both ends.**

***Edge Tone** :—Edge tones are produced when a puff of air strikes a sharp edge or an wedge of metal or wood. These tones are produced by the vortices (See Art. 195 General Physics) generated in between the edge and the slit through which air puff is injected. On both sides of the edge the eddies of different types will be formed—anti-clockwise on one side and clockwise on the other—the distance between the centres of both the types being the same. An eddy of one system is formed midway between two eddies of the other system. When an air puff strikes against an edge, the series of vortices would move forward and disappear at a distance. Fresh vortices then appear at the rear and they continue to move forward in the same way. When an anti-clockwise vortex passes by an edge, it sucks in air and produces a compression pulse on the same side. Similarly when a clockwise vortex arrives at an edge it produces rarefaction pulse on the same side. As the train of eddies proceed on in the forward direction, alternate compressions and rarefactions are produced in the medium at regular intervals giving rise to the musical tones known as *edge tones*.

(a) **Closed Organ Pipe** —As air is blown through the pipe (Fig. 59), it strikes the edge, and a slight upward deviation of the air blast produces a compressed wave which travels to the closed end (which is a rigid wall), and so the air near the end is compressed to a pressure greater than the atmospheric pressure. This compressed air forces back the air behind it in order to return to atmospheric pressure, and in so doing it starts a compressed wave which returns along the pipe. Thus a compressed wave is reflected from the closed end as a compressed wave, and returns to the mouth. But the mouth being open, and the air free to expand, the pressure of the compressed wave pushes the sheet of air outside and so the layers of air relieve themselves from a strained state and as a result there is reversal of the type of the wave and so a wave of rarefaction starts inside. The wave of rarefaction again comes back to the mouth as a rarefied wave after being reflected at the closed end. This is again reflected as a compressed wave at the mouth, which is a free end, and is also intensified by the compressed wave directed inwards by the blast of the air outside. This being the way of the vibrations, out of the various frequencies set up by the impact of the air blast with the lip C of the pipe, the air-column inside the pipe takes up only those with which it can resound, and pulses pass up and down the length of the pipe, the result being the production of a musical note and the pipe is found to *speak*.

The result of the reflected pulse meeting with the direct one *is a stationary longitudinal wave set up inside the pipe, the nodes and the antinodes occurring at definite places. The air at the open end is free to move inwards or outwards with the maximum freedom and, therefore, is a seat of* **antinodes.** *The closed end being a rigid wall, the air in contact with it has the least freedom of movement and so the closed end is always a* **node.**

(b) Open Organ Pipe.—In an open pipe, when a compressed wave reaches the far end, the air at that point is for an instant at a pressure greater than ordinary atmospheric pressure, and the mouth of the tube being open, the air there can vibrate with the utmost freedom and so suddenly expands, into the surrounding air. Thus the pressure diminishes so quickly that it falls somewhat below the pressure of the surrounding air, which causes a sudden rarefaction at the end of the pipe. This sets up a rarefied wave which passes back along the pipe. This rarefied wave is reflected back as a wave of compression at the other free end. Within the tube, the reflected pulses meet with the direct ones blasted into mouth from outside and the result is the formation of a stationary longitudinal wave having nodes and antinodes at definite intervals. Both the open ends of the tube are seats of antinodes, the air there being free to move either inwards or outwards. *For the fundamental tone emitted by the tube, there* should be one node between those two antinodes.

89. Fundamentals of a Closed and of an Open Organ Pipe of the Same Length :—

Closed Pipe.—In the simplest mode of vibration in the case of a *closed organ pipe*, there is a node at the closed end and an antinode at the open end [Fig. 60(*a*)]. *In a stationary wave the distance between two consecutive nodes, or two consecutive antinodes, is equal* *to one-half the wavelength* : so in this case the length of the tubes is one-fourth of the wavelength, *i.e.* the wavelength is *four times* the length of the tube. Under this condition, the closed pipe produces its fundamental tone.

Let n_1 and λ_1 represent frequency and wavelength of the fundamental tone given by a closed organ pipe of length l. Hence $\lambda_1 = 4l$, and $V = n_1\lambda$, where V is the velocity of sound.

$$\therefore \quad n_1 = \frac{V}{\lambda_1} = \frac{V}{4l}.$$

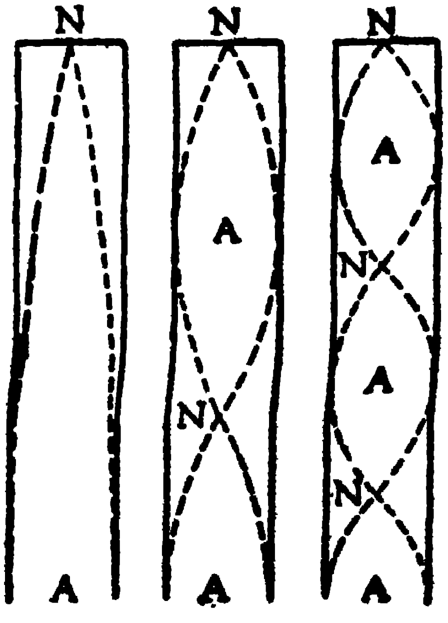
Open Pipe.—In the case of the fundamental of an *open pipe*, i.e. a pipe open at both ends, there is an antinode at each end of the pipe with a node in the middle [Fig. 61(a)]. So the length of the pipe is half the wavelength. If n' and λ' be the frequency and wavelength of the fundamental tone for the open pipe, we have $\lambda' = 2l$. Again, $V = n'\lambda$.

$$\therefore \quad n' = \frac{V}{\lambda'} = \frac{V}{2l} = 2n_1.$$

Hence, *the pitch of the fundamental of an open organ pipe is twice, i.e. one octave higher than that of a closed organ pipe of the same length.*

N.B.—If an open pipe, while giving out a note, is suddenly closed, *the pitch of the note at once decreases* and the sound emitted becomes less sharp. If an organ pipe is closed at one end by a movable shutter, the pitch of the note emitted by the pipe is found to rise on slowly opening the shutter and to fall as *the shutter is gradually closed.*

(a) Overtones (or Harmonics) of Organ Pipes.—Production of harmonics depends to some extent on the nature of excitation of the tube. If the air is blown more and more powerfully, the nature of the stationary waves remains the same



Frequency

n_1 (a) $3n_1$ (b) $5n_1$ (c)

Fig. 60—Closed Pipe

no doubt but the number of nodes and antinodes is increased, *i.e.* higher and higher harmonics are also produced.

(i) **Closed Pipe.**—In the case of a closed pipe, the closed end is always a node and the open end always an antinode [Fig. 60(*a*)]. The next possible node of vibration, after the fundamental is to have *one* intermediate node and *one* antinode [Fig. 60(*b*)], *i.e.* the length of the pipe l is three-fourth of the wavelength λ_2 ; so in this case, $\lambda_2 = \frac{4}{3}l$.

If n_2 be the frequency of the node, $n_2 = 3V/4l$. Hence $n_2 = 3n_1$, where n_1 is the frequency of the fundamental.

For the *next* higher overtone, there will be *two* intermediate nodes and *two* intermediate antinodes alternately placed [Fig. 60(*c*)].

In this case, $\lambda_3 = \frac{4}{5}l$; and the corresponding frequency, $n_3 = 5V/4l$. Hence, $n_3 = 5n_1$, and so on. In the case of a closed pipe, therefore, ***only harmonics proportional to the odd natural numbers are present and this makes the quality of the note given out by a closed pipe lacking in fullness.***

Harmonics of a Closed Pipe

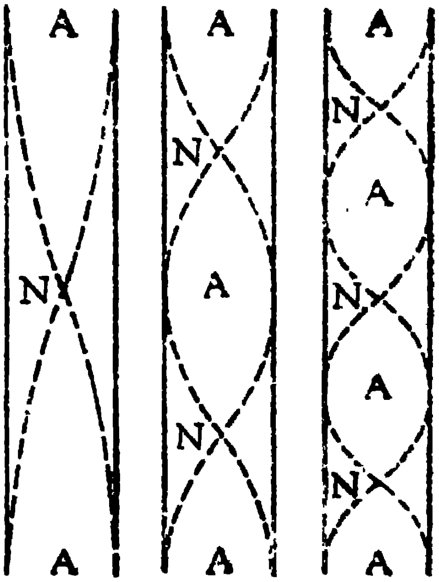
No.	Wavelength in air	Frequency of the note	Relation with the Fundamental
1	$4l$	$n_1 = \frac{V}{4l}$	fundamental
2	$\frac{4l}{3}$	$n_2 = \frac{3V}{4l}$	$n_2 = 3n_1$
3	$\frac{4}{5}l$	$n_3 = \frac{5V}{4l}$	$n_3 = 5n_1$
&c.	&c.	&c.	&c.

Therefore in a closed pipe the possible frequencies of vibration are the ratio 1 : 3 : 5, etc.

(ii) **Open Pipe.**—We have already seen that in the case of the fundamental of an open pipe, there is an antinode at each end, and a node in the middle [Fig. 61(a)]. If n' be the frequency of the fundamental, $n' = V/2l$.

For the *next* overtone, there will be *two* intermediate nodes and *one* intermediate antinode between them [Fig. 61(b)]. In this case $\lambda'' = 2l/2 =$ the length of the pipe, and frequency, $n'' = V/l = 2n'$, *i.e.* it stands an octave higher than the fundamental.

In the *next overtone*, there will be *three* intermediate nodes and *two* intermediate antinodes [Fig. 61(c)]. In this case $\lambda''' = 2l/3$, and the frequency $n''' = 3V/2l = 3n'$ and so on.



n' $2n'$ $3n'$

(a) (b) (c)

Fig. 61 – Open Pipe

Hence, in the case of an open pipe, both *odd* and *even* harmonics are present.

Harmonics of an Open Pipe

No.	Wavelength in air	Frequency of the note	Relation with the fundamental
1	$2l$	$n' = \frac{V}{2l}$	Fundamental
2	l	$n'' = \frac{V}{l}$	$n'' = 2n'$
3	$\frac{2l}{3}$	$n''' = \frac{3V}{2l}$	$n''' = 3n'$
&c.	&c.	&c.	&c.

Therefore in an open pipe the frequencies of the fundamental and overtones are in the ratio of 1 : 2 : 3 : 4, etc.

It should be noted that the *note given out by an open pipe* is an *octave higher* than given out by a closed pipe of the *same length*, and that, owing to the presence of all harmonics proportional to the natural numbers in an open pipe, *the quality of the note given out by an open pipe is richer in harmonics and sweeter than that given out by a closed pipe*.

***89. (a) Mathematical Analysis of the Vibration of the Air-column within a closed organ pipe :—**

Let us consider the closed organ pipe to be so placed that its axis lies along the X-axis. The imperfection in reflection from a rigid wall is here supposed to be very small. Let a plane wave moving in the positive direction of X-axis is partially reflected from the rigid wall of the closed tube in the $Y-Z$ plane at a distance d from the origin. Let m be the fraction reflected so that for perfect reflection $m=1$. As a result of reflection at the rigid wall, the reflected wave will superpose on the incident wave and stationary vibrations will be set up in the organ pipe. There will be a *pressure antinode* and a *displacement* (and *velocity*) **node** at the reflecting surface. The displacements in the incident and the reflected waves may be written as,

$$\left.\begin{aligned} y_1 &= a \sin \frac{2\pi}{\lambda}(x-ct-d) \quad \text{......(incident)} \ldots \\ \text{and } y_2 &= am \sin \frac{2\pi}{\lambda}(x+ct-d) \quad \text{......(reflected)} \ldots \end{aligned}\right\} \quad (1)$$

Hence the resultant displacement is

$$y = y_1 + y_2 = a(1+m) \sin \frac{2\pi}{\lambda}(x-d) \cos \frac{2\pi}{\lambda} ct$$

$$- a(1-m) \cos \frac{2\pi}{\lambda}(x-d) \sin \frac{2\pi}{\lambda} ct. \quad \ldots \quad (2)$$

In the case of perfect reflection the second term in (2) vanishes ($\because$ $m=1$) and we have,

$$y = 2a \sin \frac{2\pi}{\lambda}(x-d) \cos \frac{2\pi}{\lambda} ct. \quad \ldots \quad \ldots \quad (3)$$

This represents the equation of a stationary wave. [See Art. 37(*b*)]. Hence the particle velocity,

$$\dot{y}=\frac{dy}{dt}=-\frac{4\pi ac}{\lambda}\sin\frac{2\pi}{\lambda}(x-d)\sin\frac{2\pi}{\lambda}ct \qquad \ldots \quad (4)$$

From (3) and (4), it is evident that at any instant t, y and $\dot{y}$ are minimum when $\sin\frac{2\pi}{\lambda}(x-d)$ is zero ;

i.e. when $\frac{2\pi}{\lambda}(x-d)=\pi s$, where s is zero or an integer.

But as $d>x$ except at the rigid wall where $d=x$, $(x-d)$ is negative Hence the condition for minimum y and $\dot{y}$ is $\frac{2\pi}{\lambda}(x-d)=-\pi s$;

$$\text{or,}\quad x-d=-\frac{\lambda s}{2}.$$

$$\therefore\quad x=d-\frac{\lambda s}{2},$$

when $s=0$, $x=d$; when $s=1$, $x=d-\frac{\lambda}{2}$; when $s=2$, $x=d-\lambda$; when $s=3$, $x=d-\frac{3\lambda}{2}$ and so on. So at the position of the rigid wall as also at distances of each $\frac{\lambda}{2}$ from it towards the open end, there will be positions of minimum displacement and velocity. These are positions of ***displacement*** and ***velocity nodes*** which are at distances of $\frac{\lambda}{2}$ from each other.

Again we know (from Art. 38, Note) that the pressure variation Δp is given by—$E\frac{dy}{dx}$.

$\therefore$ From (3), the pressure variation in the **stationary wave** is given by,

$$\Delta p=-E.\frac{4\pi a}{\lambda}\cos\frac{2\pi}{\lambda}(x-d)\cos\frac{2\pi}{\lambda}ct \qquad \ldots \quad (5)$$

Since $\cos\frac{2\pi}{\lambda}(x-d)$ is maximum, when $\sin\frac{2\pi}{\lambda}(x-d)$ is **minimum** at any instant, we have,

Δp is maximum when $\frac{2\pi}{\lambda}(x-d)=-\pi s$.

Hence at the positions of *nodes*, the pressure variation is maximum. So the *nodes* are really the positions of ***pressure antinodes***.

Again y and $\dot{y}$ are maximum at any instant when $\sin\frac{2\pi}{\lambda}(x-d)=1$,

$$\text{or, when } \frac{2\pi}{\lambda}(x-d)=-\frac{\lambda(2s+1)}{2} \quad (\because \; d>x)$$

or, when $x=d-\frac{\lambda(2s+1)}{4}$, where s is zero or an integer. If $s=0$, 1, 2, 3 etc.

$$x \text{ becomes } d-\frac{\lambda}{4}, d-\frac{3\lambda}{4}, d-\frac{5\lambda}{4}, d-\frac{7\lambda}{4} \text{ etc.}$$

These are the positions of ***displacement and velocity* antinodes** as measured from the rigid wall. As $\cos\frac{2\pi}{\lambda}(x-d)$ has zero value at these points, they evidently indicate the positions of ***pressure nodes*** where pressure variation is minimum. The distance between successive antinodes is also $\frac{\lambda}{2}$.

When $s=0$, y is min. at $x=d$ *i.e.* at the closed end there is a node. Again when $s=0$, y is max. at $x=d-\frac{\lambda}{4}$, which is the open end of pipe for the fundamental mode of vibration.

Hence the length of the tube is $l=\frac{\lambda}{4}$. For the next mode of vibration, the node will appear at the rigid wall as usual and an antinode at the open end. But in between these there must be another pair of node and antinode. So the length of the tube l is equal to that of the second antinode from the fixed wall, *i.e.* $l=\frac{3\lambda}{4}$. In the next possible mode $l=\frac{5\lambda}{4}$ and so on.

Since reflection at the rigid wall is not quite perfect, the second term in equation (2) gives rise to another weak set of stationary vibrations and there is superposition of these two sets of stationary waves with a phase difference of $\pi/2$. As m is nearly unity, the second set creates weak antinodes at the position of the nodes of the first set and vice-versa.

The maximum velocity at the antinodes for these two sets of stationary waves are in the ratio,

$$\frac{v_1}{v_2}=\frac{1+m}{1-m}.$$

(b) Mathematical Analysis of the Vibration of the Air-column within an open pipe :—

In the case of an open organ pipe, imperfect reflection occurs at the gas-gas interface. Let this reflecting interface be at a distance d from the source. The reflecting interface in the present case is a *displacement* and *velocity antinode* and a *pressure node.*

Hence the displacement equations for the incident and reflected waves will be,

$$\left.\begin{array}{ll} y_1=a\cos\frac{2\pi}{\lambda}(x-ct-d) & \ldots\ \text{(incident)} \\ \text{and } y_2=am\sin\frac{2\pi}{\lambda}(x+ct-d) & \cdots\ \text{(reflected)} \end{array}\right\}\quad(6)$$

$\therefore$ The resultant displacement is

$$y=y_1+y_2=a(1+m)\cos\frac{2\pi}{\lambda}(x-d)\cos\frac{2\pi}{\lambda}ct$$

$$+a(1-m)\sin\frac{2\pi}{\lambda}(x-d)\sin\frac{2\pi}{\lambda}ct \quad \cdots \quad (7)$$

In the case of perfect reflection, the second term reduces to zero and we have,

$$y=2a\cos\frac{2\pi}{\lambda}(x-d)\cos\frac{2\pi}{\lambda}ct \quad \ldots \quad \ldots \quad (8)$$

The expression for y and Δp can be deduced in the same way as has been followed above.

By a similar analysis it can be shown that at $x=d,\ d-\frac{\lambda}{2},\ d-\lambda,$ etc. *displacement* and *velocity antinodes* (or pressure nodes) appear and at $x=d-\frac{\lambda}{4}, d-\frac{3\lambda}{4},$ etc. *displacement and velocity nodes* occur.

Here since at the open ends two antinodes appear, in the fundamental mode of vibration they must have a node in between. Hence the length of the tube should be equal to the distance between two consecutive antinodes i.e. $l=\frac{\lambda}{2}$.

In this case also a partial reflection will give rise to a superposition effect as discussed in the case of a closed organ pipe.

(c) Energy in Organ Pipes :—

The energy of the stationary waves between two successive nodes or antinodes can be calculated either from eqn. (2) or (7) above. Using (7), the kinetic energy is given by,

$$T=\tfrac{1}{2}\rho\int_{d-\frac{\lambda}{2}}^{d}\dot{y}^2.\,dx \text{ (where } \rho=\text{density of the medium)}$$

$$=\frac{2\pi^2a^2c^2\rho}{\lambda^2}\int_{d-\frac{\lambda}{2}}^{d}\left\{-(1+m)\cos\frac{2\pi}{\lambda}(x-d)\sin\frac{2\pi}{\lambda}ct+\right.$$

$$\left.(1-m)\sin\frac{2\pi}{\lambda}(x-d)\cos\frac{2\pi}{\lambda}ct\right\}^2.dx$$

Remembering that the product term vanishes on integration and that $\nu=\frac{c}{\lambda}$, we have by integration,

$$T=\tfrac{1}{2}\pi^2\nu^2a^2\rho\lambda\{(1+m)^2\sin^2 2\pi\nu t+(1-m)^2\cos^2 2\pi\nu t.$$

The potential energy for the same volume is $V=\frac{1}{2}\rho\omega^2\int_{d-\frac{\lambda}{2}}^{d}y^2.dx.$

$$\therefore \quad V = 2\pi^2\nu^2a^2\rho \int_{d-\frac{\lambda}{2}}^{d} \Big\{ (1+m) \cos \frac{2\pi}{\lambda}(x-d) \cos 2\pi\nu t +$$

$$(1-m) \sin \frac{2\pi}{\lambda}(x-d) \sin 2\pi\nu t \Big\}^2 . dx.$$

On intergation,

$$V = \tfrac{1}{2}\pi^2\nu^2a^2\rho\lambda\{(1+m)^2 \cos^2 2\pi\nu t + (1-m)^2 \sin^2 2\pi\nu t\}$$

$$\therefore \quad \text{Total Energy} = E = T + V = \pi^2\nu^2a^2\rho\lambda(1+m^2) \qquad \ldots \quad (9)$$

This is the energy of an open pipe vibrating in its fundamental mode where two successive antinodes occur. In a closed pipe vibrating in its fundamental mode, the total energy will be only half of that given in (9), because its length is only $\frac{\lambda}{4}$.

90. Effects of Temperature and Moisture on the Pitch of an Organ Pipe :—The pitch of any sound (which depends upon the frequency of vibration) is given by the relation, $V = n\lambda$; so anything which changes the velocity V will also change the frequency or wavelength or both. In an organ pipe the length which determines the wavelength does not change appreciably with changes of temperature. The velocity increases with temperature and so it follows that a rise of temperature of an organ pipe increases the frequency and so the pitch of the note emitted by it. The presence of moisture diminishes the density of air in the pipe and so it increases the velocity and consequently the pitch of the note emitted also rises.

91. Position of Nodes and Antinodes in an Open Organ Pipe :—The position of the nodes and antinodes in an open organ pipe can be demonstrated by the following experiments :—

Experiments.—(1) An open organ pipe, constructed with one of its sides of glass, is taken (Fig. 62). A small ring covered with a piece of stretched thin paper, and suspended by strings, like a scale pan, is covered with some dry sand granules. This is gradually lowered into the pipe, while the pipe is gently blown to give out its fundamental tone. It will be seen that the particles remain undisturbed in the middle position of the pipe indicating a node, *i.e.* a place of minimum

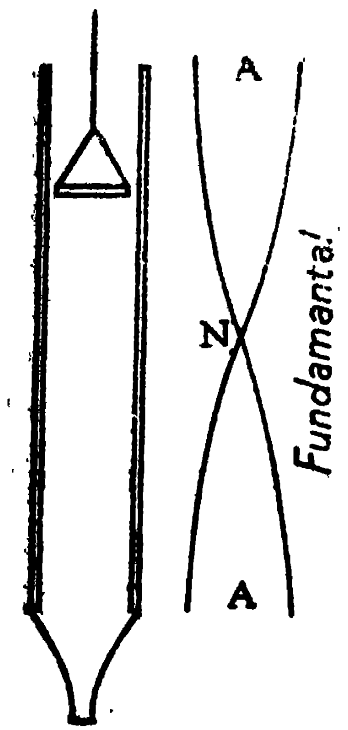


Fig. 62

agitation of air-particles, and that at the top and bottom of the pipe, the sand particles dance vigorously, indicating the positions of antinodes, *i.e.* places of maximum agitation of air-particles.

Blowing the pipe more strongly in order to have other overtones and noticing the dancing of the sand granules, other positions of nodes and antinodes can be discovered.

N.B.—It should be noted that an antinode will occur whenever there is free communication between the inner and the outer air. Hence, *by opening a hole* in the wall of an open pipe, an antinode is created there in addition to the two antinodes at the two ends of the pipe, and the column of air will vibrate by satisfying the conditions already stated. Thus the note emitted by the pipe is at once changed. From this, the reason of having different notes from ordinary bamboo or tin flutes, or from instruments such as clarionets, piccolos, etc. by opening and closing holes in the tube of the instrument, is clear.

(2) Manometric Flame Method.—Another method of studying the variations in pressure at the nodes and the antinodes in an organ pipe was devised by König in Germany and is known as the monometric flame (or capsule) method.

A circular aperture is made at any desired point in the

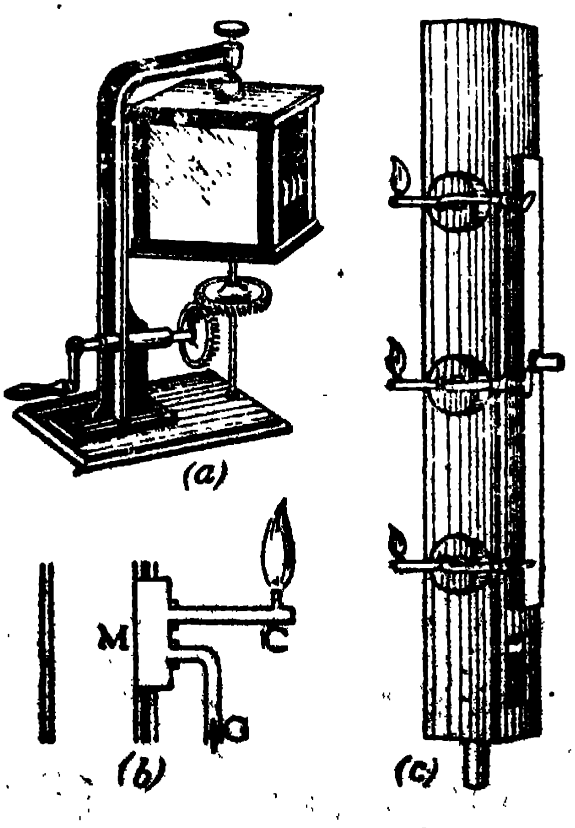


Fig. 63—Manometric Flame.

wall of an organ pipe and is then covered with a stretched diaphragm of the rubber. A piece of metal *M* in the form of a capsule is fitted on the aperture so that the membrane constitutes one side of a small chamber fitted with two narrow pipes through one of which *G* [Fig. 63(*b*)] coal gas is led which escapes through the other pipe *C* terminating in a pinhole jet where the gas is burnt. Any vibration of the air inside the pipe throws the membrane *M* in contact with it into a similar state of vibration and which again causes a corresponding variation in the pressure of the gas in the chamber and thus a corresponding change takes place in the length of the flame. If the change in pressure be periodic, the length of the flame also varies periodically. But the change in pressure being very rapid, the alterations in the length of the flame cannot be followed by the eye due to persistence of vision. To render them distinct, the light is received on a cubical box having plane mirrors on its four sides [Fig. 63(*a*)] which may be rotated rapidly about a vertical axis in front of the flame, and the successive steps of the flame are seen by looking at the reflection of the flame in the rotating mirror. When the flame [Fig. 63(*c*)] burns steadily, a continuous band of light will appear on the rotating mirror. So when the manometric flame is at an *antinode*, where there is no variation of pressure of the vibrating air-column, the membrane will not be agitated and so the flame is quite steady, and a long band of light will appear on the mirror. When, however, the flame is at a *node*, where there is the maximum change of pressure, the flame jumps up and down with a frequency equal to that of the membrane and the reflection in the rotating mirror presents a broken-up toothed appearance.

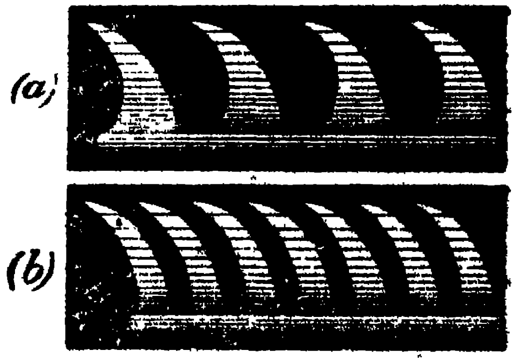


Fig 64

Fig. 64 represents the appearance of the flame in the revolving mirror produced by different tones. Fig. 64(*a*) represents that due to an organ pipe blown gently, and Fig. 64(*b*) that due to the pipe blown hard having double the frequency.

Comparison.—The manometric flame method is applied in *comparing the frequency of two organ pipes.* When a capsule is applied at the node in each pipe and the corresponding flames are examined side by side, it will be found that *n* teeth in one image will occupy the same length as *n′* teeth in other. So the frequencies of the two pipes are evidently in the ratio *n* : *n′*.

92. Determination of the Velocity of Sound by the Resonance of an Air-Column :—A vibrating tuning fork *F* is held close to the top of a glass tube which is vertically placed in a long cylinder almost full of water (Fig. 65). On gradually raising or lowering the tube a particular length of air column in the tube will be found when the sound will be strongly reinforced. Thus it is an arrangement for a closed pipe of adjustable length. Adjust the position of the tube when the intensity of the sound becomes maximum. In that position the frequency of vibration of the air-column agrees with that of the fork, and the fork and the air-column in the tube are then said to be in resonance. It should be noted that the pitch of the sound heard is independent of the diameter of the tube and its material, glass or metal. The action may be explained as follows : —

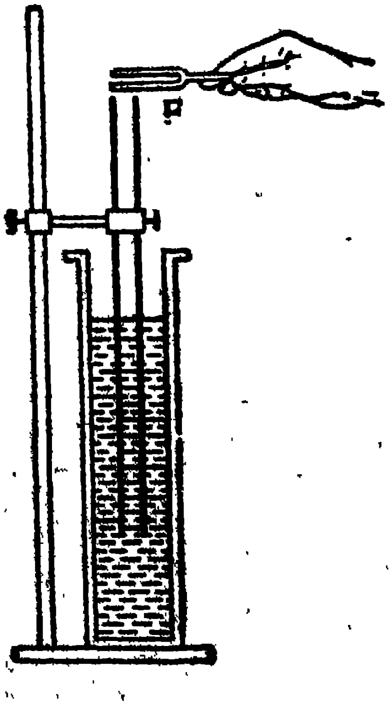


Fig. 65

Each movement of a prong of the fork towards the mouth of the tube compresses the air in front of it, and thus sends a compressed wave down the tube. The compressed wave, on reaching the surface of water, which is a denser medium, is reflected back as a *compressed* wave. The reflected compressed wave on reaching the open end of the tube is relieved from the strained condition by moving sideways and it is again reflected but this time, as a *rarefied wave* which starts down the tube. Now, if the prong reaches the extreme downward position at the same instant and begins to move upwards, a wave of rarefaction will proceed downwards into the tube. The reflected rarefied wave will thus *coincide* with the rarefied wave started down the tube due to the backward motion of the fork and so will be *reinforced*. Again the reinforced waves will be reflected back from the closed end (water surface) as rarefied waves, which will reach the open end just when the prong begins to move down. So the waves of compression formed by the reflection of the rarefied wave at the open end is helped by the fresh compressed wave sent by the prong. This shows that the fork and the air-column of the tube agree in motion (i.e. their time-periods are the same), and so *resonance* is produced.

From the above it is evident that when resonance is produced, the wave travels over *twice* the length of the air-column in the time taken by the prong to make half a vibration. Thus in a complete vibration of the prong, the wave travels over *four times* the length (l_1) of the air-column AN (Fig. 66). We have, therefore, $l_1 = \lambda/4$, or $\lambda = 4l_1$, where λ is the wavelength, and l_1, the length of the air-column. But, if V be the velocity of the sound, and n the frequency of vibration of the fork we have, $V = n\lambda$; or, $V = 4l_1 n$.

In fact, the antinode is not exactly at the mouth of the tube but is a little outside the tube, the distance depending on the diameter of the tube. Lord Rayleigh has shown that this correction is $0{\cdot}6r$, and thus the effective length of the vibrating air-column is $l_1 + 0{\cdot}6r$, where r is the radius of the tube, and $0{\cdot}6r$ is called the *end-correction*.

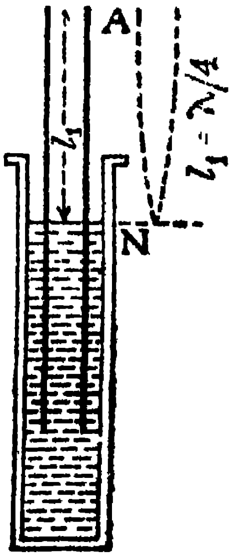


Fig. 66

Hence,

$$V = 4n(l_1 + 0{\cdot}6r).$$

Thus, from the resonant air-column, the velocity of sound can be determined by knowing the frequency of the fork.

Temperature Correction.—If the temperature of air in the tube is $t°C$, the velocity of sound at $0°C$, can be found from the relation,

$$V_t = V_0\sqrt{1 + \frac{t}{273}}\ ; \quad V_t = V_0\sqrt{\frac{T}{273}},$$ where T is the temperature in the absolute scale corresponding to $t°C$.

N.B.—In order to obtain the velocity of sound in dry air the result corrected for temperature ***should also be corrected for moisture*** contained in the air (*vide* Art. 58(3)].

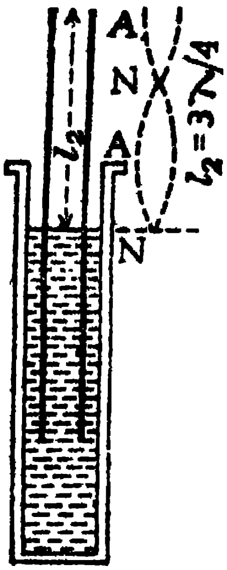


Fig. 67

The end-correction can be avoided in the following way.—In the first position of resonance, l_1 (Fig. 66) $=\lambda/4$, but if the tube be sufficiently long, then by raising the tube further out of water a second position of resonance, of weaker intensity, may be obtained where the length of the resonance air-column l_2 (Fig. 67) $=3\lambda/4$.

Since, in the first case, $\frac{\lambda}{4}=l_1+0{\cdot}6r$, and in the second case, $\frac{3\lambda}{4}=l_2+0{\cdot}6r$, we have,

$$\frac{3\lambda}{4}-\frac{\lambda}{4}=\frac{\lambda}{2}=l_2-l_1. \quad \therefore \quad V=n\lambda=2n(l_2-l_1).$$

By this means the wavelength can be determined eliminating the end-correction, whatever may be its exact numerical value.

Evaluation of End-Correction—By the method mentioned above the end-correction, may also be evaluated. Let this correction be x. For the first position of resonance, $\frac{\lambda}{4}=l_1+x$, and for the second position of resonance, $3\lambda/4=l_2+x$.

Hence $3(l_1+x)=l_2+x$; or, $x=\frac{1}{2}(l_2-3l_1)$. Hence x is determined.

The *velocity of sound in a liquid* was determined by means of organ pipes immersed in the liquid by Wertheim. The tube was made to speak by injecting a stream of the liquid in the pipe. The frequency of the note emitted was determined by one of the standard methods and the wavelength from the length of the tube. Hence the velocity was calculated. The value in water was not in agreement with the standard data. But introducing a correction for the variations of the sectional area due to the passage of compressions and rarefaction, the result agreed fairly well with the standard values.

93. Stationary vibrations in a string and in an air-column compared :—The stationary vibrations in a string are transverse in nature whereas those in an air-column are longitudinal. The sound produced in both cases is musical in nature. The frequency of vibration of a given length of a string can be altered by altering the tension and it varies inversely as the radius of the string ; but the frequency of vibration of a certain length of air-column is fixed and it does not depend on the radius of the tube. The frequency in the former case does not depend on the moisture and does very slightly depend on temperature so that a stringed instrument once tuned remains so during playing. The frequency of the sound emitted by an air-column varies with temperature and moisture and as such the pipe-instruments require frequent tuning during the operation. In strings damping effect is small and hence the resonance becomes very sharp, while in an air-column damping effect being very pronounced, the resonance is less sharp. In a vibrating string the whole series of harmonics can be produced but the note emitted is not pure, different harmonics being simultaneously excited. In air-columns, all harmonics can be produced only in open cylinders—the closed cylinders producing only odd harmonics. The note emitted is generally pure. If strongly excited a few harmonics may, however, be produced at the same time.

94. Longitudinal Vibration of Rods :—When a thin rod of metal or glass, firmly clamped at its middle point, is rubbed lengthwise with a piece of resined leather, or a wet linen, it is set in longitudinal vibration, that is, in planes parallel to its axis, and it gives out a shrill note. The rod is alternately elongated and shortened in its course of movement and the vibration takes place exactly in the same manner as the stationary vibration of an open pipe sounding its fundamental.

The free ends of the rod, being the positions of maximum vibration are ***antinodes***, whilst, for the simplest mode of vibration (fundamental) there will be a ***node*** in the middle where it is clamped. Evidently the length of the rod is half the wavelength (distance between two consecutive antiondes).

The velocity of the sound in the rod is given by, $V = \sqrt{\frac{Y}{\rho}}$, where

Y is the Young's modulus of elasticity and ρ, the density of the material of the rod. Again we have, $V = n\lambda$, where λ, the wavelength, is in this case, equal to twice the length l of the rod.

$$\therefore \quad V = 2nl \text{ ; or, } n = \frac{V}{2l} \text{ ; or, } n = \frac{1}{2l}\sqrt{Y/\rho}.$$

Thus knowing the velocity of sound in the rod, the frequency, or the pitch of the sound emitted can be calculated. Again, if the pitch of the sound is determined by comparison with a sonometer wire, the velocity of sound is known from the relation, $V = 2nl$. Thus this also provides a method of determining *the velocity of sound in a solid rod.*

If the rod be free at both ends, two antinodes will be formed at the ends with a node in between. Here also $\lambda = 2l$ and the fundamental frequency will be same as n above. In both these cases for the higher modes of vibration, $l = \lambda$, $\frac{3}{2}\lambda$, etc. and the frequencies of the overtones are given as,

$$n_1 = \frac{1}{l}\sqrt{Y/\rho}, \; n_2 = \frac{3}{2l}\sqrt{Y/\rho}, \text{ etc.}$$

When the rod is clamped at one end and free at the other the vibration will be similar to those in a closed organ pipe. Here the frequencies of the fundamental and the overtones are given by,

$$n = \frac{1}{4l}\sqrt{Y/\rho}, \; n_1 = \frac{3}{4l}\sqrt{Y/\rho}, \; n_2 = \frac{5}{4l}\sqrt{Y/\rho}, \text{ etc.}$$

The above frequency relations are valid only when the rod is so thin that its diameter is small compared to half the length of the wave.

As has been already stated the velocity and hence the frequency of a longitudinal wave is very large in comparison with that of a transverse wave. The two velocities cannot be made identical under any practical circumstances.

95. Kundt's Dust-tube Experiment :—The velocities of sound in different gases were determined by Kundt by using longitudinal vibration of rods. The velocity of sound in a rare gas is usually determined in the laboratory by this method.

The apparatus consists of a metal or glass rod which is clamped exactly at its middle point C and has a cardboard disc D firmly fixed at its end within a long glass tube AB in which it can move without touching its walls. The other end of the tube AB is closed by an adjustable piston B (Fig. 68).

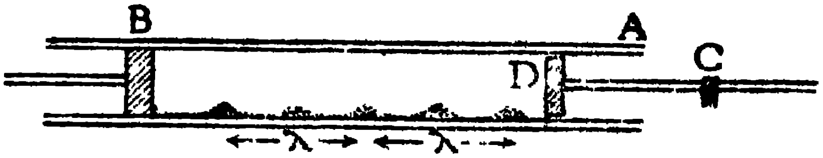


Fig. 68—Kundt's Dust-tube Experiment

Before fixing the tube in position it is thoroughly dried by blowing hot air through it, and then some dry lycopodium powder is evenly spread along its sides. The rod is now stroked (rubbed lengthwise) with a piece of resined cloth or leather, if it be metal, or with a cloth moistened with methylated spirit, if it is of glass, causing it to vibrate longitudinally. Waves are emitted by the disc D which is moving backwards and forwards with the frequency of the note emitted by the rod and thus setting up vibrations in the air within the tube. These waves started from the disc D are reflected back by the surface of the piston B and thus stationary waves having fixed nodes and antinodes are set up in the tube. The position of the adjustable piston B is carefully adjusted until a resonance is produced, when the fundamental note emitted by the rod coincides with a harmonic of the enclosed air-column within the tube.

When resonance is reached, the fine lycopodium powder is seen to be thrown into a state of violent agitation when the powder will be seen to fly away from the *loops* (antinodes), the places of maximum displacement of air-particles, and will collect at the *nodes*, forming a number of closely packed striae. These are the places of minimum displacement of air-particles. In general several nodes and loops will be formed within the tube as shown in the diagram. If l be the mean distance between two consecutive nodes or antinodes, the wavelength λ of the longitudinal vibration of air is $2l$, and if n be the frequency of the note emitted by the rod, it is also the frequency of vibration of the air in the tube, as the rod and the tube are in resonant vibration, and the velocity of the sound in air, $V=n\lambda=n\times 2l$. Now for the simplest mode of vibration of the sounding rod, a node is formed at the middle where it is clamped and two *loops* are formed at the two ends. So, if l' be the length of the rod, the wavelength λ' of the longitudinal vibration within the rod is $2l'$ and V' be the velocity of sound in the rod, $V' = n\lambda' = n \times 2l'$; so we have,

$$\frac{V}{V'} = \frac{n \times 2l}{n \times 2l'} = \frac{l}{l'} = \frac{\text{length between two consecutive loops or nodes}}{\text{length of the rod}}$$

The above relation provides a method of calculating V or V' when one of them is known ; and if frequency n be found by means of a sonometer and a standard fork, then *velocity of sound in air; and also in the rod, can both be determined.* If the Young's modulus and the density of the material of the rod be known, then $V' = \sqrt{Y/\rho}$.

Hence $V = \frac{l}{l'} \sqrt{Y/\rho}$. This relation may also be utilised for determining velocity of sound in a gas by the Kundt's tube method.

Velocity in different Gases.—To compare velocities of sound in two gases, first fill the tube with one of the gases and find out the average distance l_1 between two nodes formed at resonance. Repeat the experiment with the other gas and let the distance in this case be l_2 ; then if V_1 and V_2 are the respective velocities in the two gases, we have $\frac{V_1}{V_2} = \frac{n \times 2l_1}{n \times 2l_2} = \frac{l_1}{l_2}$.

For comparing the velocities of sound in air and in a gas denser

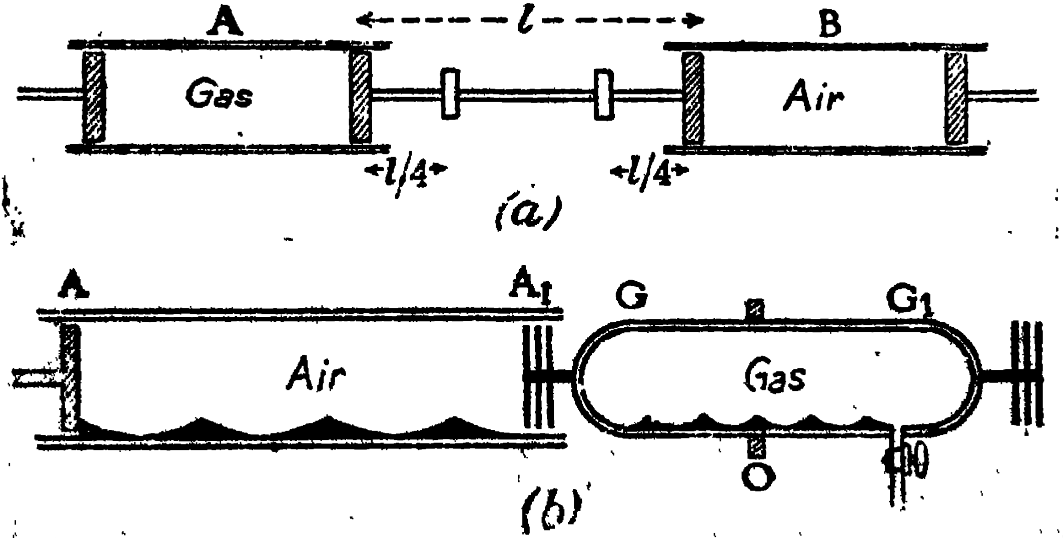


Fig. 69

than air, the quicker method will be to take two Kundt's tubes *A* and

B containing the gas and air respectively. They are as usual fitted with adjustable pistons required for resonance adjustments. These tubes are interlinked at the inner sides by means of a thin metal rod clamped at two points at distances $l/4$ from the ends. This exciting rod is provided with two loose-fitting pistons at the two ends inserted somewhat inside the two tubes A and B [Fig. 69(a)]. When the rod is excited, wavelength of the fundamental mode of vibration will be $\lambda = l$. But this does not occur in our calculations. When resonance is set up in the two tubes and lycopodium powder is found to form fixed heaps at the nodal positions in both of them, then from the relation $l_1/l_2 = \frac{V_1}{V_2}$, the velocities can be compared. When the gas is lighter than air, it is better to take it in a closed tube as shown in Fig. 69(b). The tube GG_1 is provided with metal discs at both the ends to maintain the symmetry about its point of support O in the middle. One set of discs is introduced in broader Kundt's tube AA_1 containing air. Both tubes have lycopodium powder sprinkled in them. When the tube GG_1 is excited by a wet piece of linen or any other suitable device and both set for resonance, then the velocities in the two gases can be compared from the relation, $V_g/V_a = \frac{l_g}{l_a}$, where l_g and l_a represent the distances between the successive heaps of powder in the two tubes.

Determination of the ratio of the two specific heats of a gas.— For determining the ratio of the two sepcific heats of a gas the velocity of sound in the gas in question is to be determined by the Kundt's tube method as indicated above. Let the velocity be V_g.

$$\text{Then } V_g = \sqrt{\frac{\gamma P'}{\rho}}\text{; or, } \gamma = \frac{V_g^2 \rho}{P} \qquad \ldots \qquad \ldots \qquad (1)$$

where ρ and P are the density and the pressure of the gas at the temperature at which V is determined.

Since $\frac{P}{\rho T} = \frac{p_0}{\rho_0 T_0}$, where T and T_0 are the values of above temperature and 0°C in the absolute scale, and p_0, ρ_0 refer to 0°C, we can rewrite the expression (1) as $\gamma = \frac{V_0^2 \rho_1 T_0}{p_0 T}$.

Thus γ can be determined by the Kundt's tube method.

Rayleigh and Ramsey determined γ for argon and helium by this method.

The velocity of sound in liquids have also been successfully determined by this method by Green in 1923.

96. Electrically maintained Tuning Forks :—Instead of repeated bowing or hammering, a tuning fork can be conveniently made to vibrate continuously at a constant frequency by means of an electric device. This is done either by electromagnet or with the help of a thermionic valve.

(1) Maintained by an electromagnet.—Here a small electromagnet *M* is placed in between the steel prongs of the fork. One end of the coil of the electromagnet is connected to the contact piece at *C* as shown in Fig. 70. The other end is connected to the binding screw *S*, at the stem of the fork through a battery *B* and a key *K*. When the key is closed the circuit is completed through the body of the fork and the contact piece *C*. Thus *M* is magnetised. So both the prong will be simultaneously drawn inwards due to attraction. As a result, the electrical contact at *C* will be broken and the current will cease to flow through *M*. So the prongs will be released and they will move outwards simultaneously again. Again the contact at *C* is established and the state of affairs will be repeated over and over again and thereby the vibration of the fork will be maintained continuously.

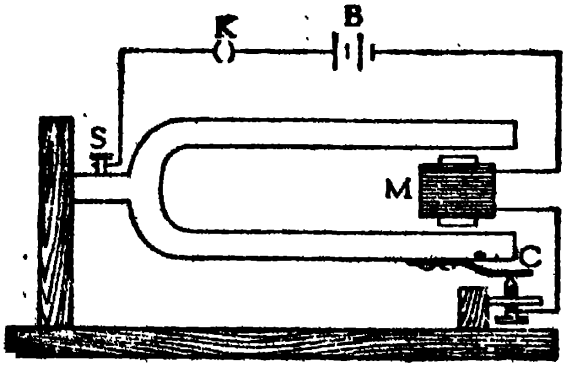


Fig. 70

(2) Maintained by a valve.—The forks maintained by electromagnets cannot be so done at frequencies higher than 100 or 200 cycles per second. So for maintaining the vibration at higher frequencies valve-maintained forks are used. Two telephone receiver magnets along with the coils are fitted either inside or outside the two prongs of a steel fork as shown by M_1 and M_2 in Fig. 71. The coil of M_1 is included in the anode circuit and that of M_2 in the grid circuit of a triode. A small vibration of the prongs' changes the flux through the grid coil and as a result the grid voltage undergoes fluctuation. This in its turns varies the anode current and hence the current through the coil M_1 which pulls the prong against it. The two prongs evidently act together so that when the direction of winding of the coils M_1 and M_2 are properly adjusted, the pull due to one of them increases the pull due to the other. So the prongs gradually vibrate more vigorously. This state of motion continues till the energy supplied by the high tension battery *H.T.* is partly utilised in overcoming the large air resistance due to large amplitude of vibration and partly in heating up the coils.

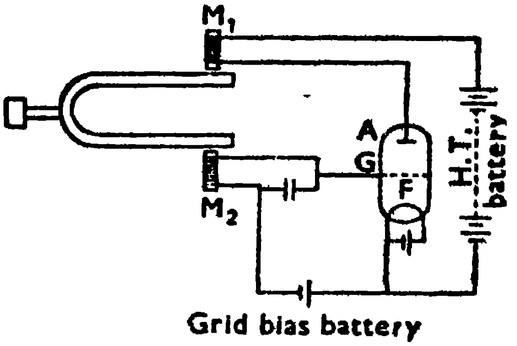


Fig. 71

The valve-maintained forks are often used as standards of time in modern measurements.

Note.—In Electricity, in Acoustics and in the accurate estimation of time-intervals, reliable **frequency standards** are often needed. For frequencies of the order of 1 or 2 per second, pendulum clocks, arranged with contacts to give electrical impulses every half-swing, would serve the purpose. But for frequencies ranging from 10 to 1000 per second, electrically operated or valve-maintained tuning forks are generally used. Above this frequency quartz oscillators driven electrically are used (See Art. 134).

97. Heat-maintained Vibrations :—

(a) **Trevelyan's Rocker.**—It consists of a triangular copper prism one edge of which is cut out into a groove with two rounded edges as shown in the section (*a*) in Fig. 72. This prism is provided with a handle having a spherical knob at the end to facilitate the movement. If the copper prism C be heated and placed on a triangular or semi-circular lead prism L the edge of which is kept in a horizontal plane as in Fig. 72(*b*), the prism C will go on rocking back and forth and a musical sound will be produced. The copper prism is to be heated moderately in a flame. The lead prism is well-cleaned and just at the moment of placing C on L, it is better to polish the edges of the groove by a fine sand paper. A critical temperature favours the rocking. It may sometimes be necessary to tap C at the start to have an effective rocking attended with the characteristic humming sound. The explanation of this rocking may be given from simple theory. When a hot edge is in contact with the lead block, the heat is conducted on to lead and a hump is set up at the point of contact which rises the edge of C slightly. This makes the other edge touch the lead block. Here again a hump is produced at the contact position and the edge is pushed out a bit. This state of alternately rocking up the edges continues till the temperature of C falls sufficiently so that the heat transferred on to L is not sufficient to set up the humps. In the Trevelyan's rocker the narrower the groove in C, the higher will be the frequency of the note produced. If the groove be wider, it is rather difficult to set up the rocking and the sound emitted is of much lower frequency.

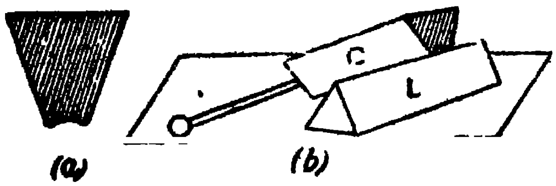


Fig. 72

(b) Howling tube.—The howling tube consists of a wide metallic tube wherein a piece of wire gauge is placed across the tube about a quarter of the way up. The gauze is made red-hot by a flame and then left. A strident sound is soon produced which continues until the gauze cools down. The harshness and loudness of the note is due to the presence of strong overtones. The heat transfer from the gauze which supplies the energy of oscillation, takes place at the maximum rate just before the compression is greatest, since the air then streams upwards with maximum velocity towards the node at the centre of the tube. This heat supply helps the expansion of the air which occurs in the next phase of the cycle. The transfer of heat almost stops during this part of the motion, as the hot air is forced down over the gauze. Thus the energy supply becomes periodic and the note is maintained.

(c) **Singing flame.**—When a jet of coal gas or hydrogen burns inside an open tube, under certain definite conditions, a musical note is found to be produced. This arrangement is known as the singing flame. It was first noticed in 1777 by Higgens. To make the flame sing the lengths of the gas tubes should be slightly less than half of the air tubes.

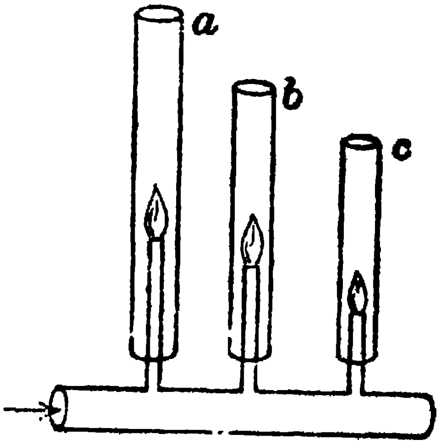


Fig. 73

The phenomenon has been extensively investigated and an explanation has been put forward by Lord Rayleigh. According to Rayleigh the phenomenon depends on the intermittent heat supply by the jet flame.

It has been found that the flame burns most intensely and consequently supplies heat to the air at the maximum rate when the air round the flame is in a state of maximum compression, when the air is just on the point of expanding. Here the flame becomes periodic because there is a reaction on the pressure of the gas-supply as also a variation in the air-supply. It is found that the flame does not sing unless the length of the tube supplying the gas is correct in relation to the natural frequency of vibration of the vertical outer tube. The flame sings best when placed at the position of greatest pressure change (*i.e. node*) in the vertical tube. It does not respond when placed near an open end as it is an antinode of the vibrating air-column. Richardson verified Rayleigh's theory of this intermittent energy supply in 1923.

This singing flame emits almost a pure tone free from overtones.

Examples

(1) A closed pipe 4 ft. long and filled with a gas resounds to a given fork. If an open pipe resounding to the same fork and containing air be 5 ft. long, what

would be the velocity of sound in the gas? The velocity in air at the temperature of the experiment is 1120 ft./sec. *(C. U. 1946)*

Ans. When filled with the gas, $l = \frac{\lambda}{4} = 4$ ft.

$$\therefore \quad \lambda = 16 \text{ ft.}$$

$$\text{But } n\lambda = V. \quad \therefore \quad n = \frac{V}{16} \qquad \dots \qquad \dots \qquad \dots \quad (1)$$

When filled with air, an open organ pipe resounds to the same fork when $l = \frac{\lambda}{2} = 5$ ft. or when $\lambda_1 = 10$ ft.

$$\therefore \quad n = \frac{V_a}{\lambda_1} = \frac{1120}{10} = 112 \qquad \dots \qquad \dots \qquad \dots \quad (2)$$

Hence from (1), $112 = \frac{V}{16}$, or, $V = 1792$ ft. per sec.

(2) *A cylindrical pipe of length 28 cm. closed at one end is found to be at resonance when a tuning fork of frequency 864 is sounded near its open end. Deduce the value of end-correction. Velocity of sound in air is 340 m/sec.*

Ans. Length of the wave set up in the pipe $= \lambda = \frac{V}{n} = \frac{34000}{864} = 39{\cdot}35$ cm.

For the fundamental mode, $\frac{\lambda}{4} = 28$. This is absurd. Hence considering the next mode, the corrected length of the pipe $= \frac{3}{4}\lambda$.

$$\therefore \quad (28 + x) = \frac{3}{4} \times 39{\cdot}35, \text{ where } x = \text{end-correction.}$$

$$\therefore \quad x = 1{\cdot}51 \text{ cm.}$$

(3) *In a resonance column experiment the first two successive positions of resonance occurred at lengths 15·4 cm. and 48·6 cm. respectively. If the velocity of sound in moist air at the temperature of the experiment be 34000 cm. per sec., calculate the frequency of the source employed and the value of the end-correction. Calculate also the length of the air-column of the next resonance.*

Ans. Let x be the end-correction and n, the frequency of the source.

$$\text{Then, } 48{\cdot}6 + x = \frac{3}{4}\lambda.$$

$$\text{and} \quad 15{\cdot}4 + x = \frac{\lambda}{4}.$$

$$\therefore \quad 33{\cdot}2 = \frac{\lambda}{2}; \text{ or, } \lambda = 66{\cdot}4.$$

$$\therefore \quad n = \frac{V}{\lambda} = \frac{34000}{66{\cdot}4} = 527{\cdot}1.$$

Again, $x=\frac{66\cdot4}{4}-15\cdot4=16\cdot6-15\cdot4=1\cdot2$ cm.

Third resonance occurs when $l+x=\frac{5}{4}\lambda=\frac{5}{4}\times66\cdot4=83$ cm.

$\therefore\ l=83-1\cdot2=81\cdot8$ cm.

(4) *A tuning fork held over a resonance tube shows resonance with air-columns 24 cm. and 74·1 cm. long. Find the frequency of the fork, the end-correction and the room temperature, if the velocity of sound is 340 metres/sec. at room temperature and 330 metres/sec. at 0°C.* (*Utk. U. 1952*)

Ans. Let x and n be respectively the end-correction and the frequency of the fork.

Then, $24+x=\lambda/4$ and

$$74\cdot1+x=\frac{3\lambda}{4}.$$

Hence, $50\cdot1=\lambda/2$: or, $\lambda=100\cdot2$ cm.

$$\therefore\ n=\frac{V}{\lambda}=\frac{34000}{100\cdot2}=339\cdot3 \text{ vibr. per sec.}$$

$$x=\frac{100\cdot2}{4}-24=25\cdot05-24=1\cdot05 \text{ cm.}$$

Again, $\frac{V}{V_0}=\sqrt{\frac{T}{T_0}}$. $\therefore\ \left(\frac{340}{330}\right)^2=\frac{T}{273}$.

$$\therefore\ T=273\times\frac{34\times34}{33\times33}=289^\circ\cdot8\ A.$$

Hence the room temperature $=(289\cdot8-273)=16^\circ\cdot8\ C.$

(5) *An open organ pipe 2·5 ft. long produces 5 beats per second with a similar but slightly shorter pipe and also with a tuning fork at 0°C. Find the number of beats produced between the fork and the shorter pipe at 22°C. (Velocity of sound at 0°C. is 1100 ft./sec. and at 22°C., 1144 ft./sec.)*

Ans. The frequency of the fundamental for the pipe is,

$$n=\frac{V}{2l}=\frac{1100}{5}=440 \text{ per sec. at } 0^\circ C.$$

If the pipe is shorter the frequency will be higher.

Hence the frequency of this shorter pipe $=440+5=445$ per sec.

The frequency of the fork will be also 440 ± 5, *i.e.* either 445 or 435 per sec.

The shorter organ pipe has a length $l_1=\frac{V}{2n_1}=\frac{1100}{2\times445}$ ft.

Hence its fundamental frequency at $22^\circ C.=\frac{1144}{2l_1}=\frac{1144\times445}{1100}$.

$=463$ per sec. (approx.)

So the number of beats with the fork will be either $(463-445)=18$; or $(463-435)=28$.

(6) *A Kundt's tube is excited by a brass rod 150 cm. long and the distance between the successive nodes in the tube is 13·6 cm. What is the ratio of the velocity of sound in brass to that in air?*

Ans. λ_1 (in the rod) $=2\times150=300$ cm.

λ_2 (in air) $=2\times13{\cdot}6=27{\cdot}2$ cm.

Now, $$n=\frac{V_1}{\lambda_1}=\frac{V_2}{\lambda_2}.$$

$$\therefore \quad \frac{V_1}{V_2}=\frac{\lambda_1}{\lambda_2}=\frac{300}{27{\cdot}2}=11{\cdot}03.$$

(7) *A rod 140 cm. long clamped in the middle is in resonance with an air-column 50 cm. long. When the rod is set into vibration the powder is arranged in 5 loops. Find the velocity of sound in the rod if that in air be 330 metres/sec.* (*Guj. U. 1955*)

Ans. λ_1 (in the rod) $=2\times140=280$ cm.

λ_2 (in air) $=2\times\frac{50}{5}=20$ cm. ($\because$ there are 5 loops).

$$\therefore \quad \frac{V_1}{V_2}=\frac{280}{20}=14\,;\quad \text{or,}\quad V_1=330\times14=4620 \text{ m/sec.}$$

Questions (Chapter VIII)

1. Describe in details an experimental arrangement for determining the velocity of sound in gases at different temperatures and pressures.

How does the velocity vary with pressure? Give reasons for your answer. (C. U. 1935)

2. Briefly explain the theory and method of finding the velocity of sound in air by resonance column. How this velocity can be converted to that at N.T.P. (*cf.* C. U. 1922)

3. If the velocity of sound in dry air at S.T.P. is 331 metres per sec. find the velocity in air, saturated with moisture, at 18°*C* and 75 cm. of mercury pressure. (Given, saturation aqueous tension at 18°*C* is 1·55 cm. of mercury and the ratio of the densities of aqueous vapour and dry air is 1 : 1·6)

[*Ans.* 343 metres/sec.]

4. Explain fully Kundt's experiment for determination of velocity of sound in a gas. (All. U. 1936)

5. Explain under what conditions interference of sound waves is possible and hence account for the dust figures produced in a Kundt's tube. Show how the above method is employed to determine the velocity of sound in solids. (Bom. U. 1927)

6. How can the ratio of the specific heats of a gas be determined by the acoustical method in a laboratory ? Describe the method in detail.

7. Describe an experiment to demonstrate the presence of nodes and antinodes in a sounding organ pipe.

8. Write notes on : (*a*) a valve-maintained fork, (*b*) a manometric flame, and (*c*) a singing flame.

9. Describe experiments illustrating the maintenance of vibration by heat and electricity. Explain in a general way the modes of maintenance. (C. U. 1943, *cf*. C. U. 1952)

10. Write short notes on singing flames. (All. U. 1951, '54)

11. Write short notes on Trevelyan's rocker. (Luck. U. 1945 ; C. U. 1957)

12. Discuss the formation of stationary waves in a pipe open at both ends, its mode of vibration and variations of displacement density and pressure along the tube. (Raj. U. 1949)

13. Explain how stationary waves are produced in a closed pipe. How do they differ from progressive waves ? (C. U. 1934, '46, '63)

14. An open organ pipe produces 8 beats per sec. when sounded with a tuning fork of 256 vibrations per second, the fork giving the lower tone. How much the length of the pipe is to be altered to bring it in accord with the fork ? (C. U. 1958)

[*Ans.* $\frac{1}{32}$*th* part of its original length.]

15. A tuning fork held over a resonance tube shows resonance with air-columns 24 cm. and 74·1 cm. long. Find the frequency of the fork, the end-correction and room temperature, if the velocity of sound is 340 metres/sec. at room temperature and 330 cm./sec. at 0°*C*. (Utk. U. 1962)

[*Ans.* (*i*) 339·3 vib./sec. (*ii*) 1·05 cm. (*iii*) 16°·8*C*]

CHAPTER IX

MUSICAL SOUND : ITS CHARACTERISTICS : MUSICAL SCALE

98. Musical Sound and Noise :—Sound may be divided into two classes : (*i*) *Musical Sound*, and (*ii*) *Noise*.

A musical sound is a continuous pleasing sound which is produced by regular and periodic vibrations : sounds produced by a tuning fork, a violin or a piano are all musical sounds.

Noise is a general term including all sounds other than musical sounds. It is discordant and unpleasant to the ear.

The essential *difference* between a musical sound and a noise, generally speaking, lies in the fact that in the former case the vibrations are *regular* and *periodic* ; while in the case of a noise the vibrations are *irregular* and *non-periodic* in character. It is, however, difficult to draw up a clear line of demarcation between a musical sound and a noise ; for, in practice, musical sounds too are seldom free from irregularities of vibration ; while, on the other hand, in noises sometimes there is also regular periodicity of the motion. Sometimes noise is accompanied by musical vibrations as in the clang of a bell. Moreover, the difference is only subjective. The same sounds may appear to be musical or noisy to different persons and under different conditions. Therefore, the difference is more artificial than real.

Characteristics of Musical Sound.—Musical sounds may be said to differ from one another in the following three particulars :—

(1) *Intensity* ; (2) *Pitch* ; (3) *Quality* or *Timber*.

99. Intensity :—It is the measure of *volume* of a note. It depends on the energy contained per unit volume of the medium through which sound waves pass. It may also be measured by the energy which passes per unit area placed normal to the direction of propagation of the sound. It is a characteristic of all sounds whether musical or not.

Loudness depends upon *the square of the amplitude* or *the extent of vibration* of the sounding body. When the body vibrates with greater amplitude, it sends forth a greater amount of energy to the surrounding medium, and hence, *energy received by the drum of the ear* is also greater. So the sound becomes louder. Loudness though commonly supposed to be same as the intensity is really the physiological aspect of it and is purely subjective in nature. Both loudness as well as intensity are found to vary directly as the square of the amplitude. This is why they are often used to mean the same thing. Though loudness depends on intensity it is not a linear function of intensity received by the ear drum. A sound of a definite intensity may appear to be of different loudness to different observers. So it depends on the physiological power of judgment of intensity of a man.

(*i*) The energy E of a body of mass m vibrating with velocity v and amplitude a is given by,

$$E = \tfrac{1}{2}mv^2 = \tfrac{1}{2}m\left(\frac{2\pi a}{T}\right)^2 = \frac{2\pi^2 ma^2}{T^2}. \quad \therefore \quad E \propto a^2.$$

Therefore, *the intensity of a note*, which depends upon the energy of the vibration, *is proportional to the square of the amplitude of the vibration.*

(*ii*) The intensity of a sound is *inversely proportional to the square of the distance* of the observer from the source of the sound **(Inverse Square Law).**

Thus, the energy received by the observer at a distance of 2 metres from the source is only one-fourth of the energy which the observer would receive, when at distance of 1 metre from the source.

[Suppose it is required to compare the intensities of the sound at two points A and B, distant r_1 and r_2 from a source of sound from which the total sound energy emanating per second uniformly all around is E. Draw two spheres with the source as centre and with radii r_1 and r_2 respectively. The amount of energy flowing per second per unit area at A normal to the surface of the sphere $= I_A =$ intensity at $A = E/4\pi r_1^2$.

Similarly, the intensity at $B = I_B = \dfrac{E}{4\pi r_2^2}$. $\therefore$ $\dfrac{I_A}{I_B} = \dfrac{r_2^2}{r_1^2}$. That is the *intensity* at a point is inversely proportional to the square of the distance.]

(*iii*) The intensity of a sound depends upon *the density of the medium* in which the sound is produced. It is seen that the greater the density of the medium, the greater is the intensity of the sound heard.

It is seen that some effort is to be made to make oneself heard by another in aeroplanes or balloons when flying high up from the surface of the earth as the density of air therein is small. For the same reason the sound is more intense in carbon dioxide than in air.

(*iv*) The loudness of a sound depends upon the *size of the vibrating* body.

If the size be larger, then a larger volume of the medium is put into vibration, and a greater amount of energy will pass per unit area. So the sound heard will be louder.

(*v*) The loudness of a sound is increased by the *presence of resonant bodies.*

The sound of a tuning fork, or a vibrating string in air, is much intensified when placed on a sounding-box which undergoes forced vibration.

100. Comparison of Intensity (Rayleigh Disc method) :—

For the comparison of intensities of two sounds, Rayleigh devised a method based upon the fact that a light disc tends to set itself perpendicularly with respect to the direction of an air stream whether direct or alternating. When the disc is suspended by a fibre so as to make an angle with the direction of air inflow in a resonator, the deflection of the disc is proportional to the energy of the sound waves received by the resonator.

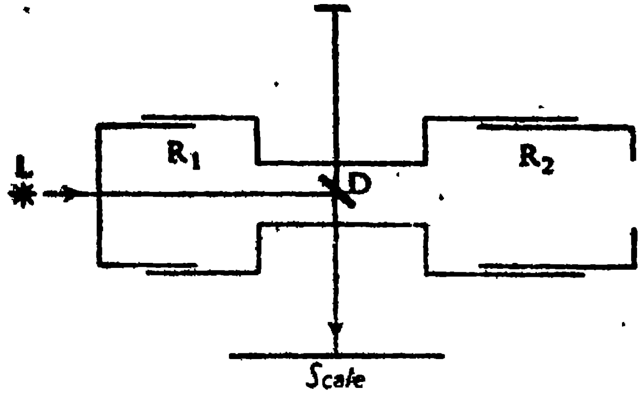


Fig. 74

König deduced a relation connecting the couple acting on the disc and the mean velocity of the air stream. This couple G is given by,

$G=\frac{4}{3}\rho a^3 V^2 \sin 2\theta$, where ρ=density of air, V is the velocity of the stream if steady or its R. M. S. value if alternating, a is the radius of the disc and θ is the angle between the direction of stream and normal to the disc.

If due to this couple the deflection is ϕ then ϕ is proportional to G.

$$\therefore \quad \phi \propto V^2.$$

But since the intensity is proportional to the kinetic energy $\frac{1}{2}mV^2$, we can say,

$$\phi \propto I.$$

Thus knowing ϕ_1 and ϕ_2, corresponding to two sounds of intensities I_1 and I_2, the latter quantities can be compared.

The actual Rayleigh disc is made of mica 1 cm. in diameter having a light mirror on its surface. The disc (Fig. 74) is suspended by a quartz fibre and is set at 45° to the axis of the resonator R_2 in the undeflected position, since for this value of θ, the couple G is maximum. For observing the deflection a spot of light from a source L is made to fall on a scale S after reflection at the mirror attached to the disc D. To increase the accuracy of the experiment the disc is kept inside the neck of a double resonator R_1 and R_2 of Helmholtz. When the open end of R_2 is pointed towards the source of sound and the lengths of R_1 and R_2 properly adjusted for resonance, the air flow round the disc causes it to deflect. So deflection ϕ_1 is noted. Similar operation is repeated for the second sound and the deflection ϕ_2 recorded. Now, $\phi_1/\phi_2=I_1/I_2$. It should be noted that the diameter of the Rayleigh disc should be small compared to the wavelength of the incident sound wave.

This method has been extensively used for comparing the intensities of sounds emitted by different sources.

101. Units of Intensity and Loudness : Bel : Phon : Sone :—

As has already been mentioned the intensity of a sound is a purely physical quantity whereas the loudness refers to the sensation it produces. A relation between the loudness and its stimulus (*i.e.* the intensity) has been formulated by Weber. According to this *law of Weber, the increase in intensity* (ΔI) *necessary to produce the minimum* *perceptible increase in loudness (ΔL) is directly proportional to the original intensity and the increase in loudness.*

$$\therefore \quad \Delta I \propto \Delta L \times I;$$

$$\text{or,} \quad \Delta L = K\frac{\Delta I}{I} \text{ where } K \text{ is a constant.}$$

Fechner wrote the equation as.

$$dL = K\frac{dI}{I}$$

and on integration derived,

$$L = K\log_e I = C\log_{10} I, \text{ where } C \text{ is another constant.}$$

This relation is known as *Weber-Fechner law.* From this law it is evident that if I is doubled, the loudness L is not doubled. But, for each doubling of the intensity, the loudness increases by the same amount.

For all practical purposes the absolute values of intensities are intentionally avoided because relative values have more practical significance. Absolute intensity which is but an energy flux per cm.^{2} is generally expressed in watts/cm.^{2}. The threshold of audibility for a note of frequency 1000 c.p.s. corresponds to an intensity (I_0) equal to 10^{-16} watts/cm.^{2} which is the standard of intensity. Relative intensity is generally expressed in *units of intensity level.* The intensity level of a sound is the ratio of its intensity to the standard intensity (I_0) as defined above. A sound having an energy 10 times that of I_0 is said to have an intensity level of 1 **bel** or 10 **decibels.** The scale is really derived from Weber-Fechner law and is logarithmic in nature.

$$\text{Intensity level in bels} = \log_{10}\frac{I}{I_0}.$$

So when I is 100 times I_0, the intensity level of I will be 2 bels or 20 decibels. Since the maximum audible intensity is 10^{14} times the minimum audible intensity the extension of the range of audible intensity is 14 bels.

Phon is the unit of equivalent loudness. For measuring the loudness of a sound in phons, another pure tone of frequency 1000 c.p.s. is taken. The intensity of the latter is gradually adjusted till the pure tone and the original sound appear equally loud. If at this stage, the intensity level of the pure tone be n decibels, the loudness equivalent of the original sound is supposed to be n phons.

Sone is a new unit recently introduced to measure the loudness of a sound. A pure tone of 1000 c.p.s. 40 decibels above a normal listener's threshold, produces a loudness of 1 *Sone*.

102. Pitch :—The pitch of a note is that physical cause which enables us to distinguish a shrill (acute or sharp) sound from a dull (flat or grave) sound of the same intensity sounded on the same musical instrument. It depends on the frequency of vibration of the emitted sound. The higher the frequency the more shrill is the sound and we say that the sound rises in pitch. As pitch is directly proportional to frequency, it is customary to express the pitch of a note by its frequency.

The pitch is a fundamental property of musical sound and a noise has no definite pitch.

103. Determination of Pitch.—The pitch of a musical note is determined by the frequency of vibration of the source of the note. Determination of frequency may be made by the following methods :—

(1) **Savart's Toothed Wheel.**—This consists of four toothed wheels of equal diameter mounted concentrically on a spindle fitted to a whirling table (Fig. 75). The number of teeth on each wheel conforms to a certain ratio, *e.g.* 20, 30, 36, 48.

A thin metal plate or a card-board *C* is clamped in front of the wheel so that it lightly presses against the teeth of one of the wheels *W* when it is in motion, and a sound formed by a series of taps is heard. On increasing the speed of rotation a musical sound is produced, the pitch of which depends on (*a*) *the number of taps made in a given time*, and (*b*) *the speed of rotation*.

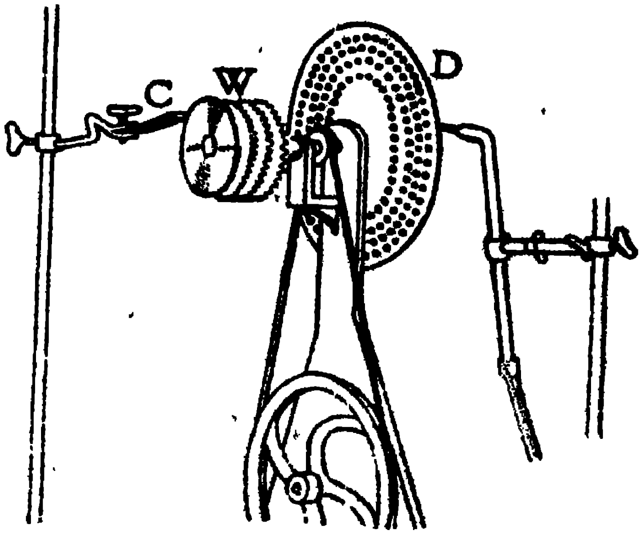


Fig. 75 - Savart's Toothed Wheel (*W*) and Seebeck's Siren (*D*)

To determine the pitch of a note, the speed of rotation of the wheel is gradually altered while the card is lightly pressed against a particular wheel until the note emitted by the wheel is in unison with the given note. Now if m be the number of teeth in the wheel used, and n the number of revolutions per second, the frequency N of the note is given by, $N=$ number of taps made per sec. $=m\times n$.

(2) **Seebeck's Siren.**—Seebeck's siren or Puff siren consists of a circular metal disc D (Fig. 75) through which a number of equidistant small holes have been drilled along concentric circles of varying diameters. The disc is mounted on a whirling table. A stream of air blown through a narrow tube ending in a nozzle, by means of foot-bellows, is directed to pass through the holes in one of the rings. As the disc rotates, the stream of air through the tube, is alternately stopped and allowed to pass through the holes, producing a series of puffs at regular intervals. To determine the pitch of a note, the rotation of the siren is adjusted until the note produced by the siren is exactly in unison with the given note. Now if m be the number of holes in the ring used, and n the number of revolutions per sec. made by the whirling table, the frequency N of the given note is given by, $N=$ number of puffs made per sec.

$$=m\times n.$$

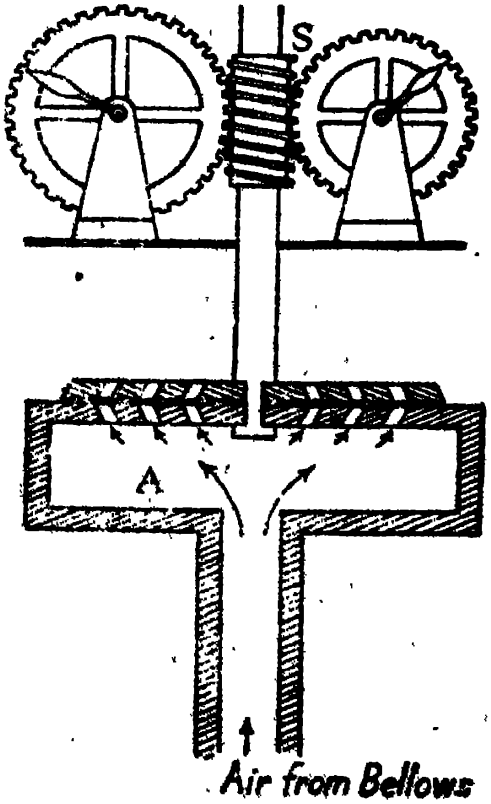


Fig. 76—Cagnaird de la Tour's Siren.

(3) **Cagnaird de la Tour's Siren.**—This is a much improved form of siren by which the pitch of a note can be fairly and accurately determined. In this siren (Fig. 76) a current of air is blown through a pipe into a wind-chest A, from which it issues through a ring of equidistant holes cut in the circular top of the wind-chest. Another disc having holes exactly corresponding with the holes in the top of the box is supported on the top of the wind-

chest and very close to it, in such a way that it can rotate freely about a vertical axis. The two sets of holes are drilled so as to slant in opposite directions, as shown in Fig. 76, so that the pressure of the air at the time of escaping through the holes causes the upper disc to rotate, the number of rotations being counted by a speed-counter S geared to the axle of the disc.

Every time the holes in the two rows coincide at the time of rotation of the upper disc, a jet of air escapes from each hole in the upper disc and, if there are m holes in each of the discs, there will be m puffs for one revolution of the disc. Of course, each puff will consist of m separate jets, but as they take place simultaneously, they are regarded as a single puff. Now, if n be the number of revolutions of the disc per second the frequency N of the note emitted is equal to $m \times n$.

Note.—In 1903 Northey devised a modified form of de la Tour siren for powerful long-range fog-signalling. This is known as the **Diaphone**. Even at a distance of 4-5 miles the intensity is found appreciably high with a diaphone.

(4) **Resonance of Air-Column**.—From the relation, $V=4nl$, in Art. 92 if V and l are known, the frequency n can be determined.

(5) **Method of Beats.**—The frequency can also be determined by the method of beats, as explained in Art. 44.

The method of beat is best applied in determining the frequency with the help of a *Tonometer*. Scheibler's tonometer consists of a series of tuning forks whose frequencies range over an octave. They are arranged in the ascending order of frequency—the frequency difference between a fork and its adjacent one on either side is generally four. The tonometer may be used for tuning a note to any desired pitch. The frequency of a note emitted by a fork is determined by the number of beats it gives with the two successive forks of the tonometer.

(6) **Sonometer.**—By tuning a vibrating string on a sonometer to unison with a given note, the frequency of the note can be determined by the formula, given below. If the tension T, and the mass

per unit length, *m*, of the vibrating string, are known, the frequency of the note is given by

$$n=\frac{1}{2l}\sqrt{\frac{T}{m}}, \text{ (\textit{vide} Art. 78).}$$

(7) Direct or Graphical Method (Duhamel's Vibroscope).—The frequency of a vibrating fork can be determined by the graphical method. A sheet of smoked paper is wrapped round a cylindrical drum which can be rotated uniformly by means of a handle attached to it (Fig. 77). A thin metal stylus is attached to one prong of the tuning fork which is so arranged that it can vibrate parallel to the axis of the drum and the stylus just touches the smoked paper. As the drum is rotated, the stylus will trace a wave line on the paper. If at the time of the vibration of the fork, two points can be marked on the wave line on the smoked paper at an interval of half-a-second, or one second the frequency of the fork can be determined by actually counting the number of complete vibration between the points.

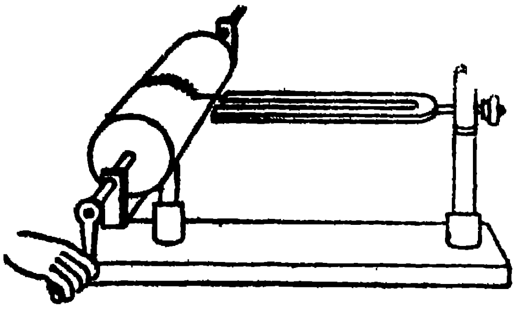
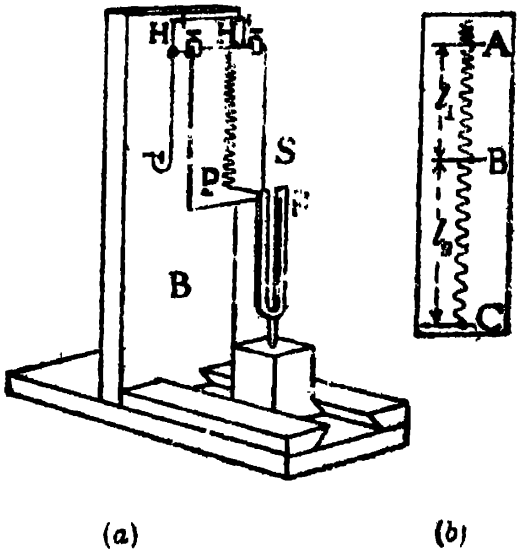


Fig. 77—The Duhamel's Vibroscope.

In order that the amplitude of the waves traced out by the stylus may not decrease owing to effects of friction, in actual practice, the fork is excited and its vibrations are maintained electromagnetically. The tracing time is recorded by means of an electric pendulum which is so arranged as to produce a spark at the expiry of a rated interval. Corresponding to successive sparks, spots are made by the end of the stylus on the wavy line traced on the smoked paper. The number of vibrations made by the fork in the rated interval is given by the number of complete waves found between any two consecutive spots and hence the frequency of the fork can be computed.

(8) Falling Plate Method.—In this experiment an arrangement is so made that a plate may fall freely under gravity. A glass plate *S* blackened preferably by camphor-smoke is suspended vertically by means of a thread from two hooks fixed on a vertical piece of wood *B*, as shown in Fig. 78(*a*). A tuning fork *F* to one prong of which a very light stylus *P* is fixed is clamped in front of the plate in such a way that the stylus just touches the smoked plate during the fall. The fork is set into vibration by striking it with a violin bow and then the plate is released by burning the thread between the hooks. As the plate falls under gravity, the stylus draws a wave-trace [Fig. 78(*b*)] of steadily increasing wave-length upon the smoked glass.



(*a*) (*b*)

Fig. 78—The Falling Plate Method

Ignoring waves at the beginning which are very crowded (and not suitable for counting), choose two lengths *AB* and *BC* *having the same number of complete waves.*

Let the velocity of the plate at the point $A = u$, and the time required by the plate to fall through the distance *AB* or $BC = t$. Calling $AB = l_1$ and $BC = l_2$, we have $l_1 = ut + \frac{1}{2}gt^2$. The velocity at *B* being $(u + gt)$, we have $l_2 = (u + gt)\,t + \frac{1}{2}gt^2$.

$$\therefore \quad (l_2 - l_1) = gt^2\ ; \quad \text{or,} \quad t = \sqrt{\frac{l_2 - l_1}{g}} \qquad \ldots \qquad (1)$$

If *n* be the frequency of the fork *F* and *m* the number of complete wave between *AB* or *BC*, we have $nt = m$.

$$\therefore \quad n = \frac{m}{t} = m \div \sqrt{\frac{l_2 - l_1}{g}}, \text{ from (1).}$$

N.B. The method has the disadvantage that by attaching the stylus to the prong, the frequency of the fork is altered. Also there may be some friction between the plate and the stylus by which the free rate of fall of the plate is affected.

(9) **Stroboscopic Wheel Method.**—A stroboscopic wheel

W is simply a metallic disc, having a number of equidistant rectangular radial slots *S* arranged along the rim, mounted vertically on a horizontal axle *A* which is mechanically driven at a known speed (Fig. 79). The fork *F* under test is placed on one side of the wheel, the plane of vibration of the prong being parallel to the plane of the wheel and the longer side parallel to the longer axis of the slot when the latter is vertical. The fork is run electrically and a strong light is focussed on a prong facing the slots. The wheel is set to motion and observation is made horizontally from the other side of the wheel.

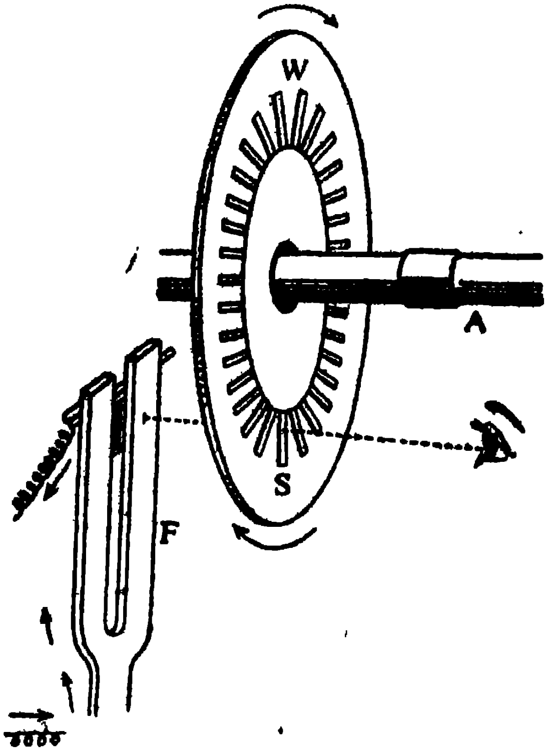


Fig. 79

The speed of the wheel is gradually increased till the interval, in which one slot is replaced by the next, becomes roughly equal to half *period* of the fork, when the prong would appear to oscillate slowly. Next, when the said interval is adjusted exactly equal to half *period* of the fork, by suitably altering the speed of the wheel, the prong would appear to remain *stationary*. This is what is called the stroboscopic principle.

Knowing the number of slots on the wheel, and the rate of motion of the wheel, the period of the fork and thus the frequency of the fork can be determined. If n be the number of slots and m the number of revolutions of the wheel per second, then the frequency of the fork N is given by $N = \frac{nm}{2}$.

(10) Phonic Wheel Method.— The principle adopted in this method was first utilised by Lord Rayleigh for the absolute determination of the frequency of a tuning fork. By the method described below, which is a mere simplified version of Rayleigh's method, quick but not so very accurate, the determination of the frequency of a tuning fork can be done. The apparatus consists of an iron wheel provided with a number (n) of equidistant studs on the periphery. It can be rotated about a horizontal axis. Two electromagnets M, M_1 (Fig. 80) are placed on diametrically opposite sides of the wheel with their cores almost touching the studs. The magnets are *intermittently* magnetised by the current from a circuit which electrically maintains the tuning fork. The frequency of this magnetisation and demagnetisation is therefore identical with that of the tuning fork which during its vibration automatically makes and breaks the current circuit. The phonic wheel W is rotated by hand while the tuning fork is vibrating. At a particular speed of rotation the wheel will just be synchronised with the periodicity of the electromagnet and it will continue to rotate. At this speed of synchronisation the number of studs appearing before a magnet per second will be equal to the frequency of excitation of the electromagnet which is same as that of the tuning fork. So counting this number of studs coming before a magnet over a considerable length of time, the frequency of the fork can be very accurately determined.

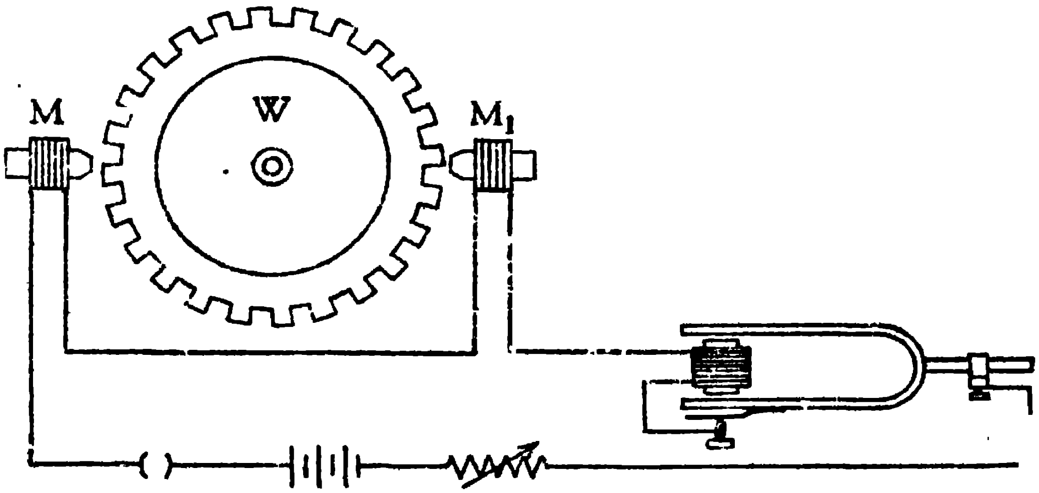


Fig. 80

Rayleigh's Method.—Lord Rayleigh in 1877 determined the absolute frequency of a tuning fork by means of a phonic wheel. The schematic diagram of the arrangements is shown in Fig. 80(a). If the fork to be standardised (not shown in the figure) be of approximate frequency n per second (actually 128 in Rayleigh's experiment), then select another auxiliary fork F_2 of almost the same frequency such that beats may be produced between the two. A third fork, F_1, to be used as an electrically maintained interrupter fork, is chosen so that its frequency (32 in Rayleigh's experiment) is a submultiple of that of F_2, say $\frac{1}{m}$th of the frequency of F_2, m being an integer (actually 4). To determine the frequency of F_1 accurately, an iron phonic wheel of N equidistant studs (actually 4) is chosen such that N is again a submultiple (say, $\frac{1}{r}$th part) of the frequency of F_1 approximately. The wheel can be rotated about a horizontal axis. The electromagnets M, M are placed on diametrically opposite sides of the wheel, the cores of which should be so adjusted that they almost touch the studs when they come against the cores. M, M are intermittently magnetised by the current from a circuit which is periodically maintained by the electrically maintained interrupter fork, F_1. The speed of the wheel W is adjusted till it is just synchronised with the periodicity of the electromagnet. The rotation of the wheel will then continue. Now the time of revolution of the wheel is directly compared with that of the pendulum P of a standard clock. For this purpose, the wheel is

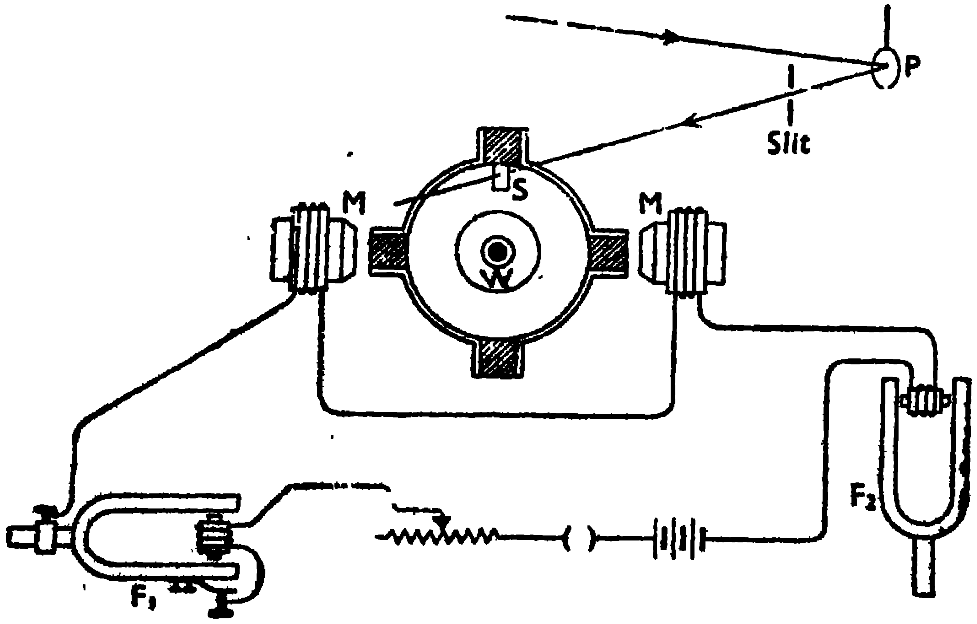


Fig. 80(*a*)

provided with a narrow slit, S. The bob of the pendulum is provided with a silvered bead upon which light from a lamp is concentrated, Immediately in front of the bead a screen provided with a vertical slit is placed. The light reflected from the bead is seen intermittently through the slit S in the wheel. Since the number of studs in W is $1/r$th of the frequency of F_1, when the speed is properly adjusted for its minimum at synchronisation, only r views of the pendulum can be secured through the slit S. "Any deviation of the period of the pendulum from a precise multiple of the period of intermittence shows itself as a cycle of changes in the appearance of the flash of light, and an observation of the duration of this cycle gives the data for a precise comparison of frequencies."

Supposing the clock to be correct, let the wheel lags behind or is in advance of the clock by a cycles per sec. (actually $\frac{1}{40}$ in Rayleigh's experiment). Since the period of this cycle is the time required for the wheel to lose or gain one revolution upon the clock, the frequency of revolution is $r \pm a$ (r was 8 in original experiment). The frequency of the driving fork F_1 is then given by $N(r \pm a)$. Hence the auxiliary fork F_2 has a frequency $mN(r \pm a)$ as it is driven by F_1. If b be the number of beats per second between the auxiliary fork and the fork to be standardised, then the frequency of the latter is given by $n = mN(r \pm a) \pm b$.

In obtaining an accurate result, the clock should previously standardised with reference to the movement of stars.

Note.—All the above methods for the determination of frequency relate only to *frequencies of medium values.* High frequency sound waves (ultrasonics) are studied by special methods referred to in Chapter XII.

104. Quality or Timber :—The *quality* or *timber* is that characteristic of a musical note which enables us to distinguish a note sounded on one musical instrument from a note of the same pitch and loudness sounded on another instrument.

A musical note consists of a mixture of several simple tones ; of these the one having the lowest frequency, called the fundamental, is relatively the most intense. Its frequency determines the pitch of the note. *Notes of the same pitch and loudness sounded on two different musical instruments differ in quality from each other owing to* *the difference in the number of other tones (or overtones) besides the fundamental, their order of succession, and their relative intensities.* Any difference in respect of these factors introduces a difference in the wave-form of a sound. So it may simply be said that the quality of two sounds will differ if their wave forms differ (*vide* Fig. 81). Now even if two sounds are similar in respect of these factors, a change in their wave-form occurs, if the phase relations between the overtones present in the two sounds are different. Helmholtz found experimentally that *when any change in wave-form is due to difference in the phase-relationships of the overtones, the quality of the two sounds do not differ.* That is, quality will differ when the wave-form differs only on account of difference in respect of the number of overtones order of their succession and their relative intensities. Helmholtz investigated on the physiological effects of overtones also. He found that a note possessing the fundamental and the first few overtones not exceeding the sixth is very pleasing to the ear, while a note in which the fundamental has mixed up in it more overtones than the sixth and which are relatively more intense, produces a metallic and harsh effect.

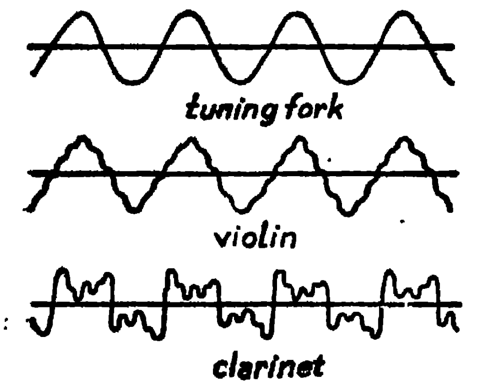


Fig. 81

Since the quality of a note depends on the number of overtones, their order and their relative intensities, two notes, similar in pitch and loudness, but differing in quality, will have different wave-forms, though the wavelength and amplitude of their fundamentals may be identical, and so their pitch and loudness are also the same. So the nature of the displacement curve of a note represents its quality.

105. Analysis of sound :—A trained ear may to a certain extent correctly analyse the component tones of a sound. Regarding the intensity and quality of the note, a rough estimate can only be formed by such aural analyses because the predominance of the fundamental tone automatically makes the judgment of the observer rather biased to it and the proper attention is scarcely paid to the harmonics, particularly, the finer ones. So for a thorough analysis of sound, instruments, particularly devised for this purpose, are generally used. Helmholtz's resonator, phonodeik, cathode ray oscillograph, heterodyne analyser, etc. come under this category of instruments. We shall here describe only those instruments which are simpler in principle and construction.

(a) Analysis by Helmholtz resonators.—As has been shown in Chapter V, (Art. 53) the fundamental frequency of a Helmholtz resonator is given by,

$$n=\frac{C}{2\pi}\sqrt{S/Vl}$$

where C=the velocity of sound, S=cross-sectional area of the neck, l=length of the neck and V=the volume of the air in the resonator. So a resonator of a particular size should respond to a particular frequency only. To find out the components of a complex sound, therefore, Helmholtz had to use a series of resonators of different capacities and of different areas of the neck. The resonators were previously calibrated. The relative intensities could also be roughly found by these resonators. But the main difficulty with these resonators is that for analysis of sound even over a short range of frequency a large number of them will be needed. So adjustable resonators have been constructed. In these resonators the volume of the air can be continuously varied by attaching a sliding piston at one end of the cylindrical cavity so that it serves the purpose of the closed end. With the change in the position of the piston the volume of the cavity changes and hence also the natural frequency of the resonator. So by adjustment it can be made to respond to any frequency. If a complex note falls on a resonator, the latter responds on adjustment and this will be detected by the ear placed in contact with the pip of the resonator. Knowing the natural frequency of the resonator, that of the responding component is also found. In this way different components can be analysed. The detection will be more accurately done by means of a manometric flame arrangement (*vide* Ch. VIII) attached to the pip end. A better method of detection by means of a hot wire microphone is described below.

(b) Hot wire Microphone.—The relative intensities of the overtones can only be roughly estimated with the help of Helmholtz resonators. But with the help of a hot wire microphone—this estimate becomes more accurate. It was first used by Tucker for the reception of sound waves during sound ranging.

The principle of the instrument lies in the change of the steady resistance of an electrically heated wire when sound waves fall on it. The instrument consists of a fine wire grid of platinum W held in position by a glass rod (g) (Fig. 82). The two ends a, b of the wire are connected to two annular foils M of silver attached to two faces of a circular mica sheet N. To increase the sensitivity of the arrangement this microphone element is mounted at the neck of an adjustable Helmholtz resonator. A steady current is made to flow through the wire and it forms an arm of a balanced Wheatstone's bridge. When the resonator is tuned to a particular overtone of a complex sound, the alternating draught in the neck causes a change in the resistance of the wire (actually a decrease in resistance) and the bridge is unbalanced. This change in resistance δR is directly determined by the Wheatstone's bridge arrangement as usual. This δR is prop. to V^2, where V is the velocity of the vibrating particle in the neck. Since the intensity is proportional to V^2, δB becomes a measure of the intensity if the instrument is previously calibrated.

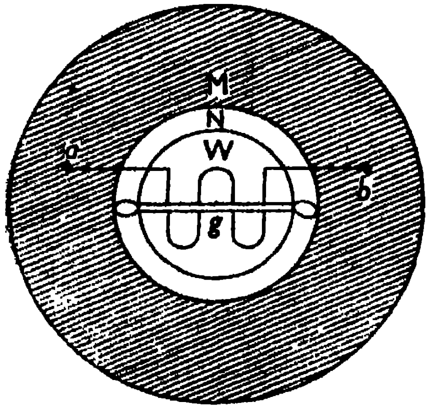


Fig. 82

106. Fourier's Theorem :—For a proper estimate of the quality of a sound, we require in addition to its wave-form, the order of succession of its overtones and their relative intensities. To have a complete knowledge of the overtones and there relative intensity Fourier's theorem is generally taken help of. So here we shall briefly deal with the said theorem and the mode of Fourier analysis.

Fourier's theorem **states that any** ***single-valued periodic function*** .

can be expressed as a summation of sinusoidal components of frequencies which are multiples of the frequency of the function. This summation is known as the *Fourier's series* and the division of the periodic function into the various simple harmonic components is known as *Fourier analysis.*

Let x be a periodic function $f(t)$ of the time t. Then according to Fourier's theorem it can be expressed as,

$$x=a_0+a_1 \cos \omega t+a_2 \cos 2\omega t+a_3 \cos 3\omega t+\cdots$$
$$+b_1 \sin \omega t+b_2 \sin 2\omega t+b_3 \sin 3\omega t+\cdots \quad \ldots \quad (1)$$

The constants $a_0, a_1, a_2 \ldots$, $b_1, b_2 \ldots\ldots$ can be obtained either by integration as shown below or by the graphical method as will be indicated in the next article. The periodicity of the function is $T=2\pi/\omega$.

Integrating the terms in (1) with respect to t over a complete period $T=2\pi/\omega$, we get, $\int_0^{2\pi/\omega} x dt=a_0 \int_0^{2\pi/\omega} dt=a_0 \frac{2\pi}{\omega}$;

$$\text{or,} \quad a_0=\frac{\omega}{2\pi}\int_0^{2\pi/\omega} x dt \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

since the sine and cosine terms vanish on integration over a complete period (*vide* any book of Integral Calculus).

Again multiplying the terms in (1) by cos $n\omega t$ and integrating over a complete period, we have.

$$\int_0^{2\pi/\gamma} x \cos n\omega t.dt=a_n \int_0^{2\pi/\omega} \cos^2 n\omega t\, dt+b_n \int_0^{2\pi/\omega} \sin n\omega t \cos n\omega t.dt$$

$$=a_n\frac{\pi}{\omega}+0=a_n\pi/\omega.$$

$$\therefore \quad a_n=\frac{\omega}{\pi}\int_0^{2\pi/\omega} x \cos n\omega t.dt \quad \ldots \quad \ldots \quad \ldots \quad (3)$$

Since n is any integer, the expression (3) gives the values of a_1, a_2, a_3 ... . etc.

Similarly, multiplying the terms in (1) by sin $n\omega t$ and integrating over a complete cycle, we get,

$$b_n=\frac{\omega}{\pi}\int_0^{2\pi/\omega} x \sin n\omega t\, dt \quad \ldots \quad \ldots \quad \ldots \quad (4)$$

Hence the value of b_1, b_2, b_3..... etc, are also known from expression (4).

If the expression for x be given it is possible to evaluate the values of a_0, a_n and b_n from relations (2), (3) and (4). But usually the graphical method is adopted for the purpose of actual analysis.

The ordinary limitations of Fourier's theorem, namely that the displacement must be *continuous* and *single-valued* and that *it must have a definite value*, do not concern the acoustical problem. This is because the displacement of a particle in a sound wave will always have a single and definite value. It can never assume more than one value, neither can it be infinite.

107. Graphical method of analysis :—In order to analyse a complex sound wave into its S.H. components, it is essential to get a time-displacement curve (*i.e.* the wave-form) of the sound by a suitable method. This can be conveniently had with the help of a phonodeik or a cathode ray oscillograph (*vide* General Physics, Chapter on S.H.M.). We shall here consider the phonodeik method of getting the wave-form of a complex sound wave.

The Phonodeik was first constructed by Miller. It consists of a thin diaphragm D mounted at the apex of a horn H which vibrates due to the sound collected by the horn (Fig. 83). This vibration of the diaphragm is converted into rotatory vibration of a spindle S mounted in jewelled bearings through a platinum fibre P attached to the centre of the diaphragm. The wire is kept stretched by means of a light spring (sp). The spindle is fitted with a small mirror (m). By suitable optical arrangements, the displacement of the light is recorded on a moving photographic film. On the film we get a magnified record of the time-displacement curve of the complex sound wave.

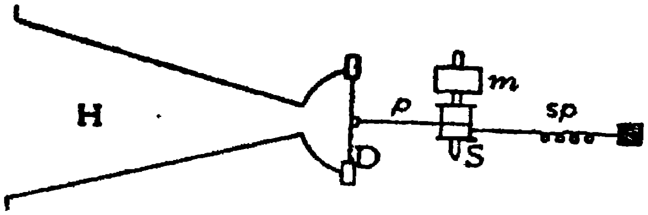


Fig. 83—Miller's Phonodeik

Once this curve is obtained a number of equidistant ordinates may be drawn on it from any reference point covering a complete period. The lengths of these ordinates give the displacements and the corresponding times are obtained from their distances along the time axis from the reference position. If six such ordinates are considered, we get six equations whose sine and cosine functions are known (since t's are known) and for which displacements (x's) are known. Hence including in the Fourier's series only six terms on the right-hand side (including a_1, a_2, a_3, b_1, b_2, b_3), we can easily find out these six unknowns (a's and b's), and hence the component harmonics. The a_0 term evidently vanishes on account of the symmetrical nature of the wave curves. Similarly, taking more number of ordinates more number of overnotes can be found out. Thus knowing the various harmonics we can find out their order of succession and also their intensities which will be proportional $(a_1^2+b_1^2)$, $(a_2^2+b_2^2)$ etc. The phases of the overtones can also be found from the relative values of a_1 and b_1, a_2 and b_2,···etc. Thus from a thorough Fourier analysis of a sound curve its quality can be properly studied.

108. Musical Scale :—Music has been developed on different lines by various races of people. But in some fundamental aspects there are always some similarities in all these distinct lines of music. In all the different branches, a series of tones have been chosen as the *musical scale.* The object of this scale is to fix up definite notes to be associated with different keys of the musical instruments like harmonium, piano, etc., wherein only a fixed number of notes are produced by the keys. But in other types of instruments like violin, no such scale need be fixed because here a skilled player can produce any number of tones whatsoever.

In the sixth century B.C. Pythagoras knew the simple ratios used in the diatonic scale. It is supposed that he might have got the same either from the Egyptians or the Indians.

109. Musical intervals :—Generally we express the pitch of a note by the number of vibrations per second. But in musical notes this absolute value of the frequency is rather immaterial because the ear can detect only the ratio in which one musical note changes into another. This ratio of the frequencies of two notes is known as the *interval* between them. If two notes have frequencies m and n, the interval between them is expressed as m/n. If these notes change their frequencies in such a manner that the ratio between them $\left(i.e. \frac{m}{n}\right)$

remains constant, the change-over is not recognised by the ear.

When the ratio between the frequencies of two notes reduces to unity, they are said to be in *unison*. When the interval is 2 : 1, it is called an *octave*. When a note is an octave higher than another, these two notes produce the most pleasing combination of the musical scale when played together. When the interval between two notes can be expressed as simple ratios (*i.e.* as ratios of small integers), they produce pleasing effect to the ear and the interval is known as the *consonant interval.* But when the interval is expressed as the ratio of two larger integers (*e.g.* 64 : 81), the notes produce a harsh or unpleasant effect and as such the interval is called a *dissonant interval.*

If three notes have frequencies m, n, p, the interval between the first and the second is evidently m/n, that between the second and the third n/p and that between the first and the third is m/p.

The interval between the first and the third is obtained by multiplying the other two intervals $\left(\because \frac{m}{n}\times\frac{n}{p}=m/p\right)$. So *any two intervals are said to be added, when their frequency ratios are multiplied.*

Certain intervals (consonant), have names :—thus 2 : 1 is the octave ; 3 : 2 is the fifth ; 4 : 3 the fourth ; 5 : 4 the major third ; 5 : 5 the minor third, etc. Thus when the fourth and the fifth are added $(\frac{3}{2}\times\frac{4}{3}=\frac{2}{1})$, an octave is produced.

Some of the important musical intervals along with their names are given in the table below :

Intervals and names

1 : 1	Unison	4 : 3	Fourth
16 : 15	Semitone	3 : 2	Fifth
10 : 9	Minor tone	5 : 3	Major sixth
9 : 8	Major tone	8 : 5	Minor sixth
6 : 5	Minor third	16 : 8	Seventh
5 : 4	Major third	2 : 1	Octave

110. Consonance and Dissonance :—Two notes which when sounded together give pleasing sensation to the ear are said to produce *consonance* or *concord*. As has been stated above, when the intervals between two notes are expressible in simple ratios of small integral numbers, *e.g.* 1 : 2, 2 : 3, 3 : 4, etc., the resultant effect is a concordant one. When two notes sounded together give an unpleasant sensation to the ear, they are said to produce *dissonance* or *discord*. Discord is produced when two notes, having their interval expressible as the ratio of two comparatively bigger integral numbers, are sounded together. This ratio may be 60 : 79, 64 : 81, etc. in case of a dissonant sound. Why this change in the ratio of the frequencies alters the musical nature of a sound had been an outstanding problem from the time of Pythagoras up to 1862 when Helmholtz gave a logical interpretation of the effect from the standpoint of beat phenomenon. According to Helmholtz, the cause of dissonance is the production of beats due to the interference of the notes. The beats produce a jerking effect on the ear-drum and are discordant just as the flickering of light is disagreeable to the eye. During the waxing of sound in the beat phenomenon the auditory nerves are to some extent fatigued and they redeem their normal conditions during the waning of sound. If the beat formation be too rapid so that before the sensitive nerves can come back to their normal states, a new waxing pulse is incident on them, the effect becomes too tiring for the nerves and a disagreeable sensation is produced in the ear.

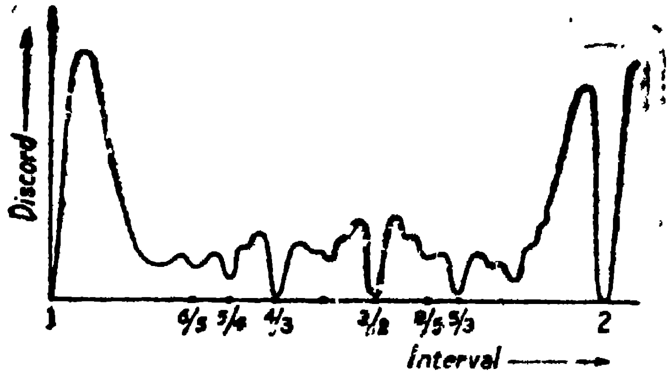


Fig. 84

Again the number of beats which produces the discordant effect is found to vary in the different parts of the musical scale. In the neighbourhood of *C* of the musical scale 33 beats per second produces the most discordant effect. The beats referred to above may occur between the two primary tones which are sounded together. They may also be due to the overtones attending each of them when played on a musical instrument. Thus dissonance also depends to a certain extent on the quality of the notes. Beats may even some times occur between the components of the combinational tone produced as a result of large amplitude primary tones or between any of these components and one of the original tones.

Helmholtz plotted discord against the intervals within an octave and got a curve which is shown in Fig. 84. To get the curve two notes originally in unison are produced. Now the frequency of one of them is kept constant and that of the other is gradually altered till it is an octave higher than the first. During the change their intervals are noted and the corresponding discords produced plotted against these intervals. As will be evident from the curve, the positions of some of the standard intervals are represented by marked depressions which indicate that at these intervals the discord is very small. So these standard intervals are pleasing to the ear. Any slight departure this way or that way from these standard intervals produces a considerable dissonance.

111. Diatonic Scale :—The musical scale which most of the European countries have been using for many centuries and which affords the simplest and the most pleasing succession of notes in an ascending order of frequencies is known as the ***diatonic scale*** (or, some times, the ***major diatonic scale***). This scale consists of eight notes, the lowest one (*i.e.* the fundamental note) being named *do* and the others ***re, me, fa, sol, la, se, do***. These correspond to the India musical notations ***sa, re, ga, ma, pa, dha, ni, sa***. They have been designated by Helmholtz respectively by the letters C, D, E, F, G, A, B, c. The note from which the scale starts, *i.e.* the lowest in the series, is known as the *tonic* or the ***key note***.

In actual musical instruments the notes one octave higher than the above ones are denoted by letters c, d, e, f, g, a, b, c' and the gradually higher scales by small letters with increased number of dashes. The scale lower than the normal scale $C \to c$ is denoted by the capital letters with suffix 1 and the next lower with suffix 2 and so on. (*e.g.* C_1, D_1, etc. and C_2, D_2...etc).

The names and the relations of notes between the ***tonic*** and its octave on the diatonic scale are given as follows :—

Name (Western system)	Do	re	me	fa	sol	la	se	do
... (Indian) ...	Sa	re	ga	ma	pa	dha	ni	sa
Symbol	C	D	E	F	G	A	B	c
Actual Frequency ...	256	288	320	341·3	384	426·7	480	512
Relative Frequency ...	24	27	30	32	36	40	45	48
Interval between C and each note	1	$\frac{9}{8}$	$\frac{5}{4}$	$\frac{4}{3}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{15}{8}$	2
Interval between each note and its predecessor		$\frac{9}{8}$	$\frac{10}{9}$	$\frac{16}{15}$	$\frac{9}{8}$	$\frac{10}{9}$	$\frac{9}{8}$	$\frac{16}{15}$

Starting from the fundamental note (tonic) *C*, the notes are generally referred to as the first, second, third, fourth, fifth, sixth, seventh and octave in succession. If the intervals between each note and its predecessor in the diatonic scale are considered, it is found that they are only three in number, namely $\frac{9}{8}$ (in cases *D* : *C*, *G* : *F*, *B* : *A*), $\frac{10}{9}$ (in cases of *E* : *D*, *A* : *G*) and $\frac{16}{15}$ (in cases of *F* : *E*, *c* : *B*). The three intervals, $\frac{9}{8}$ are known as the ***major tones***. As these are three major tones in the scale, it is sometimes referred to as the ***major diatonic scale***. The two intervals, $\frac{10}{9}$, are called the ***minor tones*** and the remaining two of ratio $\frac{16}{15}$ is called a ***semitone*** or a ***limma***. The difference between a major and a minor tone is $\frac{9}{8} \div \frac{10}{9}$ $\frac{81}{80}$ and this small interval is known as a ***comma***.

Some terms used in Music :—The pleasing effect produced by sounding two notes, which are in concord, one after another, is called ***melody***. The musical scales used in Eastern countries are set up for melodic purposes and as such contain more notes to the octave than those used in the Western countries. When two concordant notes are produced simultaneously, the pleasing effect is called *harmony*. When three notes of frequencies in the ratios 4 : 5 : 6 are sounded together (e.g. *C*, *E*, *G*) they form a concordant combination which is called a musical *triad*, and if a triad is sounded with an additional note which is the octave of the lowest note of the triad, the combination is known as the *chord*. When one musical instrument alone is played upon, (such as a flute or a violin), the performance is called a *solo*.

112. Merits and Demerits of the Diatonic Scale :—The definite superiority of the diatonic scale over all other musical scales designed so far is indicated by its long survival for centuries in the European countries. No other proposed scale is so well-arranged to provide harmonious combinations. In the diatonic scale these combinations are possible because here the intervals are mostly expressible in simple ratios only, excepting the intervals between D and C $(=\frac{9}{8})$ and between B and C $(=\frac{15}{8})$ which are rather a bit discordant.

The main defect of the diatonic scale is that it *lacks in its power of modulation*. Modern music requires frequent changes of the tonic. Such changes in the tonic or the *modulation of scale* as it is commonly called cannot be secured in the diatonic scale. But this modulation is essential to aviod the monotony of a music. When a music is played on the diatonic scale and C is taken as the keynote of frequency 256, the frequencies of the notes D, E, F, G, A, B, c are given by 288, 320, 341·3, 384, 426·7, 480, 512 respectively with their adjacent intervals $\frac{9}{8}$, $\frac{10}{9}$, $\frac{16}{15}$, $\frac{9}{8}$, $\frac{10}{9}$, $\frac{9}{8}$ and $\frac{16}{15}$. Now if that very music be played with E as the keynote on the above scale of adjacent intervals, the frequencies of E, F, G, A, B, c etc. would be 320, 360, 400, 426·7, 480 and 533·3 etc. respectively. But many of these notes are missing in the diatonic scale. Similarly, if the same music is played with any other note as the tonic, many more notes will be missing. So more provisions are to be made for these notes to effect a smooth modulation of a music. Thus the diatonic scale fails to provide for the requirements of a modern music. This necessitated a duly modulated scale for modern musical purposes generally known as the *tempered scale*.

113. Tempered Scale :—The tempered scale has been secured by fixing a number of notes between a keynote and the corresponding octave. For fixing up these notes a gradual and regular variation of pitch in the successive tones is to be ensured so that the modulation becomes smooth. The flexibility of the scale so obtained should be such that minimum discord is produced when two or more notes are simultaneously sounded. But too many keys (and hence notes) are to be avoided for the instrument to be simple both in construction as well as in use. All these conditions have been compromised in the *tempered scale* and this compromise is known as *temperament.* In the tempered scale, the notes between the tonic and its octave are twelve and in the musical instruments the four additional notes have been arranged for by inserting four black keys over and above the usual eight white keys in an octave. Various tempered scales have been so far attempted. The one, that is commonly in use, and that has been referred to above, adopts the principle of *equal temperament.* This scale is known as the *equitempered scale* or *chromatic scale.* Here each octave is divided into twelve intervals. If the interval between two consecutive notes be x, then, $x.\ x.\ x\ldots$to 12 factors$=2$.

$$\text{or, }\ x^{12}=2\ ; \qquad \text{or, }\ x=2^{1/12}=1{\cdot}05946.$$

This arrangement simplifies the playing of pianos or harmoniums.

The equitempered scale of a piano or a harmonium is shown in Fig. 85 where the keynote, *C* is taken to be of frequency 261·6 vibrations per second taking *A* (440 vibrations per second) as the standard of pitch according to the International Conference in London (1939). It will be noticed from Fig. 85 that the five black keys are inserted between all the consecutive notes except the 3rd and 4th and the 7th and 8th, *i.e.* they are placed only between the major and minor tones and not between the semitones. The interval between the consecutive notes on the scale is 1·05946 as referred to above and the frequency ratio of the original notes of the diatonic

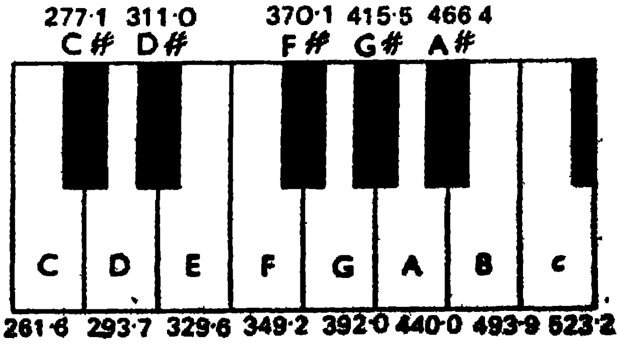


Fig. 85

scale remains unaltered in the tempered scale. The note corresponding to the first black key, *C* #, is know as *C* sharp. The second, third, fourth and fifth additional notes are named as *D* sharp, *F* sharp, *G* sharp and *A* sharp and are denoted by *D* #, *F* #, *G* #, *A* # respectively. These sharp notes are sometimes denoted as flat components of the succeeding ones. So, *C*-, *D*-, *F*-, *G*-, *A*- sharps are sometimes called *D*-, *E*-, *G*-, *A*-, *B*-, flats and are designated by symbols *Db*, *Eb*, *Gb*, *Ab*, *Bb*.

The main advantage of the tempered scale is that here any tone can be used as the keynote or tonic.

Examples

(1) A siren with 200 holes makes 132 r.p.m. It emits a note an octave lower than that of a tuning fork. Find the frequency of the latter.

(Gau. U. 1955)

Ans. The frequency of note emitted by the siren is

$$n=200\times 132 \text{ per min.}=\frac{200\times 132}{60}=440 \text{ per sec.}$$

Since it is an octave lower than that of the fork, the frequency of the fork is $2n=2\times 440=880$ per sec.

(2) The disc of a siren is making 10 revolutions per second. How many holes must it possess in order that it may produce four beats per second with a tuning fork of frequency 484 ? Which has the greater frequency, the siren or the fork ?

Ans. The number of beats per second is numerically equal to the difference of frequencies of the two notes. Hence the note emitted by the siren must have a frequency of $(484+4)=488$; or $(484-4)=480$.

The frequency of the note emitted by the siren, N=no. of holes in the siren × no. of revolutions per second. ∴ N = no. of holes × 10.

As the number of holes must be a whole number. N must be a multiple of 10. So the value 488, which is not a multiple of 10, cannot be accepted. Hence, $N=480$.

∴ 480=no. of holes × 10 ; ∴ The number of holes $=\frac{480}{10}=48$. Evidently the fork has the greater frequency.

(3) A small pointer, attached to one of the prongs of a tuning fork, presses against a vertical smoked glass plate. The fork is set vibrating and the glass plate is allowed to fall. If 30 waves be counted in the first 10 cms., find the frequency of vibration of the fork ($g=980$ cms./sec.^{2}).

Ans. We have, the distance fallen through, $S=ut+\frac{1}{2}gt^2$, but here $u=0$; $S=10$;

$\therefore$ $10=\frac{1}{2}gt^2$; or, $t^2=\frac{20}{g}=\frac{20}{980}=\frac{1}{49}$; or, $t=\frac{1}{7}$ sec.

As 30 waves are counted in $\frac{1}{7}$ second, we have frequency, $n=30\div\frac{1}{7}=210$.

Questions (Chapter IX)

1. Explain the terms :—pitch, interval, overtone and timbre. (C. U. 1926)

2. Describe three different methods of determining the frequency of a tuning fork ; which method do you regard as the most accurate and why ? (All. U. 1946)

3. What is pitch in the case of sound ? How is it accurately determined ?

4. On what do the loudness, the pitch and the quality of a musical sound depend ? Explain briefly how those may be graphically represented. How would you propose to determine the pitch of a note experimentally ? (C. U. 1915)

5. Describe some experimental method by means of which waveforms of a musical sound may be studied. (C. U. 1927)

6. Account for the fact that the quality of a note emitted by a closed pipe differs from that of the note emitted by an open pipe although the fundamental in the two may be the same. (Madras U. 1924)

7. Clearly distinguish between overtones and harmonics. Give a few instances of overtones that are not harmonics.

How can you detect the presence of harmonics in a musical note and what are their functions ?

8. Explain clearly the chief characteristics of a musical sound. Show that the intensity of sound emitted by a vibrating body is proportional to the square of the amplitude and the square of the frequency of vibration. (C. U. 1941)

9. Describe the stroboscopic method of finding the frequency of a tuning fork. (East Punj. U. 1950)

10. Write notes on analysis of sound. (Bom. U. 1954)

11. Write notes on : (*a*) Hot wire microphone. (*b*) Fourier analysis. (*c*) Rayleigh disc (*c*) Phonic wheel. (*d*) Stroboscope (C. U. 1967)

12. Explain fully what you mean by a musical scale and the interval between the musical notes. State briefly how the intervals in a diatonic scale are classified. (C. U. 1945 ; *cf.* '52)

13. What are concord and discord ? Discuss the relationship between the frequencies of two notes which cause them to be discordant when sounded simultaneously. (C. U. 1940 ; '51)

14. Explain the formation of beats when two notes are sounded together. How do you account for consonance and dissonance in combinational tones ? (C. U. 1936 ; '46)

15. (*a*) Write notes on diatonic scale. (C. U. 1955 ; Nag. U. 1955)

(*b*) What intervals are used in the diatonic major scale ? (C. U. 1952)

16. Write short note on equal tone temperament. (All. U. 1953)

17. How are musical intervals measured ? What is the physical cause of consonance and dissonance ? (C. U. 1958)

18. Sixty-four tuning forks are arranged in order of increasing frequency and any two successive forks give four beats per second when sounded together. If the last fork gives the octave of the first, calculate the frequency of the latter.

(C. U. 1946)

[*Ans.* : 252 c.p.s.]

19. Write an essay on musical scale explaining clearly the major and minor diatonic scales and tempered scale. Why is the last named scale adopted for practical purposes ? (Luck. U. 1946)

20. Describe the equitempered musical scale and compare it with the diatonic scale.

21. Distinguish between a musical note and noise. Explain the physical characteristics that determine quality, pitch and loudness of a musical sound.

(C. U. 1962)

CHAPTER X

MUSICAL INSTRUMENTS : PHYSIOLOGICAL ACOUSTICS

114. Musical Instruments :—The musical instruments can be divided mainly into *three classes*—(*a*) *Wind instruments* ; (*b*) *Stringed instruments* ; (*c*) *Percussion instruments*. Over and above these three classes there are a few unclassified types of instruments which are made of rods or bars and plates. Bells are modified plates or membranes and the theory of the same has been given in an earlier chapter. Another important instrument is a xylophone which consists of graded bars mounted side by side to give the scales when struck by a hammer, each bars emitting a note of a fixed pitch only.

(a) Wind Instruments.—The working of these instruments depends upon the vibration of an air-column. These again can be divided into two classes : (*i*) Instruments without reeds such as the flute, piccolo, etc. (*ii*) Instruments with reeds such as the clarionet, harmonium, etc. The most familiar example of the wind instruments is the organ pipe, which may be of the above two types, (*a*) one without reeds, known as the **flue pipe**, and (*b*) the other with reeds, known as the **reed pipe**.

It is already stated that only a column of air of right length may be made to respond to particular note. But in the case of a column of air contained in a pipe resonanee can be produced by making a flutter in the air at one end of the pipe. The pipe selects from the flutter (which is merely a combination of pulses of various wave-lengths) that particular pulse with which it can resound in order to produce a musical note. This is the *principle of various musical instruments* in nearly all of which the sounding part is a column of air.

The Flue Pipe.—The simplest form of this type is an ordinary organ pipe the principle of which has been already described. The note emitted by this pipe depends primarily upon the length of the pipe. The fundamental note is given out at a certain minimum blowing pressure by increasing which higher harmonics are emitted. Moreover, as an *edge tone* is produced in a flue pipe, the sharp edge should be at a critical minimum distance from the jet depending on the velocity of air issuing from the jet. The pitch of the tone falls off inversely as the distance of the edge from the jet. At a particular value of this distance, the frequency increases to the higher octave and then decreases with a further increase of distance.

In order to produce higher harmonics in a pipe the pressure of blowing should be increased.

In the Organ, there is a set of pipes of fixed pitch and the instrument is provided with a fixed keyboard as in a harmonium.

The Reed Pipe.—In this instrument the air blast impinges on a flexible metal strip [Fig. 86(i)] called the *reed*, which controls the amount of air passing to the pipe by wholly or nearly covering the aperture through which the air passes. Here the reed is properly set to turning by means of a wire which can be slided either towards the free or the fixed end of the reed. In the former case the length of vibrating reed decreases and the pitch rises and in the latter case it falls. The reed which completely closes the aperture of the pipe is called a ***beating reed***, which behaves as a stopped end of the pipe, and the other by which the aperture is nearly but not fully closed, is called a ***free reed***. Free reeds are used in harmoniums and American organs, where wind is forced into a rectangular air chamber at one side of which the reed is attached. The air presses against the reed and causes it to vibrate. A single beating reed made of cane is used at the mouthpiece of *clarionets, oboes, basson, etc.*

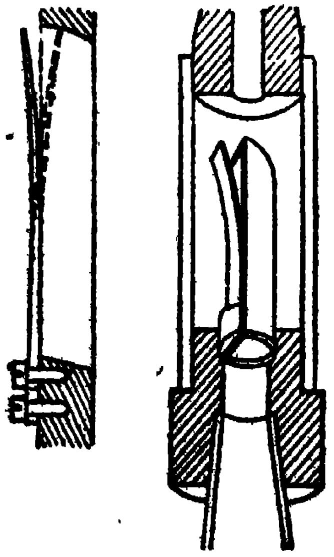


Fig. 86(i)—Reed Pipe.

Piccolo or Small Flute.—A small flute or piccolo is a cylindrical pipe with open ends having six holes bored on it [Fig. 86(ii)]. If all the holes are closed and the pipe blown with a moderate pressure, the fundamental tone as in (an open pipe) will be produced as shown in Fig. 86(*ii*)(*a*). Two antinodes at the free ends with one node in the middle will be produced. If the last hole on the right-hand side of the figure be opened and the pipe blown, the antinode at the right-hand end will be shifted on to the position of the open hole and mode of vibration will be fundamental as shown in Fig.86(*ii*)(*b*). Here due to decrease in wavelength of the note, the frequency and the pitch rises. If the holes towards the blowing end be opened one after another, the note rises in pitch gradually as the blowing end is approached. If the holes be closed but the pipe blown at a higher pressure, the octave of the fundamental will be emitted as shown in Fig. 86(*ii*)(*c*).

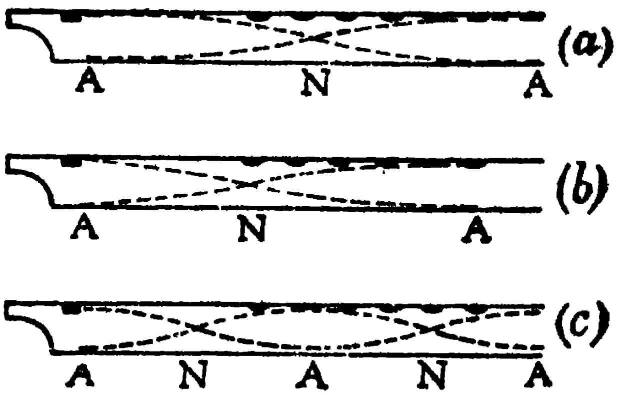


Fig. 86(*ii*)

(b) Stringed Instruments.—In this case of instruments the note is produced by the vibration of strings kept under tension. These instruments can be devided into three distinct classes according to their mode of excitation : (1) The plucked string instruments where the sound is produced by plucking the string as in guitar, *sitar*, etc. (2) The struck string instruments where the string is struck by a hammer as in a pianoforte. (3) The bowed string instruments where the string is excited by bowing as in a violin, *esraj*, etc.

When a stringed instrument is excited by plucking, theoretically an infinite series of tones will be generated. Calculations show that amplitude of a particular tone is inversely proportional to the square of the order of the tone. So the intensities of the tones of higher orders fall off very rapidly. So the sound is not rich in higher harmonics and as such lacks in sweetness. The intensity is also found to depend on the position of the plucking point as also on the total length of the string. Again it can be theoretically shown that if the length of the string be divisible into *n* equal parts and if it be plucked at any of the points of these n divisions, then the nth tone with all its harmonics will be suppressed. This is known as *Young-Helmholtz's law.*

[*Mathematical analysis of a* **plucked string** actually corroborates all the above inferences. We know that the vibration of a string can be represented by the equation, $\frac{d^2y}{dt^2}=C^2\frac{d^2y}{dx^2}$ where $C(=\sqrt{T/m})$ is the velocity of sound.

The general solution of this equation can be represented as a Fourier's series because the plucking actually gives rise to a large number of overtones over and above the fundamental. If the string be plucked at a point at a distance h from a fixed end and if l be the length of the string, then supposing the displacement of the string at the point of plucking to be k and applying the boundary conditions of the problem, the expression of displacement y at any arbitrary distance x from the former fixed end can be shown to be,

$$y=\frac{2kl^2}{\pi^2 h(l-h)}\sum_{1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi h}{l}\sin\frac{n\pi x}{l}\cos npt.$$

So theoretically an infinite number of tones are produced, the amplitude of each of which varies inversely as the square of the order of the harmonic. The amplitude (and hence intensity) depends on the total length of the string, the position of the plucked point and the extent of plucking.

Again from above expression of y we find that the nth tone disappears when $\sin\frac{n\pi h}{l}=0$; *i.e.*, when $\frac{nh}{l}$ is an integer, *i.e.* when $\frac{nh}{l}=m$ (say) ;

$$\text{or, when } h=m\frac{l}{n}.$$

Thus if the length of the string be divided into n equal parts and plucked at any one of these points of division, the nth harmonic will be absent. This is Young-Helmholtz's law.]

Again in the instruments where the strings are made to vibrate by striking, the emitted sound is also found to consist of an infinite series of tones. The intensity of the tones depends on the length of the string, the velocity of the striker and the length of contact of the same. Calculations show that the amplitude of a particular tone is inversely proportional to the order of the tone. So the intensities of the higher harmonics gradually fall but not so rapidly as in plucked strings. Hence the sound produced by a struck string is comparatively richer in overtones and as such it is much sweeter than that emitted by a pulcked string. In struck string also Young-Helmholtz's law is equally valid.

[In a **struck string**, mathematical analysis gives the expression for displacement y at any distance x from the origin as,

$$y=\frac{2c}{lp_1}\sum^{\infty}\frac{1}{n}\sin\frac{n\pi h}{l}\sin\frac{n\pi x}{l}\sin npt.$$

Where c is a constant depending on the velocity and extent of the striker. Here also Young-Helmholtz's law is found to be true and the instrument is found to produce an infinite series of tones. The amplitude varies inversely as the order of the harmonic and is found to depend on the length of the string and also on the velocity and extent of the striker.]

In instruments where the string is excited by bowing, an infinite series of tones are also produced. In the theoretical calculation of the vibrations of the **bowed string** Helmholtz did not take into account the influence of the point of bowing and showed that the intensities of the higher harmonics fall off very rapidly, the actual vibration of amplitude being inversely as the square of the order. Sir C. V. Raman made an extensive analysis of the vibrations of bowed string and showed that the vibration depends on the position of application of the bow and also on the pressure applied to the bow. It is found that the Young-Helmholtz's law is also valid in the case of a bowed string.

All types of stringed instruments are constructed on hollow wooden boxes or other types of suitable boxes like the resonant boxes of tuning forks. One end of the string or strings must be attached to a fixed peg or pegs while the other end should be tied to a turning peg or pegs for adjusting the tension on the string. The shape of the hollow boxcs should be made in conformity with the mode of excitation. Sometimes adjustable bridges are also provided on the box In some types of instruments over and above the main exciting string there are additional strings for each note of the musical scale (as in harps).

(I) **Piano.**—It is a musical instrument of the struck-string type and is widely used in Western countries. In the *key board*, each of the keys is mechanically connected with a *hammer* by a lever arrangement. The hammer has a rounded *felt-covered head* which produces a mellow or soft tone on striking. When the key is pressed the hammer strikes the string and rebounds and when the key is released a *damper* stops the vibration of the string. The hammer is generally applied somewhere about 1/7th or 1/9th of the length of the wire whereby the undesirable seventh or the ninth harmonic is suppressed. Due to the resonance of other strings, the richness of the tone is increased in a piano. The note produced by a piano is *not maintained.* So the intensity rapidly falls just after striking. The pitches of the notes produced are fixed. There are 12 notes to the octave.

The tone produced by a struck-string instrument is always richer in harmonics than that of a plucked string because the higher harmonics in a struck-string die out rather slowly.

(II) **Violin and Esraj.**—Violin and Esraj belong to the family of instruments where a stretched string is excited by bowing. Other instruments of this family are viola, double bass etc.

(i) **Violin.**—In a violin there are four stretched strings almost under equal tensions. They are of identical length. The strings are generally made of cat-gut but are occasionally loaded with wires. Near the top of the *body* of the instrument the strings are attached to four adjusting pegs of wood by which the tension can be adjusted. These strings first pass over the flat *finger board* and then over the bridge and are finally connected to the *tailpiece*. The belly of the instrument, on which the bridge is mounted is a hollow wooden box of the peculiar shape shown in Fig. 87(*i*). The central region is made narrower so that the bow can freely play on the strings. Inside the air chamber of the belly there is a wooden post connecting the back of violin with the front sound board. In violin the notes are produced by bowing. The proper notes are selected by pressing the bowed string with finger. Generally first string is bowed but for a musical glamour occasionally other strings are also excited. As there are no fixed frets on the body, the performer has a greater freedom to control the notes with his finger and at the same time requires a greater skill. The bow is made of horse-hair and is coated with rosin. When the bow is drawn across a string, the rosin carries the string with the bow for a moment whereafter the string slips back to its former position. A cyclic repetition of this operation produces vibrations of a characteristic wave form having a period nearly equal to the natural period of the string. These vibrations are intensified due to forcing caused in the enclosed air-column in the belly which is maintained at the atmospheric pressure with the help of two $\int$-shaped holes. The shape and size of the instrument is largely responsible for the quality of the notes emitted. If the wooden frame of the instrument has a natural frequency within the range of that produced by the strings, an unpleasant resonant sound **(wolf tone)** is often produced during the excitation. The notes produced in a violin extend over two octaves. A skilled performer can produce most of

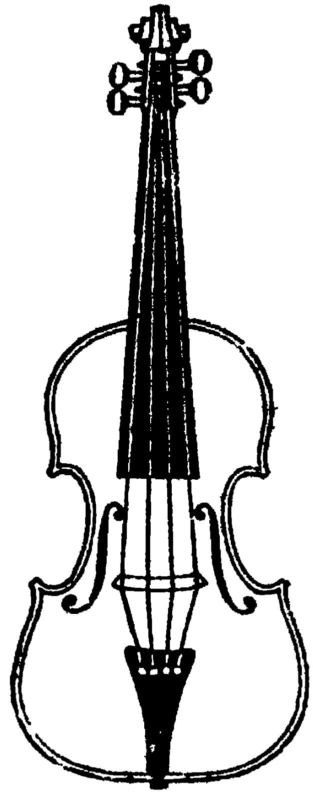


Fig. 87(*i*)—Violin.

the notes by properly controlling the positions of the nodes on the string.

(ii) **Esraj.**—It [Fig. 87(*ii*)] is also an instrument of the violin family. Like Sitar, its *body* and *belly* are also made of wood and are partially hollow inside. Near the central portion the belly is distorted as in a violin to allow for the free movement of the bow. The plane side of the body is made of wood and that of the belly consists of a stretched membrane over which a *bridge* is fitted. There are fifteen metal *frets* fixed on the body. Four main metallic strings are attached to wooden pegs near the top of the body. They are stretched over two bridges—one on the belly and the other generally of ivory fitted just above the frets. When the wires are bowed notes of about *two octaves* can be easily produced by changing the position of the finger on the frets. Fifteen resonating wires are attached to small pegs on the side of the instrument. They pass through holes at a lower part of the bridge. The quality of tone produced is very rich in harmonics. Moreover, due to the sympathetic vibration of the skin (membrane), resonating wires and the air-column, the tone produced is full of volume.

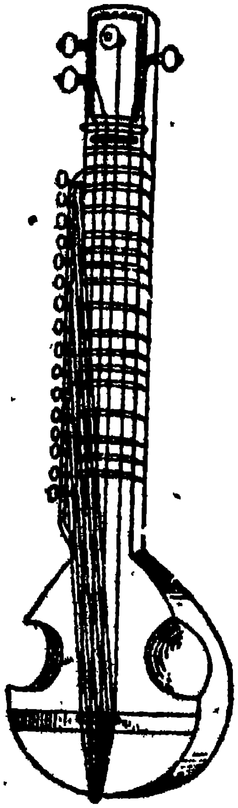


Fig. 87(*ii*)—Esraj.

(iii) **Sitar.**—One of the most commonly used stringed instrument in India is *Sitar*, which is excited by plucking. The *body* of the instruments consists of a partially hollow wooden post, the front side of which is almost flat. On this flat surface are mounted about 18 metal *frets*. The body of the sitar is attached to a hollow wooden belly which has been shape of almost a hemisphere. On the front plane surface of this *belly* is fixed a flat bridge. In the simplest form of *Sitar* [Fig. 87(*iii*)] seven metallic wires are stretched over this bridge. At the other ends they are attached to wooden pegs fitted to different positions of the *body*. Four to five of these wires extend almost throughout the length of the body. They pass over a sharp bridge fitted on the body a bit above the uppermost fret The remaining wires are of shorter lengths. The tensions are so adjusted by pegs that the strings are set to unison. So when the main string (the 1st one on the right side) is plucked others also vibrate in unison thereby adding to the richness of the sound. The auxiliary wires are called *resonant wires* which are also occasionally plucked when a higher intensity is intended. While the instrument is plucked the sounding lengths of the wire are altered by pressing the string on frets with a finger of the left hand Thus the frets enable one to excite a range of frequencies covering two octaves or even more. The air enclosed inside the body and the belly when set to forced vibration definitely improves the volume of the sound.

Other instruments of plucked string family are harp, guitar, mandolin, Indian Tanpura, etc. Some of these are fitted with frets for altering the frequencies while in some others there is no arrangement for changing the frequencies. In the plucked-string instrument, as the higher harmonics die out rather rapidly the note is not so rich as that produced by a struck-string instrument.

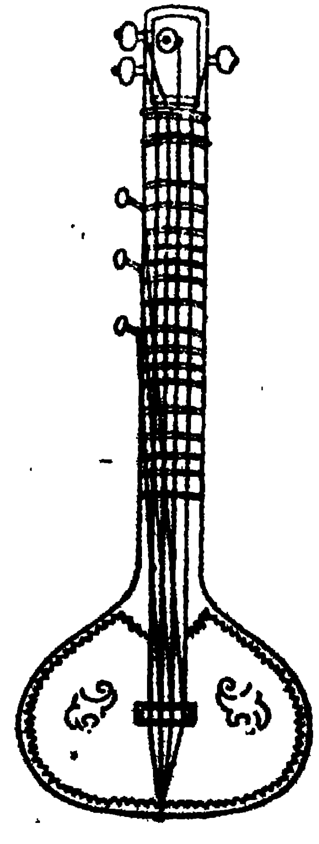


Fig. 87(144)—Sitar.

(c) **Percussion Instruments.**—Here a stretched membrane or a metal plate is fitted to a hemispherical or cylindrical bowl, which acts as the resonator. The vibration is excited by striking the membrane or the plate when a sound of a fixed pitch is produced. The pitch can be varied by the performer. This type includes kettle-drums, *tabla* (Indian), tambourine, etc. Pianoforte is sometimes classed in this group as the sound is produced by striking.

(d) **Aeolian Harp.**—The instrument consists of a number of wires stretched across a frame work. It may be put in a place where steady wind is blowing. In Art. 195 (General Physics), we have discussed the formation of eddies or vortices when a body moves through a real fluid. When a fluid moves past an obstacle, the same thing occurs here too. In air also eddies are formed just as in a liquid. So when the wind streams past the constituent wires of the harp the formation of whirls and eddies on each side of the wires gives a kind of shock to the wires and the adjacent air. If the wind blows continuously and regularly, these shocks are imparted at regular intervals and a musical sound, known as Aeolian tone, is produced. The vibration, set up in the strings due to shocks at regular intervals, gradually increases in amplitude and produces the musical sound. The strings of the harp are of different diameters. So different notes are produced when wind blows past them. *Aeolian tones* are also produced when wind blows against the sand particles in deserts.

Physiological Acoustics

115. The Ear:—The human ear (Fig. 88) consists of three parts: (*a*) the ***external ear*** (or **pinna**) by which the sound-wave is collected; (*b*) the ***middle ear*** through which the vibrations are transmitted from the external ear on to (*c*) the ***internal ear*** (or **labyrinth**).

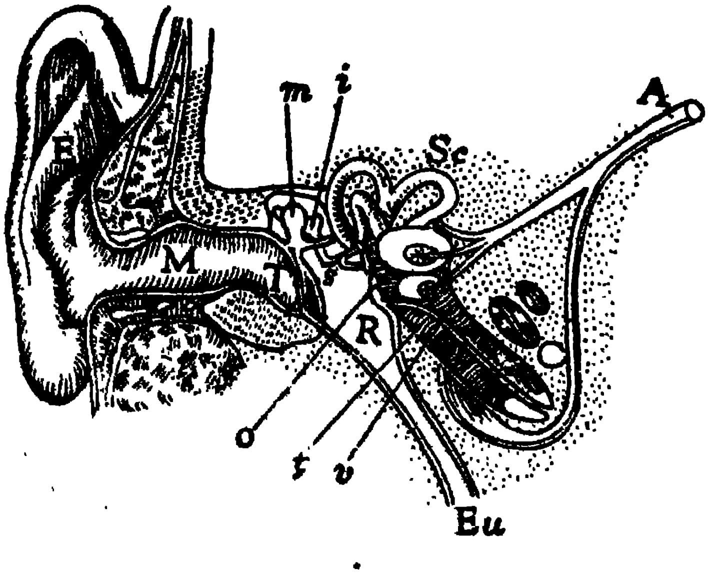


Fig. 88—Section through the Human Ear.

(a) External ear.—Starting from the outside, there is, in the first place, the *external ear E* (the part external to the head) from which extends the ear passage, called the **external auditory meatus** ***M***, down which the air-vibrations travel. This is closed at its end by a stretched membrane called the **membrana tympani** *T*, beyond which lies the cavity, called the ***ear drum*** or ***tympanum*** or the middle ear.

(b) Middle Ear.—This cavity is bounded on its outside by the tympanic membrane and on its inner side by bonny walls except at two places, the **fenestra ovalis** *O* and **fenestra rotunda** *R* where membranes are stretched. A combination of three little bones or **ossicles,** the first of which is the **malleus** *m* or the hammer bone, extends from the inside of tympanum. This bone communicates with the internal ear through two other bones, the *anvil s* (or **incus**) and the *stirup s* (or **stapes**), the base of which is joined to the *fenestra ovalis*, which separates the middle ear from one part of the inner ear. The middle ear is connected to the throat by an **eustachian tube** *Eu*. This tube is usually closed, but the action of swallowing opens a valve in this tube and serves to keep the air-pressure inside the middle ear equal to that of the atmosphere. Ear-ache is often caused when the valve does not work and due to which the outside pressure becomes greater than that inside so that the bones are pressed hard causing painful results.

(c) **Labyrinth.**—It is a complicated structure having a set of cavities. The cavities have bony walls, called the **osseous labyrinth,** and internal membrane, known as the **membranous labyrinth.** The osseous labyrinth consists of the following—(1) *Vestibule v* in the outer wall of which lies the *fenestra ovalis*. Through the inner wall of the vestibule the divisions of the auditory nerve *A* enter into the internal ear. (2) **Cochlea** *C* at the entrance to which lies the *fenestra rotunda*. It is a spiral canal like the form of a snail shell. It contains a fluid which receives and transmits vibrations to the auditory nerve. In this canal there is a membranous partition, called the **basilar membrane** which plays an important part in the act of hearing. (3) The semicircular canals *Sc* serve to maintain equilibrium and do not take part in the hearing.

The *membranous labyrinth* contains a fluid, *endolymph*, and between it and the *osseous labyrinth* is another fluid, *perilymph*.

116. How we hear :—The waves produced in the air by the vibrations of the sounding body are collected by the *pinna* and these waves passing through the *auditory meatus* strike the *tympanic membrane* which is forced to execute corresponding vibrations. These vibrations are transmitted through the three little bones in succession, the *malleus*, the *incus*, and the *stapes*, to the membrane of the *fenestra ovalis* of the inner ear. The vibrations of the fenestra ovalis start waves which pass through the perilymph and basilar membrane and set the endolymph in the cochlear canal in vibrations. These vibrations are finally transferred on to the brain through auditory nerves and the perception of sound is produced together with an interpretation of the same.

Helmholtz's resonance theory of hearing suggests that the fibres of basilar membrane act as resonators. Each of these fibres responds to a range of frequencies near its own natural frequency. Any satisfactory theory of hearing should explain the fact that the ear can respond to a wide range of pure tones extending over nearly ten octaves. Moreover, the theory must explain how beats are heard when two notes of nearly equal frequencies are sounded together. Again the fact that an ear can detect two notes of widely different frequencies when sounded together, should also be successfully explained by any proposed theory of hearing. The above facts can be explained if we assume that there is a series of resonators in the ear whose natural frequencies cover the audible range of frequencies and if these resonators be subject to damping. Due to damping a pure tone excites a groups of adjacent resonators. So in beats two such groups very near each other will be excited by two notes and a few resonators common to both the groups will produce beats. This sensation of beats is ultimately carried to the brain through the nerves.

Though no such resonators actually exist in ear, Helmholtz's theory suggests that the transverse fibres (or corti) of basilar membrane serve as such resonators. A group of such fibres are excited by a pure tone and the impulse received is carried to brain and analysed there. This theory, though not perfect, is indirectly supported by many facts.

Ohm enunciated a law regarding the function of the ear. It states that "the ear only experiences the sensation of a simple tone when it is excited by a simple harmonic vibration. It analyses every other periodic vibration into a series of simple harmonic vibrations."

117. The Human Voice :—The vocal organ can be compared to a double reed organ pipe. The voice is produced by forcing air from the lungs through the space between two stretched membranes V, V called the vocal chords. These are stretched across the top of a wind pipe, called the **trachea**, with a narrow slit between them, two edges of the slit acting as reeds (Fig. 89). This narrow slit is called the **glottis** or the **vocal slit.** The two membranes are attached to muscles by which their tension and vibration frequency can be altered. The trachea, or the wind pipe of the throat, leads to the lungs at one end, and at the front part of the throat it forms the vibrating part called the **larynx** or the *voice box.*

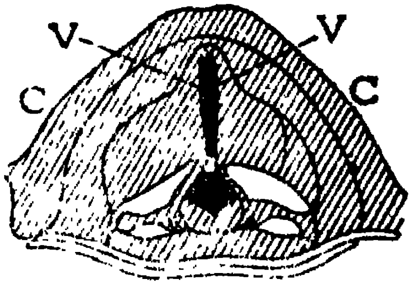


Fig. 89—Vocal Organ.

The edges of the membranes are set into vibrations like reeds by the air from the lungs and thus sound is produced, the pitch of which can be altered by altering the tension of the vocal chords and the quality of which depends upon the air-cavities of the nose, throat, and mouth, which act as resonators, the shape and the size of which the speaker can vary at will.

The vocal chords are much longer in men than those in women and children, and so the wavelength of sound emitted by a man is much longer than that emitted by a woman or a child. Thus a ***female voice is of higher pitch*** compared to a male voice.

118. Consonant and Vowel Sounds :—The **consonants** of a speech are generated by the teeth, tongue and lips. Some of them (*e.g. B, D* and hard *c*) are produced by complete stoppage of sound and some are characterised by incomplete stoppage particularly when followed by vowels. As the sound produced by complete stoppage is rather abrupt, it is rather noisy in nature and the vibration is not maintained. As the consonant sound is transient, a large number of higher harmonics are generated and hence for their proper reproduction a wide frequency range will be needed.

The **vowels** of a speech consist of continued sound and as such they are maintained. The vowels carry most of the energy in a speech. The production of vowel sound is effected by properly adjusting the cavities of the mouth, throat and nose and also by properly moving the tongue. These cavities act as resonators, the fundamental being produced by the vocal chords.

The oldest theory of production of vowel sounds attributes two tones to each vowel, each of them being independent of the pitch of the vowel. The mouth along with the tongue acts as a double resonator. How this resonator is excited is not exactly known. This resonator as well as other cavities introduce overtones having fixed frequencies for a particular vowel. Helmholtz suggested that some of the partials of the note emitted by the larynx are simplified by this double resonator, and that the overtones of the vowels have fixed frequency ratios with respect to the fundamental. Hermann considers larynx and the resonator (mouth) as a system which are coupled together. So according to Hermann, the pitch of the overtones might be slightly inharmonic in nature. According to another theory the vowel excitation is obtained impulsively by almost periodic puffs of air from the glottis. The quality of vowel here is controlled by the position of the tongue solely. A comprehensive theory of vowel sounds is still wanting.

Questions (Chapter X)

1. Describe the action of the mouthpiece of an organ flue pipe and show that an open-end pipe will produce sounds richer in overtone than a closed-end pipe. (All. U. 1944)

2. Write short notes on :—
- **(1) Principles of musical instruments. (*cf.* All. U. 1947)**
- **(2) Piccolo.**
- **(3) Stringed instruments.**

3. Write a note on the human ear. (C.U. 1947)

4. Explain with clear diagram the structure of the human ear and its working. (Luck. U. 1945)

5 What enables us to distinguish the sound of the different vowels when we hear them and how do their differences arise. (C.U. 1931)

6. Briefly account for the consonant and vowel sounds along with their theories.

CHAPTER XI

TECHNICAL ACOUSTICS

119. The Phonograph :—Long before the invention of the phonograph, Thomas Young, an English scientist, succeeded in recording sound vibrations on a rotating drum. It was Thomas Alva Edison, an American, who in 1878 invented the phonograph by which it was possible to record as well as reproduce sound vibrations.

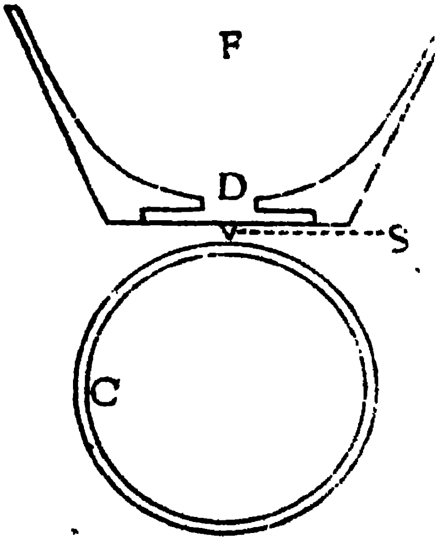


Fig. 90—The Phonograph

The *phonograph* consists of a funnel *F*, which is closed at the lower end by a thin glass or mica diaphragm *D* (Fig. 90). When sound vibrations are directed into the funnel they set the diaphragm into vibrations, and with it a pointed steel or a chisel-shaped sapphire crystal *S*, attached at the centre, also vibrates. The chisel is in contact with a cylinder *C* coated with a thick layer of paraffin wax, and at the time of vibration, cuts a groove of *varying depths* in the wax coating of the cylinder which is rotated, and at the same time moved lengthwise by a clockwork. The depth of the groove is not uniform but corresponds to the strength and complexity of the vibrations communicated to *D*. ***The cylinder is thus a faithful record of the sound vibrations directed at F.***

To *reproduce* the sound, a smooth sapphire point, attached to a similar diaphragm fitted in a frame, called the ***sound-box***, is placed at the beginning of the groove of the cylinder which is rotated and shifted sideways at the same speed as before. The sapphire point rises and falls in accordance with the height and depth of the groove, and thus the diaphragm of the sound-box reproduces exactly the movements of the diaphragm *D* of the recorder. These movements communicated to the air produce the same sound which was originally directed into the funnel F.

The materials with which the phonograph records are prepared being very soft the records do not last long and so the reproduction is not very faithful.

120. The Gramophone :—It is a machine for the recording and reproduction of sounds, vocal or instrumental, such as music, speech, etc. It is a more improved apparatus than the phonograph. The sound records are made in the form of flat discs in which spiral grooves representing sound-tracks run from the rim to the centre. The grooves are of *varying width* and *not of varying depth*, as a result of which the resistance to the movement of the needle along the furrow is much less than in the phonograph and so the reproduction of sound is much more faithful. Moreover, the discs are made of a matrix (composed of shellac, tripoli powder and other ingredients) which is much harder than the wax used in the phonograph and so they do not deteriorate with use so easily.

Recording of Sound on a Disc.—The modern method of recording is electrical. The source of sound is placed in front of a microphone by whose mechanism the current passing through it is fluctuated. This fluctuated current is amplified to a required extent by the use of thermionic valves. The amplified fluctuating current is used to actuate a cutting chisel upon a disc of wax by the principle of electro-magnetic action. This record of wax is called the **negative**. An electroplate of it is made on a copper disc by electrolysis. This electro-plate is called the 'mother shell' or the 'parent record', or the **positive**. Two 'working matrices' of two different musics are made from two such mother shells and are fixed to the top and bottom plates of a hydraulic press with their recorded surfaces facing each other. The recording material (the disc), previously warmed a little, is placed in between the two working matrices and the two records are stamped on the two faces of the recording material by pressure.

Reproduction of Sound from a disc Record.—This is done through the mechanism of a *sound-box* which has a needle, with a pointed end, rigidly screwed to the shorter arm of a lever system

(Fig. 91). The needle slides on the spiral grooves of the record, the record being made to rotate at a uniform speed with the help of an adjustable *governor*, by the action of the energy of a wound spring. The end of the longer arm of the lever is fixed to the centre of a circular mica diaphragm The diaphragm is mounted between rubber rings, called *gaskets*, and forms the front of a cylindrical metal-box, called the *sound-box*. The vibration of the needle running on the furrows set the diaphragm to motion, reproducing the recorded sound. The sound-box is connected to a metallic conical pipe called the *tone-arm*, which is capable of moving freely about a vertical axis. The tone-arm with the sound-box gradually moves to the centre of the record as the needle slides on it. The sound from the tone-arm is finally magnified through a horn which is usually housed within the cabinet. The lever system is balanced on a knife-edge forming the fulcrum. The vibration of the lever is controlled by the two springs as shown in the figure.

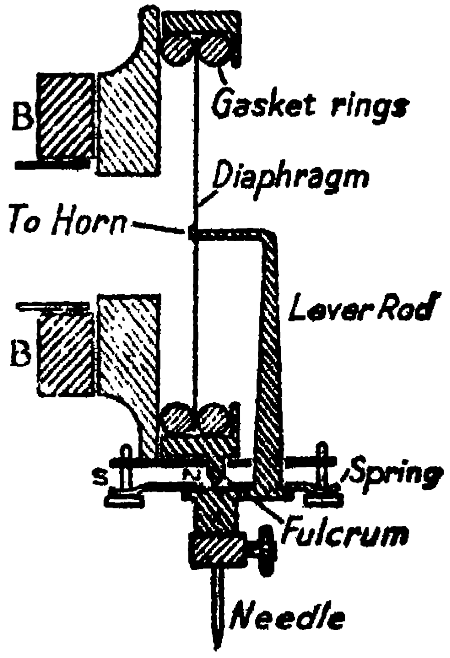


Fig. 91—The Sound box.

In the **Radio-Gramophone** the mechanical sound-box is replaced by an electric 'pick-up' by the mechanism of which a periodically modulated feeble current is obtained as the needle slides on the grooves of the record. This feeble modulated current, after suitable amplification through a combination of thermionic valves, is led through **a loud speaker** by which a voluminous sound commanding a large assembly of audience is reproduced.

The Electric Pick-up.—The pick-ups used in radio-gramophones may be of magnetic type, crystal type or even moving coil type. The ordinary pick-ups are of magnetic type. Here a horse-shoe magnet having four pole-pieces, *N*, *N*, *S*, *S*, is taken and fixed coil is fitted in between these pole-pieces as shown in Fig. 92. The needle is attached to an iron armature suitably pivoted and damped. When the needle, during its rocking motion, takes up the position shown in the figure, the lines of induction through the armature passes from upper *N* to diagonally opposite *S*. When the needle moves to the right the flux through the armature passes from lower *N* to upper *S*. So due to the sidewise movement of the needle there is a change in flux in the coil, *CC*, whose direction changes with the change in position of the needle. So the directions of the induced e.m.f.'s in the coil will also be different. The magnitude of the e.m.f. will depend on the speed of movement of the needle from side to side which is controlled by the records of the sound vibrations on the record-disc. The e.m.f. generated is then duly amplified by valves and finally reproduced by loud speaker as mentioned above.

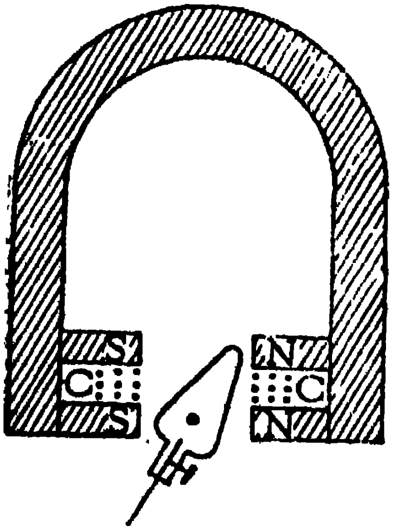


Fig. 92

The crystal pick-ups are constructed on the principle of piezo-electric properties of crystals. Due to strain produced by the motion of the needle an e.m.f. is developed in the crystal which is magnified and reproduced. The crystal pick-ups are of small weight and are quite effective in output.

121. The Telephone:—This is an electrical arrangement by which speech can be transmitted from one station to another. The *telephone* was invented by Graham Bell in 1875, and the instrument, *Bell's Telephone*, goes by his name. It consists of a long bar-magent *M* (Fig. 93), round one end of which there is a flat coil *C* of insulated copper wire. The two ends of the coil are connected to the two binding screws *T*, *T*. In front of the coil *C* there is a thin iron diaphragm *D* held firmly in a wooden cap and almost touching the end of the magnet.

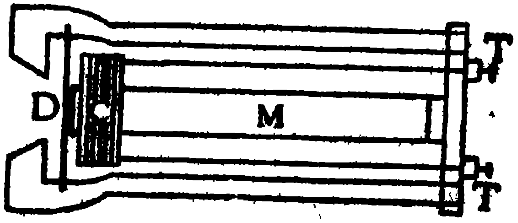


Fig. 93—The Telephone.

The conical opening in front of the diaphragm converges the sound waves directed to the diaphragm. The terminals T, T are connected to two line-wires at the distant ends of which a similar apparatus is connected.

Action.—The diaphragm, being just in front of the magnet, becomes magnetised by induction, and when the diaphragm is spoken into, it vibrates to-and-fro by the sound waves which fall on it, and so the number of lines of force passing through the coil is changed. Induced currents are thus set up in the coil. These currents travelling through the line-wires change the strength of the field in which the iron diaphragm of the distant telephone is situated and throw it into the ***same kind*** of vibration as the first diaphragm. Thus the second diaphragm reproduces the sound which caused the first one to vibrate.

The *Bell's Telephone* (Fig. 93) may be used both as a ***transmitter*** and as a ***receiver***. It is independent of any battery, but the induced current generated is so weak that it is not successful for a long distance. It has been found that Bell's telephone receivers can respond to a current of the order of 2×10^{-12} amperes. But with a weak current, the quality of sound received is poor.

A battery connected in the line will improve matters a little no doubt, but for proper reception an improved receiver is necessary. Such a modern receiver consists of a U-shaped magnet having one coil round each pole, the two coils being connected in series and the terminals finally joined to the line-wires. Both the poles are placed close to each other behind the diaphragm, so that line current will now traverse both the coils and the electro-magnetic action on the diaphragm will be almost doubled. **Headphones** are also constructed on this principle.

122. The Carbon Microphone :—*The Bell's Telephone* is not used as a transmitter now-a-days, but a transmitter working on a different principle called the **Carbon Microphone transmitter**, is used.

The ***microphone transmitter*** (Fig. 94) consists of a shallow chamber, loosely filled with granules of carbon, having two thin carbon plates, one C_1 at the front and the other C_2 at the back. The centre of a steel diaphragm D is rigidly fixed to the middle of the front carbon

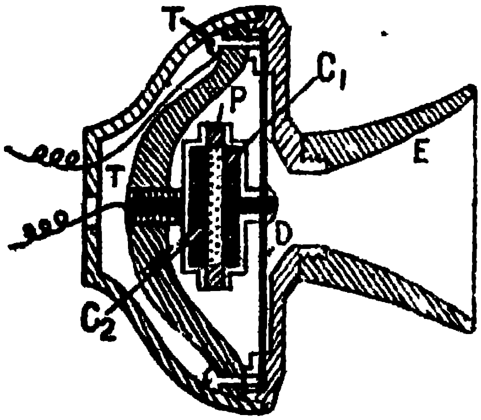


Fig. 94—The Carbon Microphone Transmitter.

plate. In front of the diaphragm is the conical or exponential mouth-piece or *horn* E which converges the sound-waves directed to it during a talk or speech.[*] The diaphragm is connected to one of the terminals T, T, the other terminal being connected to the back carbon plate. Usually, the plates C_1 and C_2 are separated at the edges by cotton wool pads P. The principle of this transmitter is that when the pressure at the surface of contact of two carbon particles varies, a large variation in the electric resistance of the contact takes place. So when a sound is received by the mouth-piece, the diaphragm sets into vibrations and these vibrations cause a pressure change in the carbon granules thereby producing a large variation of current. This variation of current, though large, does not amplify the speech because here the sound energy is not converted to electrical energy but only causes a considerable current variation is the battery circuit.

This variation of current is first stepped up by a transformer, then amplified by a suitable valve circuit and finally reproduced by a loud speaker.

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[*] **Horns or mouth-pieces.**—The mouth-pieces or horns may be conical or exponential in form The latter type is now-a-days commonly preferred because they radiate low frequencies more efficiently than the conical ones. Horns have two main functions: (*a*) intensifying and (*b*) directive. The intensification is effected by matching a sound source, such as the moving coil and diaphragm of a loud speaker system to the acoustic impedance of air. This results in a higher efficiency of energy transfer. Moreover, by special devices in the mouth and the air cavity the mechanical impedance of the diaphragm is also matched to the acoustic impedance of air. The directive property of a horn also depends on the size of the mouth. Recent works show that the directive property is much smaller than its intensifying property.

123. The Ribbon Microphone :—It consists of a very light aluminium ribbon or foil *A* corrugated horizontally and suspended in between two pole-pieces, *N* and *S*, of a permanent magnet. When the impinging sound waves strike the ribbon it is displaced perpendicularly to its length as well as to the magnetic field. So induced e.m.f.'s will be generated between its two terminals due to the change of flux in the ribbon as it moves. This e.m.f. may be properly amplified and reproduced in the usual way. Here the response is directional but the output is of an excellent quality.

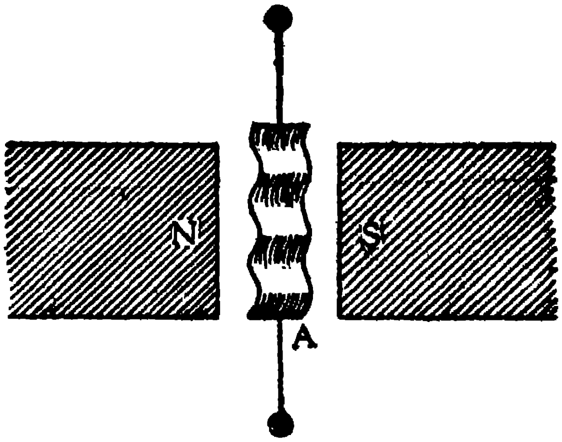

Fig. 95

124. The Condenser Microphone :—It is essentially a parallel plate air condenser, one plate (1) of which is rigid and the other (2) slightly flexible. The plate (2) is attached to the mouth-piece of the microphone. These plates are connected to the terminals of a *H.T.* battery through a high resistance *R*. When the sound concentrated by the mouth-piece *M* falls on the flexible plate (2) of the condenser, it is set to vibration thereby continually altering the distance between the plates. So the capacity also changes which causes momentary currents to flow through *R* during the vibration of the plate (2). Due to these currents pulses of p.d.'s will be generated across *R*. These p.d.'s have the same nature of variation as that of the displacement of the diaphragm plate (2) of the condenser provided variations of the capacity are very small compared to its mean value. These p.d.'s can be duly amplified and reproduced. In this type of microphone the response is uniform though weak.

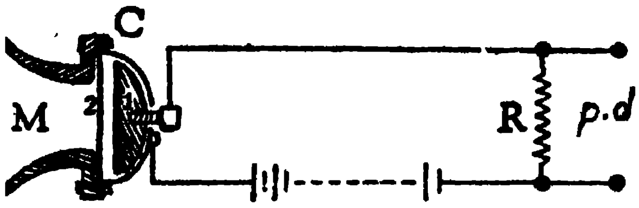

Fig. 96

125. The Moving-coil Microphone :—It consists of a pot magnet, the north pole N of which forms the central arm and is of the form of a right-circular cylinder, the south poles S forming an annular cylinder around N. The diaphragm D is small and relatively rigid. To the inner surface of it is attached a coil cc moving in the space between the pole-pieces. When due to the impinging sound waves the diaphragm vibrates, the coil cc will also vibrate in the magnetic field. So an induced e.m.f. is generated in it. This can be amplified and reproduced as usual. Here the output is considerable and of high quality.

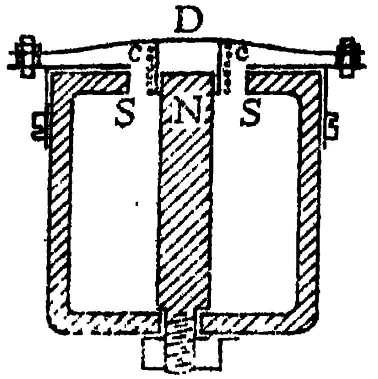

Fig. 97

126. The Crystal Microphone :—Just as in the crystal pick-ups, piezo-electric crystals like quartz or Rochelle salt are used here. As the impinging sound waves cause a variation of pressure on the crystal, a potential difference will be generated between two opposite faces of the crystal and hence on the electrodes attached to it. This is suitably amplified and reproduced. In order to increase the output more than one crystal slice is usually taken together in constructing a crystal microphone. This type of microphone has a highly directional property and the output is of an excellent quality.

127. Loud Speaker :—It is a device for transforming electrical energy into sound energy. The principle of its action is practically the reverse of that of a microphone. But a loud speaker is designed to handle considerable power in order to make the sound produced audible to a large audience. In the common types of the instrument, the alternating current from the amplifying valve circuit is passed through a small coil fixed to the middle of a diaphragm. This coil moves in a strong magnetic field due to the interaction of the current and the field. The current causes the diaphragm to vibrate with the frequency of the alternating current itself and to emit sound. To increase the efficiency sometimes a small diaphragm is used near the converging end of an exponential horn. But this

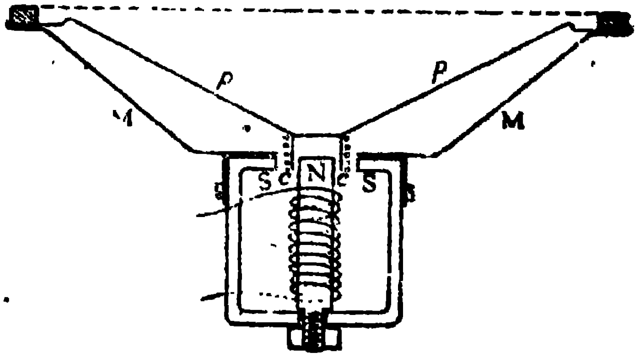

Fig. 98

horn fitted loud speakers are unsuitable for indoor purposes because of the size of the horn. So they are being speedily replaced by the type without the horn. In this type a large conical diaphragm of stiff paper (*pp*) (Fig. 98) with the *speech coil* (*cc*) at the apex is flexibly supported by a metal frame *MM*. The magnetic field is generally that due to a strong and permanent pot magnet. Occasionally electromagnets with its windings (shown by dotted coils) at the central arm may be used. The conical diaphragm is set in a large card board baffle which prevents the passage of sound from front to back and improves its low frequency response. Sometimes the cabinet in which the speaker element is fitted serves as the baffle.

128. Recording of Sound on Films :—Film recording may be done by two distinct methods. One is called the *variable density method* and the other the *variable width method*. The standard size of a cinema film is 35 mm. in width of which only 2·5 mm. is allotted for sound rcording. In the variable density method this full allotted space is utilised for recording and the record consists of a series of equal and transverse parallel lines of varying opacities.

In the earlier method based on the variable density principle, the amplified current from the microphone and amplifier circuit is made to vary the intensity of the light from a specially constructed neon lamp. The lamp illuminates a slit of constant width whose image is cast on the film. As the intensity of the lamp changes with the change of the amplified current, *i.e.* with the sound vibrations received in the microphone, the photographic records on the film will be the images of the illuminated slit of varying intensities. So on fixation, these images will have different densities or depths of silver deposits. The brightest image of the slit will be developed as the most opaque line. Now on making a positive of the film, actually the most transparent line on it represents the brightest image of the slit. So if light is allowed to pass through this positive print of the film through a silt, the transmitted light is found to follow the same nature of time variation as was the case with the light emitted by the neon lamp.

In the later methods based on the variable density principle, the amplified current from the microphone is allowed to vary the width of a slit (S_1) by an electro-magnetic-device as shown in Fig. 99. As the electro-magnets E.M. are actuated by the amplified current from the microphone, the slit formed by two strips of duralumin will change due to the fluctuation of the current. Light from a strong source of constant brilliance is first passed through this varying slit and thence through another slit S_2 of constant width, whose image is finally cast on the film with the help of the converging lenses LL as shown in Fig. 99. The varying slit here mainly functions as the modulator of the brightness of the light beam transmitted through it.

The variable width method of film recording consists in illuminating a variable length of the slit. The record on the film is an image of this slit. The amplified current from the microphone is passed through a string galvanometer or a Duddell oscillograph fitted with a reflecting mirror. A beam of light is reflected from this mirror M on to slit S and is focussed on the film. As the string of the galvanometer or the oscillograph vibrates in the magnetic field due to the amplified current which varies with the intensity and the quality of the sound received by the microphone, the mirror is rotated. This rotation causes the reflected beam to illuminate a variable length of the slit only. Hence in the film record, some portion of the space corresponding to the slit length will be exposed and some part will

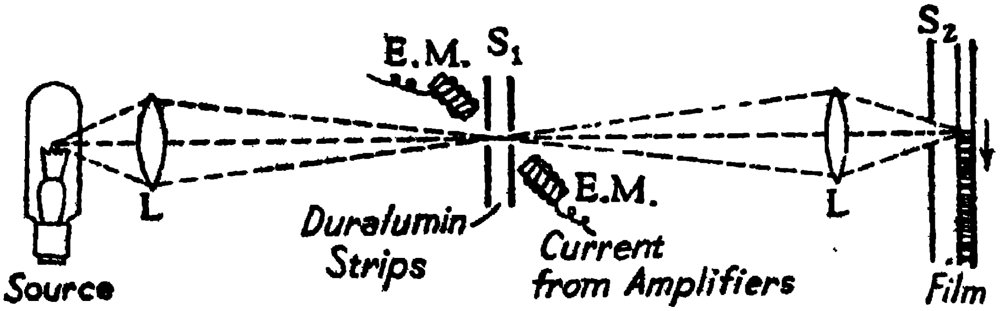

Fig. 99

remain unexposed. In the negative print of the film the opaque portion is proportional to the current in the amplifier circuit. In the positive print the transparent region will represent the sound record of a uniform density. It has the appearance of a serrated edge occupying different widths in different positions.

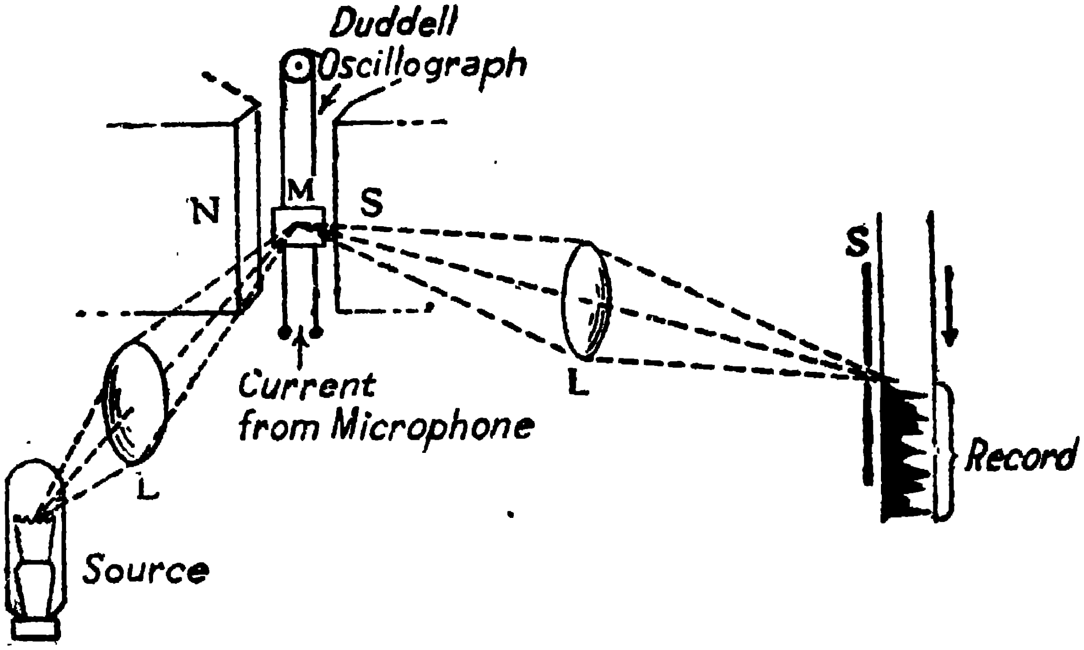

Fig. 100

129. Film Reproduction :—The method of reproduction is the same for both the variable density and the variable width type of film records. Here the light from an exciter lamp at *S* is focussed on to the sound tracks of the film by means of the converging system of lenses *L*. The light transmitted through the track falls on the photo-sensitive surface of a photo-electric cell, whereupon electrons are emitted from this surface. These electrons are collected by the

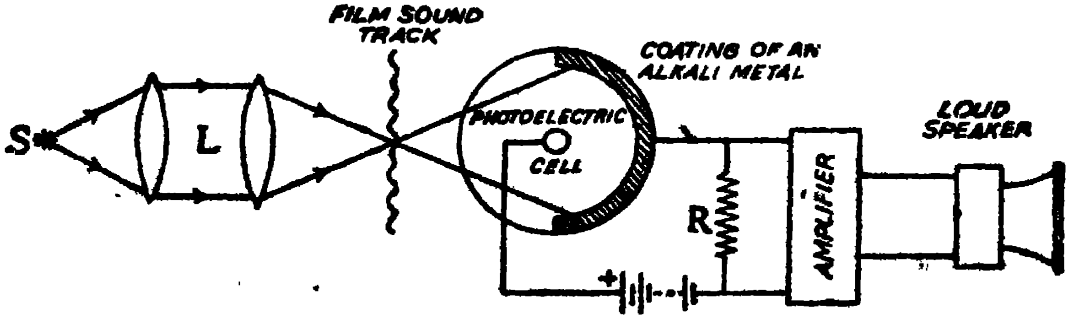

Fig. 101

positive electrode of the photo-electric cell. The sensitive surface and the collecting electrode are maintained at a potential difference by means of a battery through a high resistance *R*. Due to the emission of electrons by the sensitive surface and the collection of the same by the positive electrode, a small current variation occurs in the circuit. This causes a variation in *p.d.* across *R*, which is proportional to the light received by the photo-cell through the film. The variation of *p.d.* across *R* is then amplified by a valve circuit and this amplified *p.d.* operates the loud speaker which practically reproduces the same sound as was used for recording.

130. Magnetic or Steel-tape Recording and Reproduction :— In magnetic recording of sound a steel tape or wire or sometimes a plastic tape impregnated with iron oxide (as used in the German Magnetophone) is continuously moved between the pole-pieces of an electro-magnet whose coil is energised by the speech current from the microphone properly amplified by a valve circuit. The tape is thus longitudinally magnetised in such a manner that the variations in magnetisation exactly correspond to the variations in the speech current. These pole-pieces generally known as the ***recording head*** are fitted with knife-edges [Fig. 101 (*a*)]. This arrangement increases the strength of the magnetising field and the magnetic records become, as a result, more prominent. During recording the tape is to be moved at a uniformly high speed.

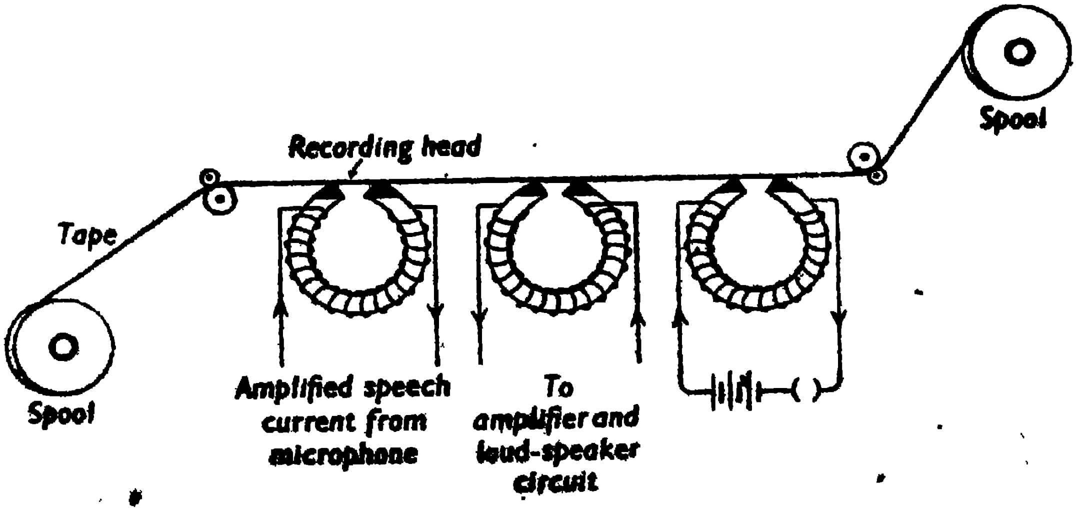
Recording head
Spool
Tape
Spool
Amplified speech current from microphone
To amplifier and loud-speaker circuit

Fig. 101 (*a*)

For reproduction, which in this case is possible even immediately after recording, the tape is moved at a uniformly high speed past a *reproducing head* exactly of the same design as the recording one. As the tape moves, its magnetisation causes changes in flux in the knife-edge pole-pieces of the reproducing unit. This induces an e.m.f. in the coil surrounding the corresponding magnet. This e.m.f. is then amplified by an electronic device and sent on to the loud speaker, which exactly reproduces the original sound practically without any distortion. The tape once used for recording may be used once again after moving it between the '*wiping head*' which is nothing but the pole-pieces of a strong direct current electro-magnet. It smooths out the non-uniformity of magnetisation in the tape and magnetises the same uniformly thereby making it ready for a fresh record. In modern tape recorders the pole-pieces of the same electro-magnet serve both as the recording as well as the reproducing head.

The reproduction in a tape recording device is as faithful as in a gramophone record or even more. This is why the method of magnetic recording is now-a-days increasingly being used in broadcasting and in film studios.

Recently suitably coated discs have also been used for magnetic recording but its reproducibility is not of the same order as that in a tape record.

131. Acoustics of Auditoriums :—

Introduction.—Before the advent of the twentieth century, the halls and lecture theatres were constructed without attaching any importance to their acoustical property. So after construction, some of them were found to be quite good from the acoustical standpoint while the others were not. That the long-drawn echoes or reverberations in a hall are the main cause of its acoustical failures was, however, realised by scientists at the end of the last century.

Reverberation.—W. C. Sabine in America, first systematically studied the different acoustical properties of a room. It was observed that the presence of various flat surfaces like walls, ceiling, etc. in a room lengthens the time of duration of sound on account of successive reflections from those surfaces The prolonged echo or the undue persistence of sound due to successive reflections is known as reverberation. Sabine showed that if the time of reverberation be properly controlled, acoustical utility of the room can be raised to a great extent. He showed that if the duration of reverberation after the source is cut off be such that within this interval a sound falls in intensity to a millionth part of its original value at the cut-off, (*i.e.* if the intensity decay be 60 decibels), the reverberations will not affect the acoustical suitability of a room. This interval is known as the ***standard time of reverberation.*** Sabine experimentally found out an empirical relation for this standard time of reverberation. In 1903, Franklin theoretically deduced an expression for the same which was found to be in agreement with Sabine's relation. Franklin's deduction may be given as follows—

Let E = the energy emitted by a source of sound per second,

V = the volume of the space occupied by the room,

$\therefore$ The volume density of energy supply per second $= E/V$.

Now, if, I = the average intensity of sound in the room,

α = the mean absorption coefficient $\left(i.e.\ \dfrac{\text{energy absorbed}}{\text{energy supplied}}\right)$

per reflection and n = number of reflections per second, then out of the total supply, the energy absorbed per second is $I\alpha n$.

$\therefore$ The rate of increase of the intensity together with the rate of absorption per second is equal to the rate of energy supply per second.

$$\therefore \quad \frac{dI}{dt} + I\alpha n = \frac{E}{V} \qquad \ldots \qquad .. \quad (1)$$

Statistically it can be shown that if C be the velocity of sound and S the surface area of the walls of the room, the number of reflections per second is $n = CS/4V$.*

So eqn. (1) can be written as,

$$\frac{dI}{dt} + \frac{C\alpha S}{4V} \cdot I = \frac{E}{V};$$

$$\text{or,} \quad \frac{dI}{dt} + \mu I = \frac{E}{V} \qquad \ldots \qquad \ldots \quad (2)$$

$$\text{where } \mu = \frac{C\alpha S}{4V}.$$

[* **Notes.** – The number of reflections suffered by the sound waves emitted by a source inside a closed room can be calculated in the following way :

If ρ be the energy density of the sound waves produced and if V be the volume of the room, then supposing that there is no loss due to absorption etc., the total energy, W, received by the walls of the room per second is $W = nV\rho$.........(1), because in each reflection the energy flux sweeps the whole volume of the room once, supposing for simplicity only the forward propagation of the waves. Here n represents the number of reflections per second.

The energy received by the walls per second can also be calculated by taking an element of wall surface ds at O, on which the energy impinges on its exposed upper side alone [Fig. 102(*a*)]. Now consider an elementary volume dv in the room, where

$$dv = r d\theta .\ dr .\ r \sin\theta d\phi ,$$

as shown in the figure. As ρ is the energy density, the energy contained in this volume or the energy flux passing through this volume $= dW = \rho r^2 . dr . d\phi .\ \sin\theta .\ d\theta .$

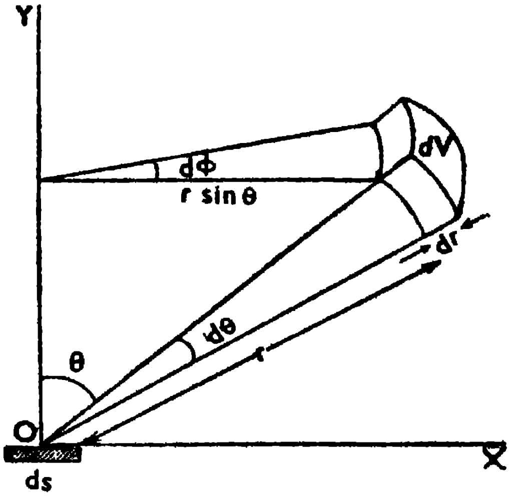

Fig. 102(*a*)

Out of this energy, the portion reaching ds is $(d\omega/4\pi) \times dW$, where $d\omega$ = solid angle contained by ds at the centre of the volume element $(ds \cos\theta)/r^2$. So the energy falling on ds per second from the upper side $= \int dW . \frac{d\omega}{4\pi}$, (integration extending over all variables)

$$= \rho .\ \frac{ds}{4\pi} \int_0^{2\pi} d\phi . \int_0^{\pi/2} \sin\theta . \cos\theta\, d\theta . \int_0^{c} dr . \quad c = \text{velocity of sound})$$

$$= \rho .\ \frac{ds}{4\pi} \times 2\pi \times \tfrac{1}{2} \times c = \rho \frac{c\, ds}{4}$$

Summing up for the whole surface area of the walls, the total energy received the walls per second is $W = \frac{\rho c}{4} \Sigma ds = \frac{\rho c s}{4}$ ... ... (2),

where S = total surface area of the room.

So equating (1) and (2),

$$n \rho V = \frac{\rho c s}{4} ,$$

Hence the number of reflections per second is $n = \frac{cs}{4V}$].

(a) Growth of intensity or energy density.—

To solve eqn. (2), let us rearrange it and write,

$$\frac{dI}{\frac{E}{V} - \mu I} = dt \text{ ; or, } -\frac{1}{\mu} . \frac{d\left(\frac{E}{V} - \mu I\right)}{(E/V - \mu I)} = dt .$$

$$\text{Integrating, } -\frac{1}{\mu} \log_e \left(\frac{E}{V} - \mu I\right) = t + k \qquad \ldots \qquad \ldots \quad (3)$$

where k is the integration constant, which can be calculated from conditions of the problem.

Now, when $t=0$, $I=0$.

$$\therefore\ k=-\frac{1}{\mu}\log_e\frac{E}{V}.$$

Substituting in (3), $\log_e\frac{E/V}{E/V-\mu I}=\mu t$;

$$\text{or, }\log_e\frac{E/V-\mu I}{E/V}=-\mu t\ ;\ \text{or, }1-\frac{\mu VI}{E}=e^{-\mu t}.$$

Hence, $I=\frac{E}{\mu V}\left(1-e^{-\mu t}\right)$ ... ... (4)

Thus intensity tends to acquire a maximum value $I_m=\frac{E}{\mu V}$ when $t\to\infty$.

$\therefore$ The intensity at any time t is given by $I=I_m(1-e^{-\mu t})$ and the curve of the intensity growth is given by Fig. 102(b).

(b) Decay of intensity.—

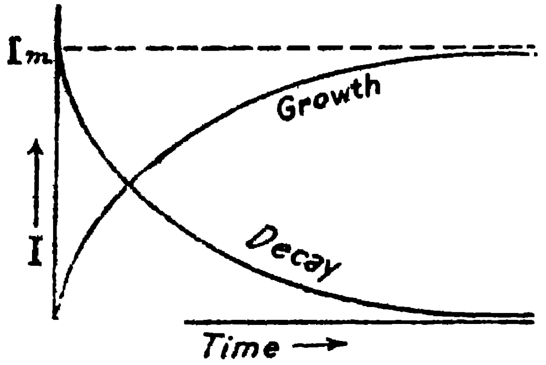

Fig. 102(b)

Here the energy supply is zero. $\therefore\ E=0$.

$$\therefore\ \frac{dI}{dt}=-\mu I.$$

Integrating, $\log_e I=-\mu t+k$.

Since at the moment of cut-off $I=I_m$ and $t=0$, $k=\log_e I_m$.

$$\therefore\ \log_e\frac{I}{I_m}=-\mu t$$

$$\text{or, }\frac{I}{I_m}=e^{-}\qquad\text{... (5)}$$

So the decay curve of intensity will be as shown in Fig. 102(b).

(c) The time of reverberation—The standard time of reverberation, is the time for the energy density (or the intensity) to decay to one-millionth of its maximum value at the cut-off. Hence from (5), the time of reverberation, T, can be obtained as follows :

$$\text{We have, }\frac{I}{I_m}=\frac{1}{10^6}=e^{-\mu T}\ ;$$

$$\text{or, }e^{\mu T}=10^{+6}\ ;\ \text{or, }T=\frac{1}{\mu}\log_e 10^{+6}$$

Putting the value of μ, $T=\frac{4V}{C\alpha S}\times 2{\cdot}303\times 6$.

Taking $C=340$ metres/sec.,

$$T=\frac{4\times 2{\cdot}303\times 6}{340}\times\frac{V}{\alpha S}={\cdot}16\frac{V}{\alpha S}.$$

If the dimensions be considered in the *F.P.S.* system,

$$T=\frac{\cdot 05V}{\alpha S}.$$

Thus T is directly proportional to V and inversely as the surface area and its absorption coefficient.

Sabine determined the absorption coefficients of different materials taking the ***absorption of an open window of unit area to be unity (perfect absorber)*** *and experimentally found that the standard time of reverberation is actually $\cdot 05V/\alpha S$ where the measurements are made in *F.P.S.* units. So the above theory is found to be correct. Recently Eyring from a different approach deduced an expression for the standard reverberation time which is found to be identical with Franklin's expression when the absorption coefficient is small.

Since in a room V and S are normally constant, mainly by choosing suitable absorbents the reverberation time is controlled. Moreover, the audience, chairs, benches, and other structures not only absorb some sound energy (*i.e.* α increases) but also ***diminish effectively the value of V and increase S.*** So T is further controlled by these factors to some extent. The value of T should not be reduced unduly (*i.e.* a room should not be over-corrected for reverberation), for in that case there will be deadening effect on the sound produced in the room. It has been found that for good audibility, the optimum time of reverberation is different for different types of acoustical performance and also depends on the size of the halls and vary generally from 1 second to 2·5 seconds in most of the cases. For good audibility of speech the time of reverberation is smaller than for music. To secure this optimum time, the wall or the floor may be covered with absorbent materials of proper thickness. Sometimes additional absorbing (or damping) structures of suitable materials are installed in the rooms. The absorbing material should be chosen according to the frequency of the sound emitted because the absorbing power depends on the frequency of the incident wave.

The acoustics of the recording and broadcasting studios involves a varieties of problems. Here a short time of reverberation is desirable and at the same time excess absorption is considered to be objectionable. It has been suggested by Olson and Massa that the effect of diminished reverberation may be secured by changing the proportion of direct sound to reflected sound. Good results are

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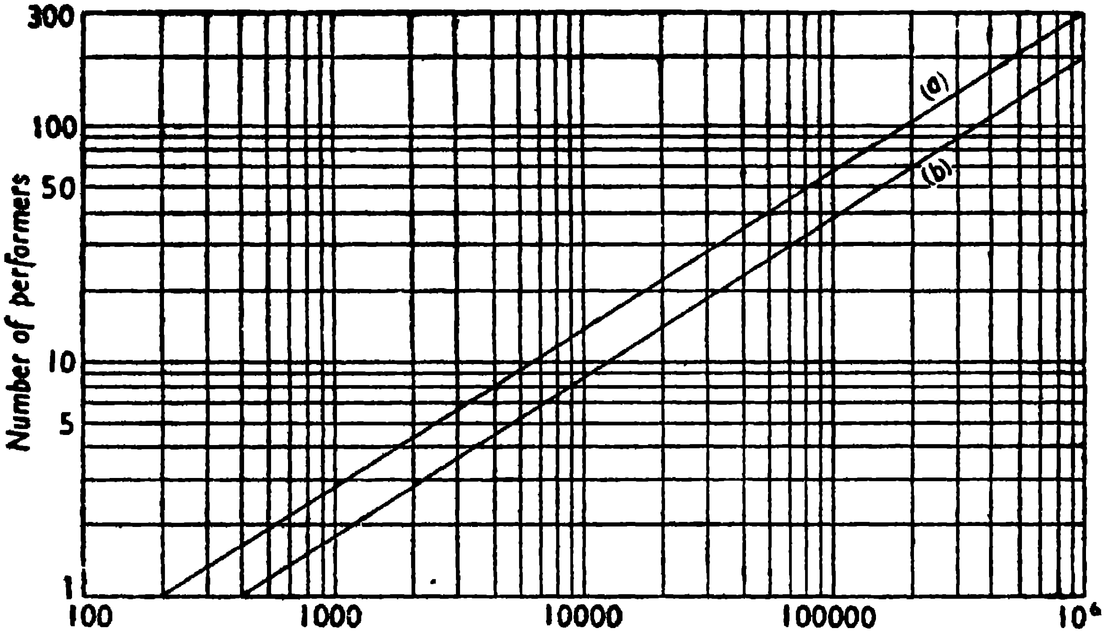
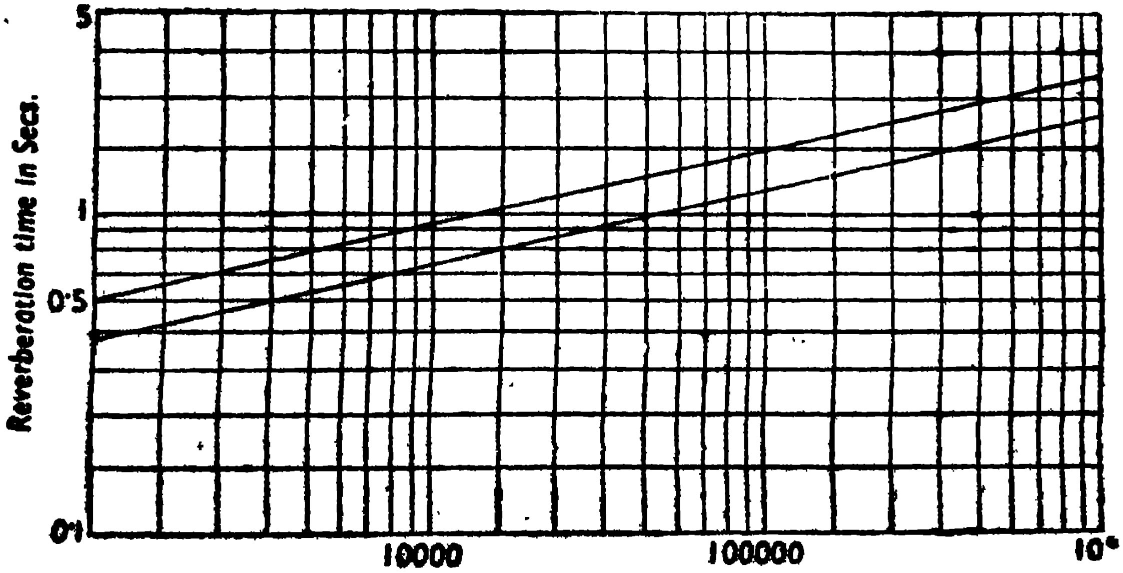
* The unit of absorption coefficient is the *Sabine*. So the absorption by 1 sq. ft. of a material of absorption coefficient 0·7 sabine is equivalent to 0·7 sq. ft. of an open window.



Volume of studio in cu. ft.
(*a*) Maximum number of performers,
(*b*) Optimum number of performers.
Fig. 102(*c*)



Volume of studio in cu. ft.

Fig. 102(*d*)—Optimum reverberation time for studios.

obtained by using a directional microphone which discriminates between the direct sound from the source and the reverberated sound. In the matter of time of reverberation at the standard frequency of 512, the design of B.B.C. is based on the graphs shown in Figs. 102 (*c*) and (*d*). The graphs in Fig. 102 *d*) show the relation between the size of the studio and the time of reverberation whereas Fig. 102(*c*) shows the relation between the size of the studio and the number of performers.

Besides the time of reverberation, the shape of the room also plays an important part in auditorium acoustics. In large rooms, the conditions of (*i*) adequate loudness, (*ii*) uniform distribution and (*iii*) absence of echoes should be secured. The rooms should also be protected against the external noise. This is effected by non-homogeneous double or multiple walls.

[*Deduction of time of reverberation from dimensional equation :*

Sabine experimentally showed that the time of reverberation depends on the volume V of the room, the effective reflecting area S and the velocity of sound C Thus, we can write,

$$[t]=[V^x S^y C^z].$$

$$\text{Again,}\quad V=L^3,\ S=L^2 \text{ and } C=LT^{-1}.$$

$$\therefore\quad T=L^{3x}L^{2y}L^{z}T^{-z}=L^{3x+2y+z}\ T^{-z}.$$

$$\therefore\quad z=-1 \text{ and } 3x+2y-1=0.$$

$\therefore\quad 3x+2y=1.$ The simplest solution is evidently $x=1,\ y=-1.$

$\therefore\quad t=\frac{KV}{CS},$ where K is to be found from experiment.]

Absorption of sound : Coefficient of Absorption :—

It is evident from above that in order to control the reverberation time to optimum value, the absorbing properties of the materials used in the room and those of the walls of the room as also its volume are of utmost importance. The absorption of sound mainly deals with the dissipation of the acoustical energy into heat energy. It is affected by porosity of the bounding surfaces and flexural vibrations. The theory of the action of pores accounts for the high coefficient of absorption for felt, rugs, carpets etc. The absorption coefficient is also found to increase with the thickness of the material upto a certain limiting value. Flexible materials if not rigidly mounted are set in vibrations and the damping forces dissipate a considerable amount of sound energy. The coefficient of absorption generally increases with frequency. But some materials are found to absorb strongly in the low frequency region, while others in the high frequency region. So a suitable absorbent may be chosen which is effective in regions of high pitch. This would increase the relative importance of fundamentals and conduce to purity of tone. Again to increase the brilliance of tones, a low frequency absorbent may be chosen which would emphasise the higher partials. There are some absorbents called the ***straight-line absorbents*** which absorb equally strongly throughout the range of audible frequencies. The straight-line absorption principle was attempted in the construction of some of the studios. But the best result has been obtained when the absorption is relatively constant only upto a frequency of 600 to 700 and then increased slightly in the higher frequency range.

As absorption is one of the most important properties involved in building acoustics, the measurements of the absorption coefficients of different materials were undertaken in various laboratories during the earlier part of the present century. By absorption coefficient α, we mean the ratio of the energy absorbed to the energy supplied. Sabine defined it in terms of the open window. The **absorption coefficient** of a substance is the ratio of the energy absorbed by unit area of the substance per second to the energy that escapes through an open window of unit area in the same time. Some of the standard methods of determining the coefficient of absorption α are given below.

(a) Determination of α by direct comparison.—

In this method, the absorbing material is spread out in a hall and its area is determined is square feet. Then the reverberation time is found out under this condition. The absorbent is then removed and some windows are opened by trial till the reverberation time once again assumes the same value.

Hence we get, $$t=\frac{\cdot 05V}{\Sigma\alpha_\omega S_\omega+\alpha S}=\frac{\cdot 05V}{\Sigma\alpha_\omega S_\omega+A}.$$

$$\therefore\quad \alpha S=A\;;\ \text{or, }\alpha=\frac{A}{S}.$$

[illegible]dow in square feet ; α, S are the [illegible] material under test ; $\Sigma\alpha_\omega S_\omega$ [illegible] walls and furniture in the room.

(b) Determination of α by varying the Power output.—

In this method, the average value of α is determined with the help of ear only by varying the power output of the source. Let two sources of power P_1 and P_2 be taken. If now the time-intervals t_1 and t_2, during which the steady value of the energy density I falls to the bare audibility in both cases be noted, then we shall have,

$$I_a = {}_1I_m e^{-\mu \times t_1} = {}_2I_m e^{-\mu \times t_2} \qquad \ldots \qquad \ldots \quad (6)$$

where I_a is the limiting audible energy density and ${}_1I_m$ and ${}_2I_m$ the maximum energy densities of the two sources.

From eqn. (5), we know $I = I_m e^{-\mu t}$,

where $\mu = \frac{C\alpha S}{4V}$ and $I_m = \frac{E}{\mu V} = \frac{4E}{C\alpha S}$, E being the energy emitted by the source per sec. *i.e.* its power P.

$$\text{Hence, } I_m = \frac{4P}{C\alpha S}.$$

$$\therefore \quad \text{From (6), } I_a = \frac{4P_1}{C\alpha S} e^{-\frac{C\alpha S}{4V}t_1} = \frac{4P_2}{C\alpha S} e^{-\frac{C\alpha S}{4V}t_2};$$

$$\text{or, } \frac{P_1}{P_2} = e^{-\frac{C\alpha S}{4V}(t_2 - t_1)};$$

$$\text{or, } \log\frac{P_2}{P_1} = (t_2 - t_1) \times \frac{C\alpha S}{4V} \qquad \ldots \qquad \ldots \quad (7)$$

From relation (7), the mean absorption coefficient α can be determined for the room in which the experiment is conducted. The absorption coefficients of different materials may similarly be determined by measuring the reverberation periods for two sources both with and without the materials in the room.

(c) Determination of absorption coefficient α by the Stationary wave method.—

It has been shown by Taylor that if b/a is the ratio of minimum and maximum amplitudes observed in the stationary wave formed by interference between the direct and reflected sound waves, the absorption coefficient is given by,

$$\alpha = \frac{4}{\left(2 + \frac{a}{b} + \frac{b}{a}\right)}.$$

Using his expression the absorption coefficients of materials were determined recently by E. T. Paris. The arrangement consists of a large earthenware-pipe K, E [Fig. 102(e)] closed at one end by the specimen S of the material, which has a backing B of a good reflecting material. At the other end a loud speaker L supplies a sound of constant intensity and frequency from a valve oscillator circuit. With the help of a calibrated hot wire microphone M the minimum and maximum amplitudes along the tube axis are determined.

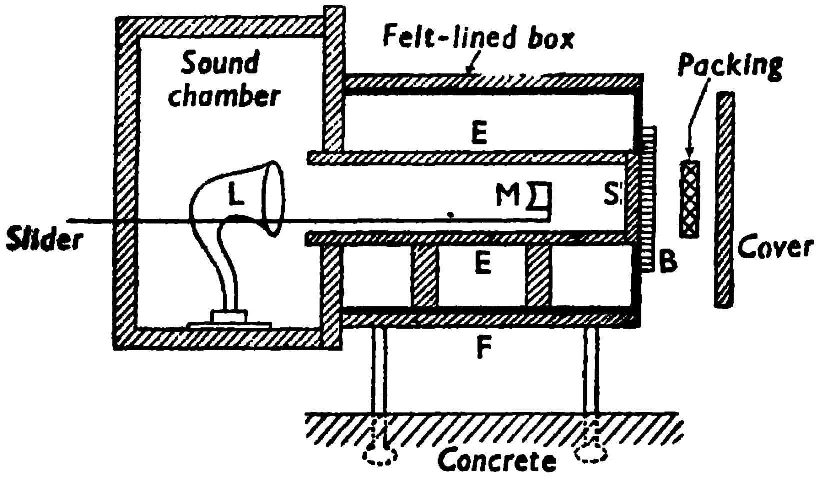


Fig. 102(e)

Hence b/a being known, α can be computed. The method is simple and direct. In a modified form it has been used in the British N.P.L.

Table of Absorption Coefficients of Important Materials :—

Open window	1·00	Brick wall	0·03
Felt	0·70	Varnished wood	0·03
Carpet	0·25 to 0·40	Plaster	0·03
Plane wood	0·06	Glass	0·027

132. A general review of the Acoustics of Buildings :—Up to the end of the last century, the building, especially the lecture theatres, halls and concert rooms, etc., were constructed without due considerations as to their utility and suitability from acoustical point of view. It was W.C. Sabine in America who for the first time made a systematic study of the acoustical properties of rooms.

It has been found that the conditions for a room to be good from acoustical standpoint are as follows : (1) There should be no echo formation in the room, (2) the loudness should be uniform and sufficient throughout the room, (3) the time of reverberation should be optimum, (4) there should be no resonance effect anywhere in the room and, (5) the room should be sufficiently soundproof. The echoes may be minimised by subdividing large plane surfaces into smaller units, avoiding pronounced curves and by using absorbing materials. The loudness may be increased by placing reflectors at suitable positions in the room. It is a well-known fact that the various flat surfaces in a room like its walls, floor, etc. reflect sound only with a small loss of energy. So whenever a sound is produced inside a room, the sound waves after reaching these flat surfaces, are reflected back. In fact, before these reflected components die out, quite a good number of reflections may occur. A listener thus gets the sound by various paths and the sound may appear to him as a ***long-drawn echo or reverberation.***

After systematic observations Sabine concluded that the choice of a correct time of reverberation is the most important factor that makes a hall acceptable as a lecture theatre or as a concert room. He defines the '***standard time of reverberation***' as the time taken by a sound to fall in intensity by a factor of a million after the stopping of the source (or as the time taken for a decay of intensity of 60 decibels). Sabine empirically showed that this standard time of reverberation for a room of volume V, of exposed surface area S, having a coefficient of absorption α, is given by $T = \cdot 05 \dfrac{V}{\alpha S}$, where the dimensions of measurements are in *F.P.S.* units. Franklin theoretically corroborated the above relation. When the time of reverberation is longer, the original sound together with the reverberated sounds will create a discordant and confusing effect and the intelligibility will be hampered. Again if this time be too small the room would turn sound-dead. So there is an optimum time of reverberation for the best intelligibility. This is different for different rooms depending on its capacity and the use for which it is designed. For average men this time varies from ·5 sec. to about 1·5 seconds.

The duration of reverberation depends on the various reflecting structures present in the room. If these are good reflectors, a longer time will be required for the decay of the successive echoes. But if the room contains a number of sound absorbers like screens, carpets, etc., the reverberation time will be sufficiently reduced. The acoustical absorbing powers of different materials have been systematically determined and it has been found that acoustic plaster, fibre boards and cabot quilt are specially suited as absorbers in acoustical engineering. So by properly inserting *'open windows'* which are perfect absorbers of sound and other suitable materials in the very structure of the room, its time of reverberation may be brought to the desired value.

Again the intelligibility of speech is affected by loudness. If sufficient absorbing device is set up in the room, the reverberation time is short and there is very little confusion of syllables even though the intensity is low. In cinema halls the reverber tion time is sufficiently reduced because the natural voices are magnified by loud speaker. If the natural voice is to be relied on, there should be always some arrangements for reflection so that the necessary loudness is secured, as has been already mentioned. While designing a concert hall the seats must be made of such absorbing materials that the reverberation time remains constant both when the hall is empty for rehearsal and when it is packed up with audience wearing garments of wool or cotton which are good absorbents. Concave reflectors, which were formerly used in big halls, are not desirable from the acoustical viewpoint as they may cause a focussing effect. As stated before it is better to have small rigid reflectors distributed throughout the hall at different inclinations. This ensures a better intensity distribution. Resonance anywhere in the room or of the air contained in the room should be heavily damped to avoid distortion. Some amount of sound-proofing may be necessary in a room. This is done by constructing massive walls, tight-fitting doors and windows, etc.

When a building is meant for acoustical purposes, the acoustical properties of its design should always be investigated by making small models and studying them with the help of high frequency sound waves.

Questions (Chapter XI)

1. **Describe the construction and operation of a gramophone. (C. U. 1951)
What is the principle of the electric pick-up in a radio-gramophone ?**

2. **Write a short note on 'loud speaker'. (Luck. U. 1949 ; All. U. 1951)**

3. **Describe different types of microphones and loud speakers that are commonly used.**

4. **Give a short account of the production and exhibition of sound motion pictures. (All. U. 1952)**

5. **Give an account of how sound is recorded and reproduced in talkies. (C. U. Hon. 1953)**

6. **Discuss and enumerate the factors which will make an auditorium suitable for good audibility, and explain the importance of the nature of the surface of the walls, ceilings, furniture and windows. (Luck. U. 1944)**

7. **Give an account of the causes of bad acoustical properties of a hall. Discuss the methods of remedying these defects. (All. U. 1948 ; Raj. U. 1948)**

8. **Write a short note on the acoustics of buildings. (East Punj. U. 1955)**

9. **What are the acoustic requirements of a good auditorium ? Discuss the theory of reverberation. (C. U. Hon. 1952)**

10. **Explain the various acoustic defects commonly found in a large hall. How are these defects removed ? (All. U. 1953)**

11. **Write a short note on the acoustics of an auditorium. (All. U. 1947 ; Bom. U. 1954)**

12. **Briefly describe two different methods of determining the coefficient of absorption of an acoustical absorbent.**

CHAPTER XII

ULTRASONICS

183. Ultrasonics :—Sound waves having frequencies above the limit of audibility of human ear are known as ***ultrasonic*** or ***supersonic*** waves. The audibility limit being about 20 kilocycles per second, sound waves of frequencies higher than about 20,000 per second come under the category of ultrasonic waves. Modern methods can produce ultrasonics of frequencies upto 5×10^{5} kilocycles per second. So the shortest ultrasonics may have a wave-length, $\lambda=\frac{V}{n}=\frac{33000}{5000\times10^{5}}$ cm $=6{\cdot}6\times10^{5}$ cm. $=6600$ Å, which is of the order of the wave-lengths of visible light. Because of their small wave-lengths, it has been possible to use ultrasonics in various practical fields. The acoustical laws valid for the audible range are also true for the ultrasonic range. Over and above this, in supersonic range newer phenomena, which are not detectable in the audible range, are observed.

Note :—Occasionally the term ***supersonic*** is used in relation to velocities higher than that of sound. The rockets, planes etc. possessing speeds greater than that of sound are commonly known as ***supersonic*** rockets and ***supersonic*** planes.

184. Production of Ultrasonics :—Ultrasonics are produced generally by any one of the ***three*** types of generators described below—

(1) Mechanical Generator.—(a) Small tuning forks having prongs only a few millimetres long can produce vibrations of frequencies up to 90,000 cycles per sec.

(b) Galton Whistle.—This instrument as designed by Edelmann gives vibrations of frequencies up to 100,000 c p.s. Here the puff of air passes through the nozzle, R, on to an annular slit c, from which it streams against a circular knife-edge D. This sets the air in vibration and the supersonic waves are produced. Here the frequency of the note can be altered by varying the volume, V, of the vibrating air-column with the help of the moving piston Z operated by a micrometer screw arrangement (Fig. 103).

(2) Magnetostriction Type Generator.—Here the principle of transformation of alternating electric current into sound is taken help of. By this method frequencies of the order of 20 to 50 kilocycles per second can be secured.

If a rod or a tube of ferromagnetic substance is placed in a magnetic field parallel to its length, the latter is slightly shortened. This is known as magnetostriction. As the effect is rather irregular in steel or soft iron, annealed cobalt or nickel rods are generally used. The best effect is produced when the alternating magnetising field due to a coil wound over the rod is adjusted to be in resonance with the natural elastic vibration of the rod. In such a case of resonant vibration, the frequency is given by $n=\frac{k}{2l}\sqrt{E/\rho}$, where $k=1, 2, 3$, etc., l=the length of the rod, E=the Young's modulus of the material of the rod and ρ=its density.

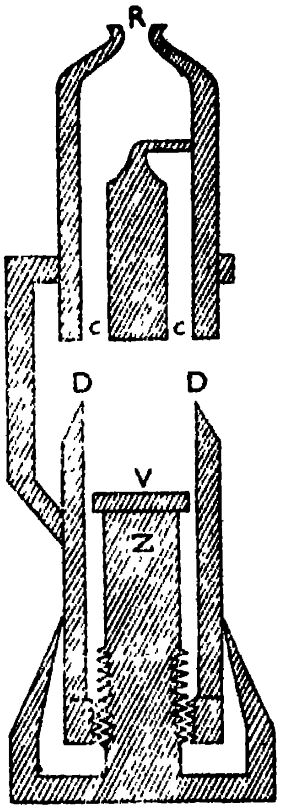


Fig 103

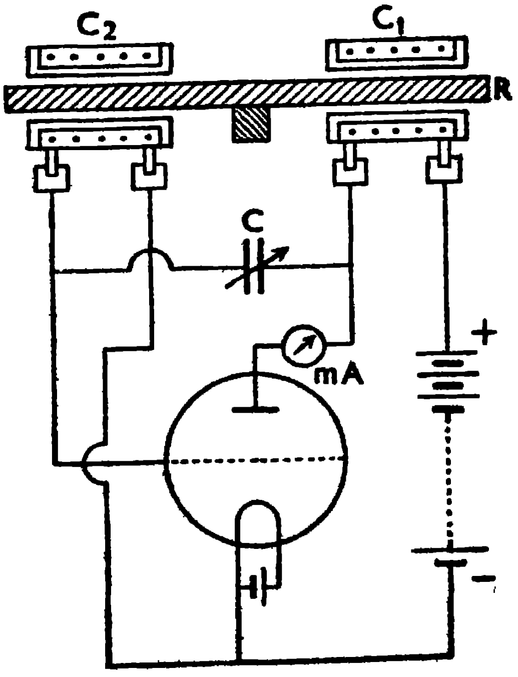


Fig. 103 (*a*)

Fig. 103(*a*) shows a simple arrangement of a magnetostriction generator or oscillator. The vibrating nickel rod *R* is clamped in the middle and rests axially in the two magnetising coils C_1 and C_2, C_1 being connected to the plate and C_2 to the grid of a valve oscillator. The rod may be permanently magnetised by a permanent magnet placed near it or by the plate current itself. The frequency of the oscillatory circuit is tuned to that of the natural frequency of the rod *P* by adjusting the variable capacitor C. This tuning is indicated by sudden change in plate current, when the oscillation sets in. The oscillations are maintained by the alternating reaction of the stresses in the bar on the state of magnetisation.

(3) **Piezo-electric Generator.**—With the help of a properly cut piezo-electric crystal high frequency electric oscillations can be transformed into mechanical oscillations. Piezo-electric crystals are unsymmetrical in shape and have the property of expansion along one axis and of contraction along the other when subjected to an electric field. It was first discovered by J. and P. Curie that when such crystals are subjected to pressure on opposite faces, electrical charges are developed on the faces at right angles to the direction of applied pressure. The converse effect which is taken help of in the piezo-electric generators is also found to be true. This effect is best shown in slices of the crystal cut in a particular direction only. Quartz is one of such piezo-electric crystals. The long axis of the crystal with hexagonal perimeter forms an optic axis [Fig. 104(*a*)]. A portion cut from the crystal by two planes perp. to this axis has the form of a hexagonal cylinder [Fig. 104(*b*)]. The lines joining opposite angles of this hexagon give the directions of possible electric axes. The final experimental slice is to be cut perpendicular to one of these axes.

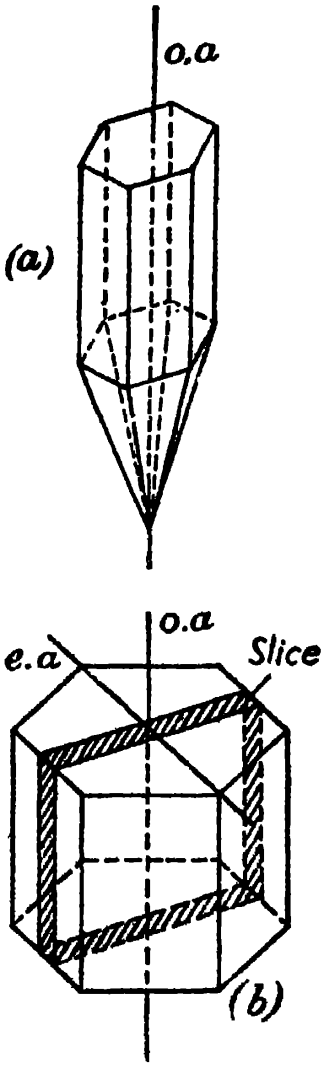


Fig. 104

The slice so obtained is to be mounted between two metal electrodes. It is then set in vibration with the help of undamped electric oscillations. These oscillations may be conveniently generated at any desired frequency and intensity by means of an electronic device. The simplest circuit consists in connecting the crystal in parallel with the condenser of the oscillating Hartley's circuit (Fig. 105). The oscillating circuit consists of a self-inductance and a variable condenser K_v, by varying which the circuit is tuned to the natural frequency of the quartz oscillation Q. This vibrating quartz generates ultrasonic waves in air whose frequency may be of the order 5×10^8 c.p.s. To set up high frequency vibrations in solid and liquids the quartz oscillators are generally taken help of.

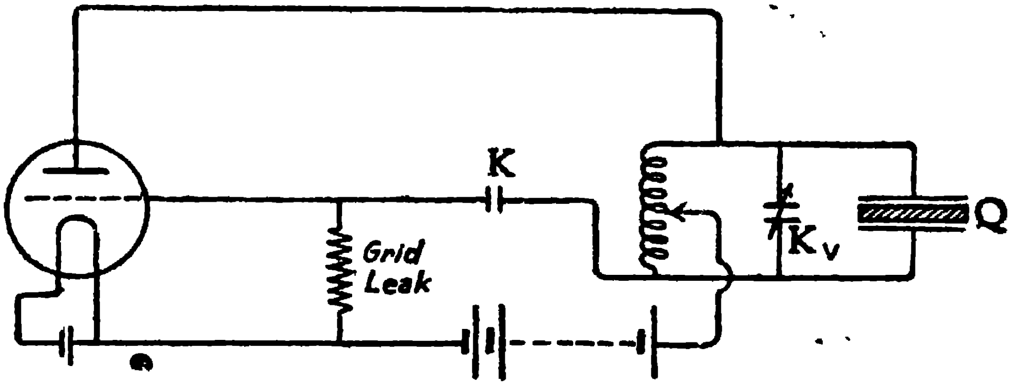


Fig. 105

135. Detection and Measurement of Ultrasonic Vibrations :—Ordinary acoustical methods cannot be directly used for the measurement and detection of supersonic waves. So special methods have been devised for the purpose. A few such methods are described below :—

(i) Mechanical Method.—The ultrasonic waves upto wave-length of a few millimeters may be studied by using lycopodium powder in a Kundt's tube.

(ii) Thermal Ultrsonic Receivers.—The transformation of sound energy into the electrical energy may be accomplished by the principle of hot wire microphones. Some ultrasonic receivers are also based on the same principle. Here the resistance of thin wires, gently heated by a current, changes when ultrasonic waves fall on the wires. The resistance change is noted by an ordinary Wheatstone's net work and the intensity of the incident waves can be calculated from that.

(iii) Electrical Sound Receivers.—This type of receiver is very sensitive and can detect waves of the highest frequencies. It consists of a crystal detector and a quartz vibrator both connected in parallel to a sensitive galvanometer. The intensity of the ultrasonic waves incident on the quartz crystal is proportional to the deflection of the galvanometer.

(iv) Optical Method.—

(a) Striation Method.—This is a method by which direct pictures of sound waves can be obtained. A circular slit S is strongly illuminated by light L with the help of a condensing lens K. A sharp image of S is produced at the point B by the objective O. A screen placed at B prevents light at B from entering into the space between the quartz oscillator Q and the reflector R. The light is neither allowed to enter the eye or the camera on the right-hand side of B. If a stationary ultrasonic wave train is generated in air in front of the objective by reflecting the waves sent out by the oscillator Q at the reflector R, definite nodes and antinodes will be formed in the medium. As a result the refractive index will change at these points. If a monochromatic light is used the pressure nodes will appear as dark bands and the antinodes as bright to an eye placed beyond B. From the photographic records of these bands their width can be measured and allowing for the magnification of the arrangement, λ can be determined from these measurements.

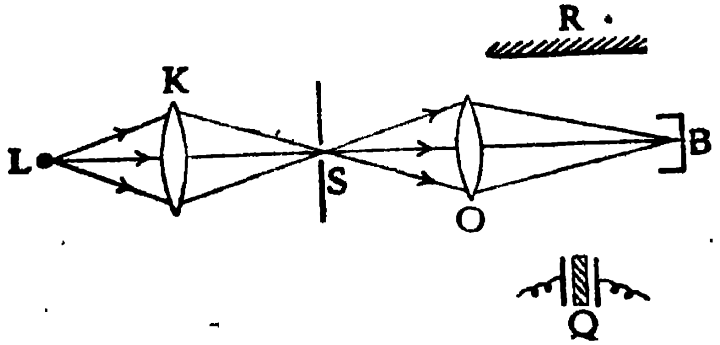


Fig. 106

(b) Diffraction Method.—Debye and Sears first showed that diffraction of light is possible in a liquid traversed by sound waves and that diffraction phenomena similar to those in a ruled grating may be observed. Raman and Nagendranath developed a complete theory of the phenomenon. This provides us with an easy and accurate method of determining the wave-lengths of supersonic waves in liquids and hence its velocity if the frequency is known.

Here the series of compressions and rarefactions which constitute ultrasonic waves in a liquid serve as an optical grating and cause diffraction of light. The method consists in allowing a parallel beam of monochromatic light to pass through a liquid at right angles to the ultrasonic waves in it. Looking along the beam a central image of the slit together with its diffraction images will be seen. If $\lambda_s =$ wave-length of the sound in the liquid, $\lambda_l =$ the wave-length of the monochromatic light and $\theta =$ the angle of the diffraction of the *n*th order, $\sin\theta = \theta = n\lambda_l/\lambda_s$. Hence λ_s can be determined. It is a very accurate method of determining λ of sound at ultrasonic frequency.

136. Supersonic Echo-Sounding :— The general principle of echo depth sounding has been described in Chapter III. But in the case of echo sounding with audible waves, the energy reflected from a pulse of audible waves is small unless the surface is sufficiently large. Moreover, in cases of submerged icebergs, the relative reflecting powers of ice with respect to sea water being very small, these cannot be correctly detected by ordinary echo sounding methods. For this reason Langevin used piezo-electric oscillators to produce high energy sound waves (supersonics) which could be successfully employed for the purpose of detecting icebergs. Frequencies of about 40,000 c.p.s. are produced by a quartz oscillator of thickness 1·8 cms. The wave-lengths of the supersonic waves so produced are of the order of 3·5 cms. The oscillator is not allowed to oscillate continuously but it emits damped waves at definite intervals which are regulated by an electric motor. The pulses reflected from the submerged icebergs or other objects are received back by the same quartz slab. These waves set the slab into vibrations thereby generating a small potential difference between its faces. The lag between this e.m.f. and the one initiating the supersonic waves is recorded by an oscillograph. The light spot on the screen of the oscillograph moves over a scale directly graduated in fathoms or metres from which the position or the depth of the reflector or the obstacle can be found.

137. Applications of Ultrasonics :—(*a*) In ultrasonic stroboscopes stationary ultrasonic waves are used to produce a high frequency modulation of light for stroboscopic purposes. (*b*) The scophony system of television for producing highly defined pictures is based on the modulation of ultrasonic waves in liquids. (*c*) In signalling, depth sounding, and in detecting sunken obstacles like icebergs, submarines, etc. ultrasonic waves are utilised. (*d*) Supersonics are also used for guiding ships from a harbour mouth. (*e*) They are also used for testing flaws or discontinuities in the structure of materials. (*f*) Two immiscible liquids may be made to form an emulsion by subjecting the same to ultrasonic vibrations. Wood and Loomis in 1927 succeeded in transforming water and oil or water and mercury into very stable emulsions with the help of intense ultrasonic waves. (*g*) They are also used in laundries for quick removal of dust particles from clothings. (*h*) The ultrasonic waves are found to increase the virulence of some types of bacteria while in some other types, the virulence is found to diminish. When exposed to ultrasonic waves yeast cells are found to lose their power of reproduction.

Questions (Chapter XII)

1. **Write a short note on ultrasonics. (Luck U. 1946 ; All. U. 1950)**
2. **Write a detailed note on 'ultrasonics', their importance and uses. (Bom. U. 1954)**
3. **Describe briefly the different methods of producing supersonic waves.**
4. **Give in short the different methods of detection of ultrasonic waves. How can the wave-lengths of these waves be measured ?**
5. **Describe the principle of supersonic echo-sounding. In what respect is supersonic echo-sounding superior to ordinary echo-sounding in the audible range ?**

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